8 Coupled Oscillations and Normal Coordinates

We have learned in some detail how a single vibrating system will behave. However, oscillators rarely exist in complete isolation. We now consider systems of interacting oscillators. This subject is of general interest because numerous physical systems are well approximated by coupled harmonic oscillators. Coupled oscillators, in general, are able to transmit their energy to each other because two oscillators share a common component, capacitance or stiffness, inductance or mass, or resistance. Resistance coupling inevitably brings energy loss and a rapid decay in the vibration, but nonresistance coupling consumes no power, and continuous energy transfer over many oscillators is possible.

We shall investigate first a mechanical example of stiffness coupling between two pendulums. Two atoms set in a crystal lattice experience a mutual coupling force and would be amenable to a similar treatment. Motion of this type can be quite complex if it is described in ordinary coordinates that describe the geometrical configuration of the system. Fortunately, as we shall see, it is always possible to describe the motion of any oscillatory system in terms of normal coordinates that are constructed from the original position coordinates in such a way that there is no coupling among the oscillators. Thus, each normal coordinate oscillates with a single, well-defined frequency. Before we take up the general analytic approach, let us illustrate the concepts of normal coordinates and normal frequencies with a very simple example: the coupled pendulum.

8.1 COUPLED PENDULUM

Consider a pair of identical pendulums of mass m suspended on a massless rigid rod of length b. The masses are connected by a massless spring whose spring constant is k and whose natural length equals the distance between the masses when either is displaced from equilibrium. The displacements from rest of the two pendulum bobs are measured in x and y, respectively, as indicated in Figure 8.1. The kinetic and potential energies for the coupled system are given by, for small displacement,

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{8.1a}$$

$$V = \frac{1}{2}K(x^2 + y^2) + \frac{1}{2}k(x - y)^2 \tag{8.1b}$$

where $K = mg/b$. It is clear that in this small displacement approximation the coupled pendulum system of Figure 8.1 is equivalent to the mass-spring system of Figure 8.2, with a spring constant K for the outer spring and k for the inner spring. Now, from the Lagrangian $L = T - V$ and Lagrange's equations $\mathrm{d}(\partial L/\partial x)/\mathrm{d}t - \partial L/\partial x = 0$, $\mathrm{d}(\partial L/\partial y)/\mathrm{d}t - \partial L/\partial y = 0$, we obtain the differential equations of motion:

$$m\ddot{x} = -Kx - k(x - y) \tag{8.2a}$$

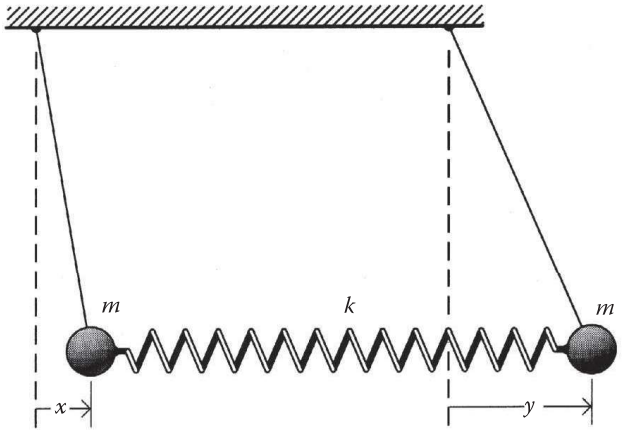

FIGURE 8.1 Two simple pendulums coupled by a spring.

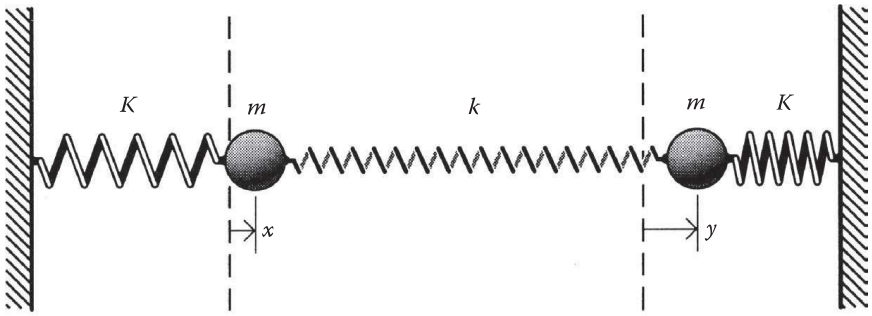

FIGURE 8.2 Equivalent mass–spring system for the coupled pendulum system.

$$m\ddot{y} = -Ky + k(x - y). \tag{8.2b}$$

It is obvious that Equation 8.2a contains a term in y, and Equation 8.2b contains a term in x. These two differential equations cannot be solved independently and must be solved simultaneously. A motion given to one bob affects the other. As a simple approach to the solution of these simultaneous differential equations, we try an oscillatory state in which both pendulums oscillate with the same frequency ω:

$$x(t) = Ae^{i\omega t} \text{ and } y(t) = Be^{i\omega t} \tag{8.3}$$

where A and B are constants. Because damping is not present, ω will be a real quantity. Substitution of Equation 8.3 into Equations 8.2a and 8.2b gives a pair of simultaneous linear algebraic equations in the undetermined amplitude A and B:

$$(-m\omega^2 + K + k)A - kB = 0 \tag{8.4a}$$

$$kA + (m\omega^2 - K - k)B = 0. \tag{8.4b}$$

Such is a system of two linear homogeneous equations whose solution is not zero only if the determinant of its coefficients vanishes

$$\begin{vmatrix} -m\omega^2 + K + k & -k \\ k & m\omega^2 - K - k \end{vmatrix} = 0. \tag{8.5}$$

Upon expanding this secular determinant, we have

$$(-m\omega^2 + K + k)^2 - k^2 = 0, \tag{8.6}$$

an equation for the determination of ω, which may be rewritten as

$$(m\omega^2 - K)(m\omega^2 - K - 2k) = 0$$

from which we obtain the characteristic frequencies (often called eigenfrequencies) for the system, either

$$\omega^2 = K/m = g/b; \quad \omega_{1,2} = \pm\sqrt{g/b} \tag{8.7a}$$

or

$$\omega^2 = g/b + 2k/m; \quad \omega_{3,4} = \pm\sqrt{g/b + 2k/m}\,. \tag{8.7b}$$

