5.3 INTEGRALS OF MOTION AND CONSERVATION THEOREMS

5.3.1 Energy Integrals

In the preceding section, we showed that if the Hamiltonian H does not depend on time explicitly, then H is a constant of the motion:

$$H = \sum_j p_j \dot{q}_j - L = h \text{ (constant).} \tag{5.13}$$

The quantity h is called the Jacobian integral of the motion. Further, if the potential energy depends only on the coordinates, and the holonomic constraints are time independent, H is also the total energy of the system:

$$H = h = E. \tag{5.14}$$

5.3.2 Cyclic Coordinates and Integrals of Motion

A cyclic coordinate is defined as one that does not appear explicitly in L. It is obvious that a coordinate that is cyclic will also be absent from H for

$$\frac{\partial H}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\sum_j p_j \dot{q}_j - L\right) = 0.$$

Combining this result with Hamilton's equations, we obtain

$$\dot{p}_j = \frac{\partial H}{\partial q_j} = 0$$

from which it follows that

$$p_j = b_j \text{ (constant).} \tag{5.15}$$

Thus, we get the same result: the generalized momentum conjugate to a cyclic coordinate is conserved, that is, it is an integral of the motion. In the following section, we will show a more general momentum conservation theorem.

When some coordinates, say, $q_1, q_2, \ldots, q_m$ ($m < n$), are cyclic, the Lagrangian of the system is of the form

$$L = L(q_{m+1}, \ldots, q_n, p_1, p_2, \ldots, p_n).$$

We still have to solve the problem of n degrees of freedom even though m of them correspond to m cyclic coordinates. But the Hamiltonian of the system is of the form

$$H = H(q_{m+1}, \ldots, q_n, p_{m+1}, \ldots, p_n, b_1, b_2, \ldots, b_m; t).$$

Thus, $(n - m)$ coordinates and momenta remain, and the problem is essentially reduced to $(n - m)$ degrees of freedom. Hamilton's equations corresponding to each of the $(n - m)$ degrees of freedom can be obtained while completely ignoring the cyclic coordinates. The cyclic coordinates themselves can be found by integrating the equations of motion $\dot{q}_j = \partial H/\partial b_j, j = 1, 2, \ldots m$. Routh has devised a procedure that combines the advantage of the Hamiltonian formulation in handling cyclic coordinates with the Lagrangian formulation. We refer the students to the book by Goldstein about Routh's procedure.

5.3.3 Conservation Theorems of Momentum and Angular Momentum

In the preceding section, we found that, for a conservative system, the Hamiltonian is a constant if it does not depend on time explicitly. This represents the constancy of the energy of the system. By examining the Hamiltonian, it is possible to establish two other conservation theorems, namely, conservation of momentum and conservation of angular momentum, just as we did in Chapter 4 by examining the Lagrangian of the system.

We first show that if the Hamiltonian H is invariant with respect to an arbitrary infinitesimal translation of the coordinates, the total momentum of the system is conserved. For simplicity, consider the Hamiltonian depending on the difference of two coordinates $\left|\vec{r}_1 - \vec{r}_2\right|$. If the whole system is translated by a small amount $\mathrm{d}\vec{r}$, then the difference of two coordinates is not affected:

$$\left|\vec{r}_1 - \mathrm{d}\vec{r} - (\vec{r}_2 - \mathrm{d}\vec{r})\right| = \left|\vec{r}_1 - \vec{r}_2\right|.$$

Hence,

$$H(\vec{r} + \mathrm{d}\vec{r}, \vec{p}) = H(\vec{r}, \vec{p}). \tag{5.16}$$

Expanding the left-hand side, we obtain

$$H(\vec{r}, \vec{p}) + \sum_{j=1}^{3N} \mathrm{d}r_j \cdot \frac{\partial H}{\partial r_i} = H(\vec{r}, \vec{p})$$

or

$$\sum_{j=1}^{3N} \mathrm{d}r_j \cdot \frac{\partial H}{\partial r_i} = 0. \tag{5.17}$$

By Hamilton's equations, Equation 5.17 reduces to

$$\sum_{j=1}^{3N} \mathrm{d}r_j \left(-\frac{\mathrm{d}p_j}{\mathrm{d}t} \right) = 0.$$

Because $\mathrm{d}\vec{r}$ is arbitrary, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_j p_j = 0$$

which is the conservation of the total momentum.

