5.7 PHASE SPACE AND LIOUVILLE'S THEOREM

Hamilton's canonical formalism is of no great assistance in getting an actual solution of the trajectories of the particles, but it is very valuable in picturing the motion of a dynamic system in phase space. The motion in phase space takes place along a surface given by $H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) =$ energy (constant). As an example, the motions in phase space for the harmonic oscillators are along the elliptical curves (Figure 5.3):

$$\frac{p^2}{2m} + \frac{x^2}{(2/k)} = E.$$

We could picture the motion of a dynamic system in an n-dimensional configuration space of the q_n's in the Lagrangian scheme. We are often interested in the motion of a group of systems having different initial conditions. At any point q_n, there are many possible curves intersecting with different values of the velocity at that point. This is not the case in the Hamiltonian scheme, where there is only one possible path through each phase point because for given $2n$ initial conditions $q_j(0)$ and $p_j(0)$, the solution of the Hamilton's equations of motion is uniquely determined.

An important property of the $2n$-dimensional phase space is described by Liouville's theorem, which states that the phase points move like an incompressible fluid. More precisely, the phase volume occupied by a set of phase points is constant.

To prove Liouville's theorem, we shall first define a few terms, such as phase volume. The product of differentials $\mathrm{d}V = \mathrm{d}q_1 \ldots \mathrm{d}q_n\, \mathrm{d}p_1 \ldots \mathrm{d}p_n$ may be regarded as an element of volume in phase space. If ρ is the density of representative points in phase space, then $N = \rho\, \mathrm{d}V$ is the number of representative points within the element volume $\mathrm{d}V$. We next consider an element of area in the q_j and p_j planes in phase space (Figure 5.4). The number of representative points moving across the left-hand edge into the area per unit time is

$$\rho \dot{q}_j \mathrm{d}p_j.$$

By a Taylor series expansion, the number of representative points moving out of the area through its right-hand edge is

$$\left[\rho \dot{q}_i + \frac{\partial}{\partial q_i}(\rho \dot{q}_i)\mathrm{d}q_i\right]\mathrm{d}p_i.$$

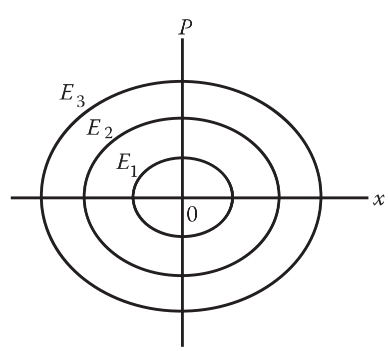

FIGURE 5.3 Phase space for the harmonic oscillators.

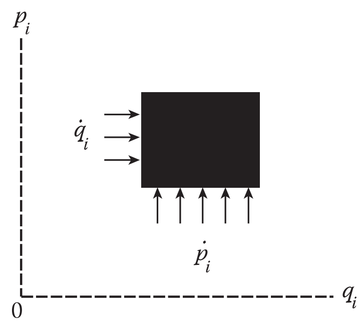

FIGURE 5.4 Element area in phase space.

Hence, the net increase in ρ in the element $dq_i dp_i$ because of flow in the horizontal direction is

$$-\frac{\partial}{\partial q_i}(\rho \dot{q}_i) dq_i dp_i.$$

In a similar way, we find the net gain resulting from flow in the vertical direction to be

$$-\frac{\partial}{\partial p_i}(\rho \dot{p}_i) dq_i dp_i.$$

The total increase in density in the element $dq_i dp_i$ per unit time is, therefore,

$$-\left[\frac{\partial}{\partial q_i}(\rho \dot{q}_i) + \frac{\partial}{\partial p_i}(\rho \dot{p}_i)\right] dq_i dp_i.$$

This should equal the net changes in ρ in $dq_i dp_i$ per unit time, which is

$$(\partial \rho / \partial t)\, dq_i dp_i.$$

Summing over all possible values of i, we find

$$\frac{\partial \rho}{\partial t} + \sum_i \left[\frac{\partial}{\partial q_i}(\rho \dot{q}_i) + \frac{\partial}{\partial p_i}(\rho \dot{p}_i)\right] = 0$$

or

$$\frac{\partial \rho}{\partial t} + \sum_i \left(\dot{p}_i \frac{\partial \rho}{\partial p_i} + \dot{q}_i \frac{\partial \rho}{\partial q_i}\right) + \rho \sum_i \left(\frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i}\right) = 0.$$

The last parentheses vanish because of Hamilton's equations of motion, leaving

$$\frac{\partial \rho}{\partial t} + \sum_i \left(\dot{p}_i \frac{\partial \rho}{\partial p_i} + \dot{q}_i \frac{\partial \rho}{\partial q_i}\right) = 0.$$

But this is just the total time derivative of ρ with respect to time t, so we conclude that

$$d\rho/dt = 0$$

or, in Poisson bracket notation,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + [\rho, H] = 0. \tag{5.52}$$

It says that the total time rate of change of ρ, the density of representative points in phase space (briefly, phase-point density), vanishes. This result is often referred to as the principle of conservation of phase-point density.