One of the characteristic frequencies is of the free pendulum $(g/b)^{1/2}$ exactly; the two pendulums move in phase. The other frequency reactivates both the pendulums and the springs (with a factor of 2); in this state, the pendulums move in opposite directions. The most general motion of the system is a superposition of these two modes of oscillation:

$$x(t) = A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t} + A_3 e^{i\omega_3 t} + A_4 e^{i\omega_4 t}$$

$$y(t) = B_1 e^{i\omega_1 t} + B_2 e^{i\omega_2 t} + B_3 e^{i\omega_3 t} + B_4 e^{i\omega_4 t}$$

where, as usual, only the real or only the imaginary part is to be taken. Only four of the eight arbitrary constants present in the preceding equations are independent. This follows because the differential equations are of the second order and are two in number. The ratio between A and B can be determined by inserting the allowed value of ω into Equation 8.4a or 8.4b:

$$\begin{aligned} &\text{at } \omega = \omega_{1,2} \qquad A_{1,2} = +B_{1,2} \\ &\text{at } \omega = \omega_{3,4} \qquad A_{3,4} = -B_{3,4} \end{aligned}$$

The complete solutions then become

$$x(t) = A_1 e^{i\sqrt{g/b}t} + A_2 e^{-i\sqrt{g/b}t} + A_3 e^{i\sqrt{g/b+2k/m}t} + A_4 e^{-i\sqrt{g/b+2k/m}t} \tag{8.8a}$$

$$y(t) = A_1 e^{i\sqrt{g/b}t} + A_2 e^{-i\sqrt{g/b}t} - A_3 e^{i\sqrt{g/b+2k/m}t} - A_4 e^{-i\sqrt{g/b+2k/m}t}. \tag{8.8b}$$

We now have only four arbitrary constants, A_1, A_2, A_3, and A_4, present in the general solution of the two second-order differential equations.

8.1.1 Normal Coordinates

It is very clear, by inspection of Equations 8.8a and 8.8b or Equations 8.4a and 8.4b that it is possible to make linear combinations of x and y such that a combination involves a single frequency. These linear combinations are, in our case, merely the sum and difference of x and y:

$$X(t) = x(t) + y(t) = 2\left[A_1 e^{i\sqrt{g/bt}} + A_2 e^{-i\sqrt{g/bt}} \right] \tag{8.9a}$$

$$Y(t) = x(t) - y(t) = 2\left[A_3 e^{i\sqrt{g/b+2k/mt}} + A_4 e^{-i\sqrt{g/b+2k/mt}} \right] \tag{8.9b}$$

where X describes one mode of oscillation with $\omega_0 = (g/b)^{1/2}$, while Y describes the other with $\omega = (g/b + 2k/m)^{1/2}$. The terms X and Y are called normal coordinates, and the two modes of oscillation, $(g/b)^{1/2}$ and $(g/b + 2k/m)^{1/2}$, are the corresponding normal modes. We see that a normal mode vibration involves only one dependent variable X (or Y) and has its own normal frequency.

Coordinates X and Y have no immediate geometric meaning. In order to learn something about the geometrical configuration of the system, we would have to transform back from (X, Y) to (x, y):

$$x = (X + Y)/2 \text{ and } y = (X - Y)/2. \tag{8.10}$$

However, the normal coordinates X and Y have some interesting and useful properties from the physical viewpoint:

(1) The equations of motion, when expressed in terms of normal coordinates, are linear equations with constant coefficients, and each contains but one dependent variable. That is, each of which has an equation of motion that is simple harmonic:

$$\ddot{X} = -(g/b)X, \quad \ddot{Y} = -(g/b + 2k/m)Y. \tag{8.11}$$

Thus, the normal coordinates are independent of each other in the sense that each normal oscillation may be excited while the other is at rest. If the boundary conditions are such that only one normal coordinate is excited and the other is zero initially, the latter will remain zero for all time.

(2) The total energy of the system, which in ordinary position coordinates is a mixed function of both x and y and their time derivatives, assumes in normal coordinates the following form:

$$E_{tot} = \frac{m\left(\dot{X}^2 + \dot{Y}^2\right)}{4} + \frac{1}{4}\left[\left(\frac{mg}{b}\right)X^2 + \left(\frac{mg}{b} + k\right)Y^2\right]. \tag{8.12}$$

The potential energy is now expressed as a sum of squares of the normal coordinates, multiplied by constant coefficients. There are no longer any cross terms present in E_{tot}. Thus, when one normal coordinate is excited while the other remains zero, there is no tendency for energy to pass from one normal coordinate to another, and the coordinate that is zero initially remains zero for all time.

From the preceding discussion, it is evident that transformation from ordinary position coordinates to normal coordinates leads to replacement of the actual system of interacting oscillators by a system of as many independent oscillators. It is very clear, by inspection of Equations 8.8a and 8.8b, that the motion of the pendulums is, in general, a superposition of the two normal modes of oscillation. To excite a single frequency (i.e., one normal mode), either A_1 and A_2 or A_3 and A_4 must be zero. These two possibilities correspond to the pendulum bobs swinging synchronously (in phase) with frequency $(g/b)^{1/2}$ or swinging anti-synchronously (out of phase) with frequency $(g/b + 2k/m)^{1/2}$ as depicted in Figure 8.3. Notice that the synchronous mode has the lower frequency and is a general result. In a complex system of linearly coupled oscillators, the mode having the highest degree of symmetry has the lowest frequency.

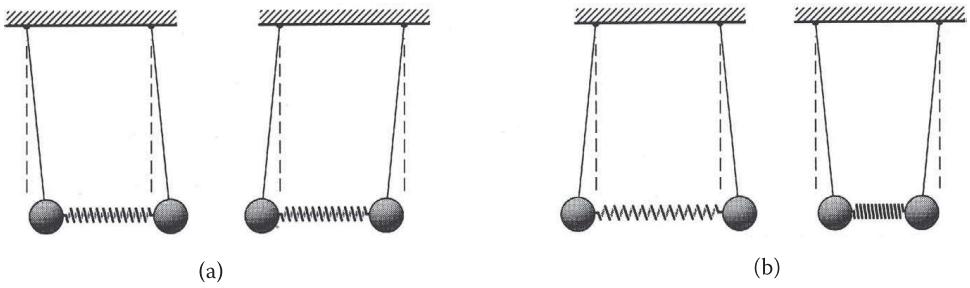

FIGURE 8.3 (a) The two pendulum swing in phase and (b) the two pendulum swing out of phase.

The general motion of the system is a linear combination of the synchronous and anti-synchronous modes (see Equations 8.8a and 8.8b). At the moment of excitation, $t = 0$, if $x = \dot{x} = 0$, $y = C$, and $\dot{y} = 0$ (i.e., we displace pendulum 2 a distance C and release it from rest). Equations 8.8a and 8.8b then give

$$A_1 = A_2 = C/4 \text{ and } A_3 = A_4 = C/4.$$

Substituting these results into Equations 8.8a and 8.8b, we find

$$x(t) = \frac{1}{2}C(\cos\omega_1 t - \cos\omega_3 t) \text{ and } y(t) = \frac{1}{2}C(\cos\omega_1 t + \cos\omega_3 t)$$

where ω_1 and ω_2 are given in Equations 8.7a and 8.7b. These equations may be put in the form

$$x(t) = C\sin\left[\frac{1}{2}(\omega_3 - \omega_1)t\right]\sin\left[\frac{1}{2}(\omega_3 + \omega_1)t\right] \tag{8.13a}$$

$$y(t) = C\cos\left[\frac{1}{2}(\omega_3 - \omega_1)t\right]\cos\left[\frac{1}{2}(\omega_3 + \omega_1)t\right]. \tag{8.13b}$$

Now, in the case of weak coupling,

$$\begin{aligned}\omega_3 - \omega_1 &= \sqrt{\omega_1^2 + 2k/m} - \omega_1 \\ &= \omega_1\sqrt{1 + (2k/m)\omega_1} - \omega_1 \\ &= \frac{k}{m}\omega_1 \ll 1.\end{aligned}$$

Thus, the first factors of the right members of Equations 8.13a and 8.13b vary slowly with time. These slowly varying factors constitute an envelope for the rapidly oscillating sinusoidal factors of argument as depicted in Figure 8.4. As the amplitude of $x(t)$ becomes large, that of y becomes smaller, and vice versa. Thus, there is a transfer of energy back and forth. At $t = \pi/(\omega_3 - \omega_1)$, pendulum 2 has come to rest, and all the energy has been transferred through the coupling to the first pendulum. It is this circumstance that gives rise to the phenomenon known as beats. The beat frequency is $(\omega_3 - \omega_1)/2$, and the frequency of the envelope of the amplitude is $(\omega_3 - \omega_1)$.

The preceding procedure can be applied to a system of oscillators with different masses; also, there is no limit to the number of oscillators constituting the system. For a system of n coupled oscillators, we will find n normal coordinates, $Q_1, Q_2, ..., Q_n$, each representing one normal mode of

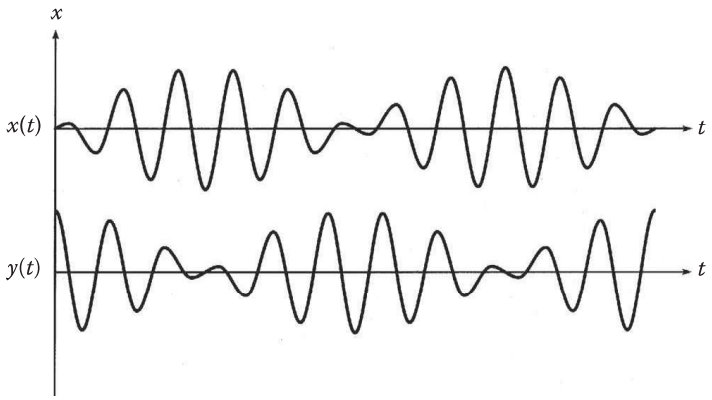

FIGURE 8.4 Oscillograph of two weakly coupled pendulums.

oscillation or one independent oscillator. We examine this general problem of n coupled oscillators in the next section.

8.2 COUPLED OSCILLATORS AND NORMAL MODES: GENERAL ANALYTIC APPROACH

For an oscillatory system with n degrees of freedom, the maximum number of normal frequencies is n. When some of the normal frequencies are identical, the system is said to be degenerate. We shall limit our discussion to the nondegenerate case.

In the simple example given earlier, the normal coordinates were introduced from symmetry consideration. Suppose it was not easy to discover the normal coordinates from symmetry consideration. How then could we plow through to a solution? Here we give a general analytic approach to this problem.

8.2.1 The Equation of Motion of a Coupled System

Consider a conservative system that has n degrees of freedom and a position of stable equilibrium where the potential energy of the system assumes a minimum value. We measure the generalized coordinates q_1, q_2,..., q_n of the system from this equilibrium position. If the motion of the system is of small amplitude and takes place about the point of equilibrium, then the potential energy V of the system can be expanded into a Taylor's series about the equilibrium configuration:

$$V(q_1,q_2,\ldots,q_n) = V_0 + \sum_i \left(\frac{\partial V}{\partial q_i}\right)_0 q_i + \frac{1}{2}\sum_i\sum_j \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 q_i q_j + \ldots. \tag{8.14}$$

Terms of higher orders of approximation have been dropped. The second sum vanishes because all the generalized forces, $(\partial V/\partial q_i)_0$, vanish at the equilibrium configuration. Also, no generality will be lost by taking $V_0 = 0$. Equation 8.14 now reduces to the quadratic form

$$V = \frac{1}{2}\sum_i\sum_j K_{ij} q_i q_j \tag{8.15}$$

where

$$K_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0. \tag{8.16}$$

Because V is measured from its minimum value, and this minimum is taken to be zero, we must have, in general, $V > 0$. That is, Equation 8.15 is a positive-definite quadratic form. The K_{ij} are constants and symmetrical ($K_{ij} = K_{ji}$) because the second derivatives are evaluated at the equilibrium position and the order of differentiation is immaterial under our assumption. The diagonal components of K_{ij} represent the force constants of the restoring force acting on a single particle when that particle alone is displaced.