Equation 5.52 means that to one following the motion of the phase points in an ensemble (a group of a large number of similar systems), the phase-point density does not change with the time. The phase-point density may change at any given place in phase space, but what may be called the "motional change" vanishes. Thus, if we consider a certain region of phase space containing a certain number of phase points, in the course of time, these points move in such a way as to occupy an equal phase volume at every instant, even though the shape of the phase volume may alter.

Liouville's theorem attains considerable importance for aggregates of microscopic particles where the concepts of statistical physics are important as in the statistical mechanics of particle systems, focusing properties of charged particle accelerators, in the study of the motion of electrons in the Earth's magnetic field, or even in stellar dynamics and galactic dynamics where stars or galaxies are treated as particles. It is often impractical to calculate an exact solution for such complex systems or even impossible because of a lack of complete information on the initial conditions. Statistical physics makes no attempt to obtain a complete solution for systems containing many particles. Instead, its aim is to make predictions about certain average properties at a given time by examining the motion of a group of a large number of similar systems. The group of similar systems is called an ensemble of systems and is to be seen as an intellectual construction to simulate and represent, at one time, the properties of the actual system as developed in the course of time. The statistical properties of an ensemble of systems can be specified at any time t by giving the density ρ in the phase space of system points per unit volume.

Example 5.7

As an illustration of Liouville's theorem, consider a system consisting of a large number of charged particles, each of mass m and charge e moving in a uniform electric field D. The Hamiltonian for such a particle is

$$H = p^2/2m - eDq = E \quad (E_1 < E < E_2)$$

where the electric field D is assumed to be in the direction of the positive q-axis. At a given time, say, $t = 0$, an ensemble representing such a system occupies the region A_1 shown in Figure 5.5. As time advances, these points will move to an adjacent region A_2 bounded by the momentum values p_1' and p_2', where

$$p_1' = p_1 + pt, \quad p_2' = p_2 + pt$$

so

$$p_1' - p_2' = p_1 - p_2.$$

Now, $\dot{p} = -\partial H/\partial q = eD$, and the phase "volume" occupied by the ensemble at $t = 0$ is the area

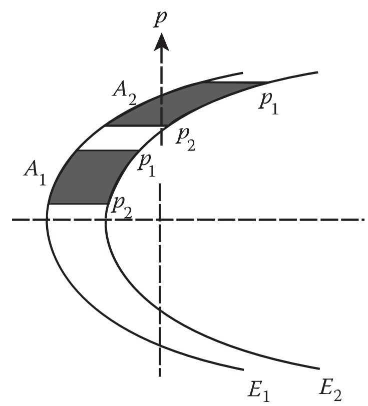

FIGURE 5.5 Motion of a "volume" in phase space.

$$A_1 = \frac{(p_2 - p_1)(E_2 - E_1)}{eD}.$$

Obviously, the phase "volume" occupied by the ensemble at time t is equal to A_1:

$$A_2 = \frac{(p_2' - p_1')(E_2 - E_1)}{eD} = \frac{(p_2 - p_1)(E_2 - E_1)}{eD} = A_1.$$

As time passes, the same set of phase points occupies an equal volume of phase space. Therefore the phase-point density must remain the same as Liouville's theorem requires. It is easy to note that the shape of the successive phase region occupied by the ensemble changes, although the phase volume is invariant.

5.8 TIME REVERSAL IN MECHANICS (OPTIONAL)

We saw earlier and in Chapter 4 that conservation of energy, momentum, and angular momentum are well established in mechanics; they are the consequences of translation invariance in time, space, and the rotational symmetry of space, respectively. These conservation laws can be extended to the whole field of physics if all the physical laws (such as electromagnetic) have the same invariance properties as in mechanics. In all experiences accumulated up to now, no evidence has been found to cause us to doubt the validity of these laws. Thus, we believe that the nature of our physical world has the fundamental symmetries represented by the invariance properties above.

Symmetries in variables other than spatial coordinates are also significant in many parts of physics, especially in subnuclear physics (or high-energy particle physics). We now take a brief look at the symmetry of physical laws under a reversal of the direction of time. The question as to how a direction of time is to be defined is in itself a difficult one. But ignoring this point for the present, we may ask whether we should expect physical laws to be invariant under the simple time reversal operation T:

$$t' = Tt = -t. \tag{5.53}$$

A little reflection shows that, apart from an intuitive feeling or attitude, there is no a priori reason for requiring all physical laws to be invariant under time reversal. In actual fact, mechanics and electromagnetic laws are invariant under the time reversal in Equation 5.53.