If the constraints are time independent, kinetic energy T can be written as a homogeneous quadratic form in the velocities

$$T = \frac{1}{2}\sum_i \sum_j m_{ij}\dot{q}_i\dot{q}_j \tag{8.17}$$

where the m_{ij} are, in general, functions of the generalized coordinates and contain the masses, or other inertial parameters, of the system. We may expand the quantity m_{ij} into a Taylor's series about the equilibrium values of the q_i in a manner similar to that shown in Equation 8.14. As the q_i are assumed to be small, we shall take the constant values of m_{ij} at the equilibrium position as an approximation and neglect all the higher-order terms in the expansion. Call these constants A_{ij}; then Equation 8.17 becomes

$$T = \frac{1}{2}\sum_i \sum_j A_{ij}\dot{q}_i\dot{q}_j \tag{8.18}$$

where $A_{ij} = A_{ji}$. It is clear that T is a positive-definite quadratic form. The Lagrangian of the system is

$$L = T - V = \frac{1}{2}\sum_i \sum_j (A_{ij}\dot{q}_i\dot{q}_j - K_{ij}q_iq_j). \tag{8.19}$$

The equations of motion follow from the Lagrange's equation $\dfrac{d}{dt}\dfrac{\partial L}{\partial q_i} - \dfrac{\partial L}{\partial q_i} = 0$

$$\sum_i (A_{jk}\ddot{q}_k + K_{jk}q_k) = 0,\ j = 1,2,\ldots,n \tag{8.20}$$

where the terms $A_{jk}\ddot{q}_k$ and $K_{jk}q_k$ are called dynamical and static coupling terms, respectively.

Equation 8.20 is a system of n second-order linear homogeneous differential equations with constant coefficients, and each of these equations involves all n coordinates. So it is desirable to separate the variables and obtain n equations, each only involving a single unknown. This means that we introduce normal coordinates.

8.2.2 Normal Modes of Oscillation

Our first step is to find a set of constants C_j. If we multiply the first of Equation 8.20 by C_1, the second by C_2, and so on, and then add the resulting equations, we obtain a new set of equations of the form

$$\ddot{\eta}_j + \omega^2\eta_j = 0,\ j = 1,2,\ldots,n \tag{8.21}$$

where η_j are linear combinations of q_j:

$$\eta_j = \sum_k h_{jk} q_k. \tag{8.22}$$

The various constants are related by the sets of equations

$$\sum_j C_j A_{jk} = \omega^2 \sum_j C_j K_{jk} = h_{jk}. \tag{8.23}$$

From the last equation, we have

$$\sum_k (\omega^2 A_{jk} - K_{jk}) C_k = 0,\ j = 1, 2, \ldots, n \tag{8.24}$$

which is a system of n linear homogeneous algebraic equations with the unknowns C_1, C_2,..., C_n. They have a trivial solution $C_1 = C_2 = \ldots = C_n$ and a nontrivial solution given by, as before, setting the determinant of its coefficients equal to zero:

$$\begin{bmatrix} K_{11} - \omega^2 A_{11} & K_{12} - \omega^2 A_{12} & . & . & . & K_{1n} - \omega^2 A_{1n} \\ K_{21} - \omega^2 A_{21} & K_{22} - \omega^2 A_{22} & . & . & . & K_{2n} - \omega^2 A_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ K_{n1} - \omega^2 A_{n1} & K_{n2} - \omega^2 A_{n2} & . & . & . & K_{nn} - \omega^2 A_{nn} \end{bmatrix} = 0. \tag{8.25}$$

It is evident that if we assume that the solution of the equation of motion, Equation 8.20, is (compared with Equation 8.3)

$$q_j(t) = C_j e^{i(\omega t - \delta)}$$

then, by substitution of q_j into Equation 8.20, the resulting equations will take the same form as Equation 8.24.

Now the determinant of Equation 8.25 is an nth-degree equation in ω that can be solved for the n roots: ω_1^2, ω_2^2,..., ω_n^2. It can be proved that these roots are all real. To accomplish this, we multiply through the jth equation of Equation 8.24 by C_j and sum over j, which gives

$$\omega^2 \sum_j \sum_k A_{jk} C_j C_k - \sum_j \sum_k K_{jk} C_j C_k = 0. \tag{8.26}$$

We notice that the double sum in the first term is the function $2T$ of Equation 8.18 with $\dot{q}_j$ and $\dot{q}_k$ replaced by C_j and C_k. The double sum in the second term is the function $2U$ of Equation 8.15, with C_j and C_k replacing q_1 and q_k. And Equation 8.26 may thus be written in the form

$$\omega^2 T(\mathrm{C}) - V(C) = 0$$

or

$$\omega^2 = T(\mathrm{C})/V(C). \tag{8.27}$$

Here, T and V are positive for all sets of values of the variables, and accordingly, ω^2 is real and positive.

For the nondegenerate case, Equation 8.25 has n different real positive roots ω_j^2, $j = 1, 2,..., n$. This means that there are n different modes of oscillation with frequencies given by ω_j. ω_j are the natural frequencies of the system and are called the characteristic frequencies or eigenfrequencies of the system. Each value of ω can then be used, in turn, in Equation 8.24 to calculate a set of C_j. For a given value of ω_j, $(n - 1)$ of the C_j can be determined in terms of the nth one from Equation 8.24. The value for C_n must then be determined arbitrarily. Each set of C_j can be considered to define the components of an n-dimensional vector, called an eigenvector, of the system. We designate C_k as the eigenvector associated with the eigenfrequency ω_k and C_{kj} the kth component of the jth eigenvector. Accordingly, the factor C_i in Equations 8.23 and 8.24 should now be replaced by C_{ir}:

$$\sum_i C_{ir}A_{ij} = \omega_r^{-2}\sum_i C_{ir}K_{ij} = h_{jr} \tag{8.23a}$$

and

$$\sum_k (K_{jk} - \omega_r^2 A_{jk})C_{kr} = 0,\ j = 1,2,\ldots,n. \tag{8.24b}$$

The general motion of the system is a linear combination of the normal modes:

$$q_j(t) = \sum_k C_{jk}e^{i(\omega_k t - \delta_k)},\ j = 1,2,\ldots,n \tag{8.28a}$$

or, passing over to the real part of this expression,

$$q_j(t) = \sum_k C_{jk}\cos(\omega_k t - \delta_k),\ j = 1,2,\ldots,n. \tag{8.28b}$$

Thus, in general, a given coordinate will depend on C_{jk} and the frequencies of all modes of oscillation at the same time. In certain special circumstances, the system may only oscillate at one of the characteristic frequencies alone. When this happens, we speak of a normal mode of oscillation.

8.2.3 Orthogonality of Eigenvectors

There is a fundamental property of the eigenvector $\vec{C}_j$ that is important in determining the values of its components C_{jk} from the initial conditions. To investigate this fundamental property of the eigenvector $\vec{C}_j$, let us return to Equation 8.24, which gives, for the sth root ω_s,

$$\omega_s^2\sum_k A_{jk}C_{ks} = \sum_k K_{jk}C_{ks}. \tag{8.29a}$$

Interchanging j and k and replacing s by r, we obtain a comparable equation for the rth root of ω_r:

$$\omega_r^2\sum_k A_{jk}C_{jr} = \sum_k K_{jk}C_{jr} \tag{8.29b}$$

where use has been made of the symmetry of A_{jk} and C_{jk}.

First, multiplying Equation 8.29a by C_{jr} and summing over j, multiplying Equation 8.29b by C_{ks} and summing over k, and then subtracting the resulting equations, we obtain

$$(\omega_r^2 - \omega_s^2)\sum_{j,k} A_{jk}C_{jr}C_{ks} = 0. \tag{8.30}$$

For the case $\omega_r \neq \omega_s$, the sum must be zero identically:

$$\sum_{j,k} A_{jk}C_{jr}C_{ks} = 0. \tag{8.31}$$

The preceding condition is generally called the "orthogonality condition." When $\omega_r = \omega_s$, the factor $(\omega_r^2 - \omega_s^2)$ in Equation 8.30 vanishes, and the sum is indeterminate. But it can be shown that the sum is, in general, positive. To show this, we return to Equation 8.18 for the kinetic energy of the system and Equation 8.28 for q_j:

$$\begin{aligned} T &= \frac{1}{2}\sum_{j,k} A_{jk}\dot{q}_j\dot{q}_k \\ &= \frac{1}{2}\sum_{j,k} A_{jk}\left[\sum_r \omega_r C_{jr}\cos(\omega_r t - \delta_r)\right]\left[\sum_s \omega_s C_{ks}\cos(\omega_s t - \delta_s)\right] \end{aligned}$$

or

$$T = \frac{1}{2}\sum_{r,s}\omega_r\omega_s\cos(\omega_r t - \delta_r)\cos(\omega_s t - \delta_s)\sum_{j,k} A_{jk}C_{jr}C_{ks}.$$

For $\omega_r = \omega_s$, this expression becomes

$$T = \frac{1}{2}\sum_r \omega_r^2\cos^2(\omega_r t - \delta_r)\sum_{j,k} A_{jk}C_{jr}C_{kr}. \tag{8.32}$$

Now T is positive and can become zero only if all of the velocities vanish, and we note that

$$\omega_r^2\cos^2(\omega_r t - \delta_r) \geq 0.$$

Thus, in general, we have

$$\sum_{j,k} A_{jk}C_{jr}C_{kr} \geq 0 \tag{8.33}$$

where the equal sign applies when $T = 0$.

As remarked earlier, $(n - 1)$ of the C_{jr} can be obtained in terms of the nth one from Equation 8.24 when the ω_r are known. The value for the nth one must then be determined arbitrarily. We now remove this indeterminacy by choosing C_{jr} so that the eigenvector $\vec{C}_r$ has unit length:

$$\sum_{j,k} A_{jk}C_{jr}C_{kr} = 1. \tag{8.34}$$

Equation 8.34 is generally called the normalization condition. The orthogonality condition 8.31 and normalization condition 8.34 are often combined into one statement by the use of the Kronecker delta symbol δ_{ij}:

$$\sum_{j,k} A_{jk} C_{jr} C_{ks} = \delta_{rs}. \tag{8.35}$$

The vector $\vec{C}_r$ defined in this way constitutes an orthonormal set: the set is orthogonal according to Equation 8.31 and has been normalized according to Equation 8.34. The orthogonality and normalization conditions of Equation 8.35 are of more than academic interest; as we shall see, they prove useful in establishing the constants resulting from the initial conditions.

8.2.4 Normal Coordinates

The η_j given by Equation 8.22 are the desired normal coordinates. In terms of these coordinates, the small oscillations of the system are described by a set of harmonic oscillators of frequencies ω_j. Each oscillation can be excited independently. Once we know C_j, h_{jk} are readily obtained from Equation 8.23a. Finally, η_j follows from Equation 8.22. In terms of normal coordinates, the equations of motion are completely separated, as shown by Equation 8.21.

The normal coordinates are often introduced through the use of Equation 8.28 instead of Equation 8.20. However, because we have normalized C_{jr} according to Equation 8.34, we need to introduce a constant scale factor α that will depend on the initial conditions of the problem to account for the loss of generality that has been introduced by the arbitrary normalization. Thus,

$$q_j(t) = \sum_k \alpha C_{jk} e^{i(\omega_k t - \delta_k)} \tag{8.36}$$

or

$$q_j(t) = \sum_k \beta_k C_{jk} e^{i\omega_k t} \tag{8.36a}$$

where $\beta_k = \alpha e^{-i\delta_k}$, the new scale factors. The normal coordinates η_k are defined by the following relationships:

$$\eta_k(t) = \beta_k e^{i\omega_k t}. \tag{8.37}$$

It is evident that η_k undergoes oscillation at only one frequency. In terms of these new quantities, Equation 8.36a becomes

$$q_j(t) = \sum_k C_{jk} \eta_k(t). \tag{8.38}$$

Equation 8.38 can be considered as the inverse transformation of Equation 8.22.

The kinetic and potential energies assume the simple form

$$T = \frac{1}{2} \sum_j \dot{\eta}_j^2, \quad V = \frac{1}{2} \sum_j \omega_j^2 \, \eta_j^2. \tag{8.39}$$

To show these, we first note that $\dot{q}_j(t) = \sum_k C_{jk}\dot{\eta}_k(t)$. Equation 8.18 becomes

$$T = \frac{1}{2}\sum_{j,k} A_{jk}\dot{q}_j\dot{q}_k = \frac{1}{2}\sum_{j,k} A_{jk}\left(\sum_r C_{jr}\dot{\eta}_r\right)\left(\sum_s C_{ks}\dot{\eta}_s\right)$$

from which we obtain, by rearrangement of summations,

$$T = \frac{1}{2}\sum_{r,s}\left(\sum_j\sum_k A_{jk}C_{jr}C_{ks}\right)\dot{\eta}_r\dot{\eta}_s$$

$$= \frac{1}{2}\sum_{r,s}(\delta_{rs})\dot{\eta}_r\dot{\eta}_s \quad \text{(with the aid of Equation 8.35)}$$

$$= \frac{1}{2}\sum_r \dot{\eta}_r^2, \qquad \text{Q.E.D.}$$

In terms of η_r, Equation 8.15 becomes

$$V = \frac{1}{2}\sum_{j,k} K_{jk}q_jq_k = \frac{1}{2}\sum_{r,s}\left(\sum_{j,k} K_{jk}C_{jr}C_{ks}\right)\eta_r\eta_s.$$

Now, multiplying Equation 8.29a by C_{jr} and summing over j, we obtain

$$\omega_s^2\sum_{j,k} A_{jk}C_{ks}C_{jr} = \sum_{j,k} K_{jk}C_{ks}C_{jr}.$$

Substituting this into the expression for U, we obtain

$$V = \frac{1}{2}\sum_{r,s}\left(\sum_{j,k} K_{jk}C_{jr}C_{ks}\right)\eta_r\eta_s = \frac{1}{2}\sum_{r,s}\left(\omega_s^2\sum_{j,k} A_{jk}C_{jr}C_{ks}\right)\eta_r\eta_s$$

$$= \frac{1}{2}\sum_{r,s}\left(\omega_s^2\delta_{rs}\right)\eta_r\eta_s = \frac{1}{2}\sum_r \omega_r^2\eta_r^2.$$

The Lagrangian L takes the simple form

$$L = T - V = \frac{1}{2}\sum_r\left(\dot{\eta}_r^2 - \omega_r^2\eta_r\right) \tag{8.40}$$

and the Lagrange's equation gives Equation 8.21:

$$\ddot{\eta}_r + \omega_r\eta_r = 0.$$

We now apply the general formulation just developed to the coupled pendulum.

Example 8.1: Coupled Pendulum Revisited

1. Eigenfrequencies
   The eigenfrequencies for the two-coupled pendulums are given by Equation 8.25. Thus, we must first calculate the quantities K_{ij} and A_{ij}. The potential energy of the system is given by Equation 8.15:

$$V = \frac{1}{2}K(x_1^2 + x_2^2) + \frac{1}{2}k(x_1 - x_2)^2 = \frac{1}{2}(K + k)(x_1^2 + x_2^2) - kx_1x_2 \tag{8.41}$$

   where we have replaced the original coordinates x and y by x_1 and x_2. From Equation 8.16,

$$K_{ij} = \left.\frac{\partial^2 U}{\partial x_1 \partial x_2}\right|_0, \quad i, j = 1, 2$$

   we obtain

$$\left.\begin{aligned} K_{11} &= \left.\frac{\partial^2 V}{\partial x_1^2}\right|_0 = K + k \\ K_{12} &= \left.\frac{\partial^2 V}{\partial x_1 \partial x_2}\right|_0 = -k = K_{21} \\ K_{22} &= \left.\frac{\partial^2 V}{\partial x_2^2}\right|_0 = K + k. \end{aligned}\right\} \tag{8.42}$$

   The kinetic energy of the system is given by Equation 8.1a:

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right). \tag{8.43}$$

   According to the general formalism, we have

$$T = \frac{1}{2}\sum_{i,j} A_{ij}\dot{x}_i\dot{x}_j. \tag{8.44}$$

   Comparing these two expressions for T, we find

$$A_9 = A_{22} = m \text{ and } A_{12} = A_{21} = 0. \tag{8.45}$$

   Substituting the results of Equations 8.42 and 8.45 into Equation 8.25, we obtain

$$\begin{vmatrix} K + k - m\omega^2 & -k \\ -k & K + k - m\omega^2 \end{vmatrix} = 0. \tag{8.46}$$

   This is exactly Equation 8.5, and the solutions are the same as those given by Equations 8.7a and 8.7b.
2. Eigenvectors
   We return to Equation 8.24a to determine the eigenvector components C_{jr}:

$$\sum_k (K_{jk} - \omega_r^2 A_{jk}) C_{kr} = 0. \tag{8.24a}$$

For $r = 1$ and $j = 1$, we have

$$(K_{11} - \omega_1^2 A_{11}) C_{11} + (K_{12} - \omega_1^2 A_{12}) C_{21} = 0. \tag{8.47}$$

Substituting $\omega_1^2 = K/m$ (from Equation 8.7a) and A_9 and K_9, and so forth, from Equations 8.42 and 8.45, we find

$$C_9 = -C_{21}. \tag{8.48}$$

The orthonormality condition can be used to determine C_9 and C_{21}. For the coupled pendulum, the orthonormality condition

$$\sum_{j,k} A_{jk} C_{jr} C_{ks} = \delta_{rs}$$

reduces to

$$\sum_{j,k} m\delta_{jk} C_{jr} C_{ks} = m \sum_j C_{jr} C_{js} = \delta_{rs}. \tag{8.49}$$

For $r = s = 1$, we have

$$C_{11}^2 + C_{21}^2 = 1/m. \tag{8.50}$$

Solving Equations 8.48 and 8.50, we find

$$C_{11} = -C_{21} = \sqrt{1/2m}. \tag{8.51}$$

For $r \neq s$, Equation 8.49 becomes

$$\sum_j m C_{jr} C_{js} = 0 = \sqrt{2m}(C_{12} - C_{22}) \tag{8.52}$$

from which we find

$$C_{12} = C_{22}. \tag{8.53}$$

For $r = s$, Equation 8.49 gives

$$C_{12}^2 + C_{22}^2 = 1/m. \tag{8.54}$$

Solving Equation 8.53 and 8.54, we obtain

$$C_{12} = C_{22} = \sqrt{1/2m}. \tag{8.55}$$

3. Normal coordinates
We first calculate the quantities $h_{jk}\left(h_{jk} = \sum_i C_{ik} A_{ij}\right)$ and find

$$h_{11} = C_{11}A_{11} + C_{21}A_{21} = \sqrt{m/2}$$
$$h_{21} = C_{11}A_{12} + C_{21}A_{22} = -\sqrt{m/2}$$
$$h_{12} = C_{12}A_{11} + C_{22}A_{21} = \sqrt{m/2}$$
$$h_{22} = C_{12}A_{12} + C_{22}A_{22} = \sqrt{m/2}.$$

The normal coordinates η_1 and η_2 are given by

$$\eta_1 = h_{11}x_1 + h_{12}x_2 = \sqrt{m/2}(x_1 + x_2). \tag{8.56}$$

$$\eta_2 = h_{21}x_1 + h_{22}x_2 = \sqrt{m/2}(-x_1 + x_2). \tag{8.57}$$

They differ from Equations 8.9a and 8.9b by a constant factor.

Example 8.2: Longitudinal Vibrations of a CO_2 Molecule

Atoms in polyatomic molecules behave as the masses of our pendulum; the method of normal coordinates may be applied to the study of molecular vibrations—often with rich rewards. The carbon dioxide molecule that has the chemical structure O–C–O provides a simple example. We can regard this system as equivalent to a set of three particles joined by elastic springs (Figure 8.5). Clearly, the system will vibrate in some manner when subjected to an external force. For simplicity, we shall consider only longitudinal vibrations, and the interaction of the oxygen molecules with one another will be neglected (so we consider only the nearest neighbor interactions). Our coordinates are x_1, x_2, and x_3. The Lagrangian function L for the system is clearly

$$L = \frac{1}{2}m(\dot{x}_1^2 + x_3^2) + \frac{1}{2}M\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2. \tag{8.58}$$

The equations of motion are found to be

$$\begin{aligned} m\ddot{x}_1 - k(x_2 - x_1) &= 0 \\ M\ddot{x}_2 + k(x_2 - x_1) - k(x_3 - x_2) &= 0 \\ m\ddot{x}_3 + k(x_3 - x_2) &= 0. \end{aligned} \tag{8.59}$$

Let

$$x_j = A_j \cos(\omega t - \alpha),\ j = 1, 2, 3. \tag{8.60}$$

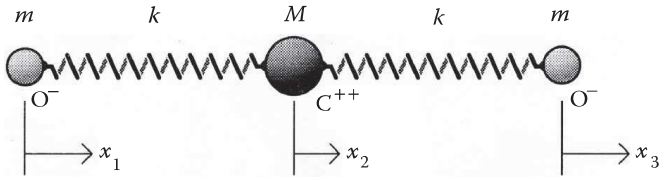

FIGURE 8.5 Linear symmetrical CO_2 molecule.

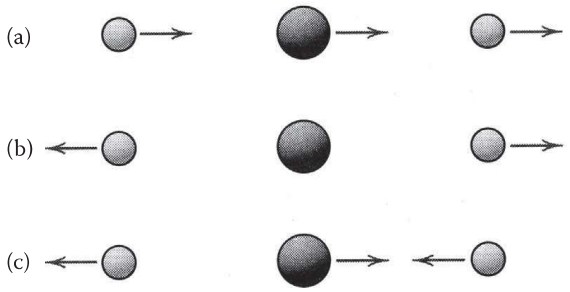

FIGURE 8.6 Longitudinal vibrations of a CO_2 molecule, (a) A pure translation of the system as a whole, (b) with the center mass at rest, the two end masses vibrate in opposite direction, and (c) the two end masses vibrate in unison, the center mass vibrates oppositely.

Substituting Equation 8.60 into Equation 8.59, we obtain

$$\begin{aligned}(-m\omega^2+k)A_1-kA_2&=0\\ -kA_1+(-M\omega^2+2k)A_2-kA_3&=0\\ -kA_2-(-m\omega^2+k)A_3&=0.\end{aligned}\tag{8.61}$$

The secular equation is

$$\begin{vmatrix}-m\omega^2+k & -k & 0\\ -k & -M\omega^2+2k & -k\\ 0 & -k & -m\omega^2+k\end{vmatrix}=0\tag{8.62}$$

or

$$\omega^2(-m\omega^2+k)(-mM\omega^2+kM+2km)=0\tag{8.62a}$$

from which we find the normal frequencies of the system

$$\omega_1=0,\quad \omega_2=\sqrt{k/m},\quad \omega_3=\sqrt{k/m+2k/M}.\tag{8.63}$$

1. Setting $\omega = 0$ in Equation 8.61, we find that $A_1 = A_2 = A_3$. Thus, this mode is no oscillation at all but is a pure translation of the system as a whole (Figure 8.6a).
2. Setting $\omega=\sqrt{k/m}$ in Equation 8.61, we find $A_2 = 0$ and $A_1 = -A_3$. Thus, the center mass M is at rest, while the two end masses vibrate in opposite directions with the same amplitude (Figure 8.6b).
3. Setting $\omega = (k/m + 2k/M)^{1/2}$ in Equation 8.61, we find $A_1 = A_3$ and $A_2 = -2A_1(m/M) = -2A_3(m/M)$. Thus, in this mode, the two end masses vibrate in unison, and the center mass vibrates oppositely with a different amplitude (Figure 8.6c).

8.3 FORCED OSCILLATIONS OF COUPLED OSCILLATORS

We shall find the transformation from the q-coordinates to the η-coordinates also useful in discussing the forced vibrations of coupled systems. Suppose that, in addition to the restoring forces that we have already considered in earlier sections, the component masses of the system are also acted upon by generalized forces $Q_1, Q_2,\ldots, Q_n$. In order to handle the resulting motion in terms of the normal coordinates, we must first find the generalized forces $R_1, R_2,..., R_n$ in the normal coordinate system during the displacements $\eta_1, \eta_2,\ldots, \eta_n$. We do this by setting up the work done in both coordinate systems. The work done, Q_iq_i by $Q_1, Q_2,\ldots, Q_n$ in displacements $q_1, q_2,..., q_n$ must be the same as the

work done by a corresponding set of generalized forces $R_1, R_2,..., R_n$ in the normal coordinate system during the displacements $\eta_1, \eta_2,\ldots, \eta_n$:

$$\sum_{i=1}^{n} Q_i q_i = \sum_{j=1}^{n} R_j \eta_j. \tag{8.64}$$

Substituting its value in terms of C_{ij} and η_j for each q_i from Equation 8.38 and equating terms in η_j, we obtain

$$R_j = \sum_{i=1}^{n} C_{ij} Q_i,\ j = 1,2,\ldots,n \tag{8.65}$$

for the jth mode of oscillation. We notice from Equation 8.65 that a force applied to a particle cannot excite a mode in which that particle does not oscillate but will be very effective in exciting a mode in which it oscillates strongly. In terms of R_j, the Lagrange's equation of motion gives

$$\ddot{\eta}_j + \omega_j^2 \eta_j = R_j. \tag{8.66}$$

We can use all the methods learned in Chapter 7 in discussing the motion of each of the normal coordinates. As an example, consider that the forces Q_i all depend on time t through a factor $\exp(i\omega t)$, where ω is arbitrary. Then R_j will likewise vary according to this exponential. To solve Equation 8.66, we assume that η_j varies in the same way, and we find, as in Chapter 7,

$$\eta_j = \frac{R_j}{\omega_j^2 - \omega^2} \tag{8.67}$$

and

$$q_i = \sum_j \frac{C_{ij} R_j}{\omega_j^2 - \omega^2}. \tag{8.68}$$

Thus, the phenomenon of resonance occurs in the forced motion of coupled systems just as it does in the case of a single forced oscillator. The essential difference is that there are now n separate resonance points, corresponding to the n frequencies ω_j of the normal modes.

The forces Q_i could be damping forces. For example, as a result of friction, there will be a force $-c_i \dot{q}_i$ acting on the component mass m_i, where c_i is a constant. Then

$$R_j = -\sum_i C_{ij} c_i \dot{q}_i = -\sum_i C_{ij} c_i \sum_k C_{ik} \dot{\eta}_k = -\sum_k \gamma_{jk} \dot{\eta}_k \tag{8.69}$$

where

$$\gamma_{jk} = \sum_i c_i C_{ij} C_{ik}. \tag{8.70}$$

The equations of motion now take the form

$$\ddot{\eta}_j + \omega_j^2 \eta_j = -\sum_k \gamma_{jk} \dot{\eta}_k. \tag{8.71}$$

These equations cannot be separated if all the overtones are excited, and in this case, we cannot have a general solution. A simple solution exists only if one overtone is excited, say, the *j*th, for then all the η's except η_j are zero, and Equation 8.71 becomes

$$\ddot{\eta}_j + \gamma_{jj} \dot{\eta}_j + \omega_j^2 \eta_j = 0 \tag{8.72}$$

which is of the same form as Equation 8.29 for a damped oscillator.

It is evident that for forced oscillations of coupled damped oscillators, we have the following equations of motion:

$$\ddot{\eta}_j + \gamma_{jj} \dot{\eta}_j + \omega_j^2 \eta_j = R_j - \sum_{k \neq j} \gamma_{jk} \dot{\eta}_k. \tag{8.73}$$

We shall not go on to treat this problem in detail.

8.4 COUPLED ELECTRIC CIRCUITS

The examples we have been taking up so far have related to coupled mechanical oscillators. We shall see that methods developed earlier apply to other coupled systems, such as electric circuits. When a *LC* circuit is brought close to another similar electric circuit, oscillations in the one can be induced in the other as shown in Figure 8.7a. Figure 8.7b and c shows two different coupling schemes. For the purposes of illustration, we shall analyze the circuit in Figure 8.7a. Kirchhoff's circuit rules require that, in traversing each circuit completely, the total voltage drop be zero:

$$L\dot{I}_1 + q_1/C + M\dot{I}_2 = 0 \tag{8.74a}$$

$$L\dot{I}_2 + q_2/C + M\dot{I}_1 = 0. \tag{8.74b}$$

Differentiating with respect to time, we obtain

$$L\ddot{I}_1 + I_1/C + M\ddot{I}_2 = 0 \tag{8.75a}$$

$$L\ddot{I}_2 + I_2/C + M\ddot{I}_1 = 0. \tag{8.75b}$$

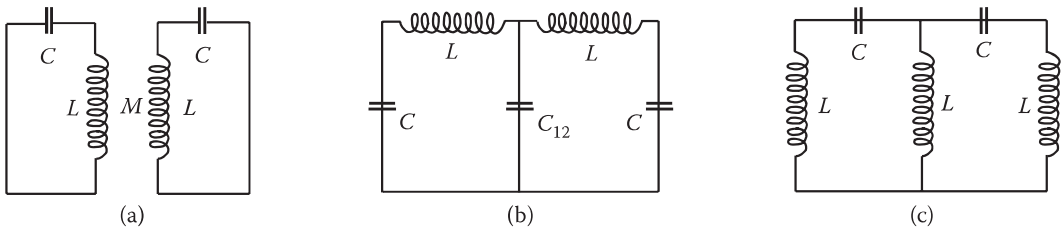

FIGURE 8.7 Coupled electric oscillator, (a) Inductive coupling, via mutual inductance *M*, (b) Capacitive coupling, via C_{12}, and (c) Inductive coupling, via L_{12}.