We can also show that if the Hamiltonian is invariant with respect to an arbitrary infinitesimal rotation of the coordinate axes, the total angular momentum of the system is conserved. Consider a vector $\vec{r}$ in the $x_1 x_2$ plane; we rotate the coordinates counterclockwise through an angle θ about the x_3 axis (Figure 5.2). In the old coordinate system, the point P is at

$$x_1 = r \cos \phi, \quad x_2 = r \sin \phi$$

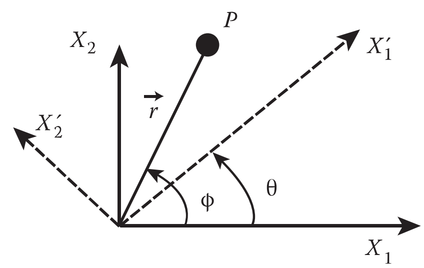

FIGURE 5.2 Infinitesimal rotation of the coordinate axes.

while in the new coordinate system, P is at

$$x_1' = r\cos(\phi - \theta), \quad x_2' = r\sin(\phi - \theta).$$

Using the trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$$

we obtain

$$x_1' = r\cos\phi\cos\theta + r\sin\phi\sin\theta = x_1\cos\theta + x_2\sin\theta$$

and

$$x_2' = r\sin\phi\cos\theta - r\cos\phi\sin\theta = -x_1\sin\theta + x_2\cos\theta.$$

For infinitesimal rotations $\delta\theta$, $\cos\delta\theta \approx 1$, and $\sin\delta\theta \approx \delta\theta$, we obtain

$$x_1' = x_1 + x_2\delta\theta, \quad x_2' = -x_1\delta\theta + x_2$$

and so

$$x_1' - x_1 = \delta x = x_2\delta\theta, \quad x_2' - x_2 = \delta x_2 = -x_1\delta\theta \tag{5.18}$$

where $\delta\theta$ has only a z component. Equation 5.18 can be written as a vector equation:

$$\delta\vec{r} = \vec{r} \times \delta\theta. \tag{5.19}$$

All vectors will transform according to Equation 5.19 under an infinitesimal rotation about an axis.

If the Hamiltonian H is invariant with respect to an infinitesimal rotation, we then have

$$H\left([\vec{r} + \vec{r} \times \delta\theta],\ [\vec{p} + \vec{p} \times \delta\vec{p}]\right) = H(\vec{r},\ \vec{p}).$$

Expanding the left-hand side, we have

$$H(\vec{r},\vec{p}) + \sum_{j=1}^{3N}\left[(\delta\theta \times \vec{r})_j \frac{\partial H}{\partial r_j} + (\delta\theta \times \vec{p})_j \frac{\partial H}{\partial p_j}\right] = H(\vec{r},\vec{p})$$

or

$$\sum_{j=1}^{3N}\left[(\delta\theta\times\vec{r})_j\frac{\partial H}{\partial r_j}+(\delta\theta\times\vec{p})_j\frac{\partial H}{\partial p_j}\right]=0$$

from which we obtain

$$\delta\vec{\theta}\cdot\left[\sum_{j=1}^{3N}\left\{\left(\vec{r}\times\frac{\partial H}{\partial\vec{r}}\right)_j+\left(\vec{p}\times\frac{\partial H}{\partial\vec{p}}\right)_j\right\}\right]=0 \tag{5.20}$$

where we have used the vector identity $(\vec{A}\times\vec{B})\cdot\vec{C}=(\vec{C}\times\vec{A})\cdot\vec{B}$.

Using Hamilton's equations, Equation 5.20 can be further simplified:

$$\delta\vec{\theta}\cdot\left[\sum_{j=1}^{3N}\left\{(\vec{r}\times\dot{\vec{p}})_j+(\vec{p}\times\dot{\vec{r}})_j\right\}\right]=-\delta\vec{\theta}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{3N}(\vec{r}\times\vec{p})_j=-\delta\vec{\theta}\cdot\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}=0$$

where $\vec{L}$ is the total angular momentum of the system. Because $\delta\vec{\theta}$ is arbitrary, we have

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}=0,\quad\text{or}\quad \vec{L}=\vec{\alpha}\ \text{(a constant vector)}.$$

5.4 CANONICAL TRANSFORMATIONS

As shown in the previous section, there is some advantage in using cyclic coordinates. However, in general, it is impossible to obtain more than a limited number of such coordinates by means of coordinate transformations. On the other hand, we can employ a more general class of transformations that involve both generalized coordinates and momenta. If the equations of motion are simpler in the set of new variables Q_j and P_j than in the original old set q_j and p_j, we then have a clear gain. We will not be able to consider all possible transformations but only the so-called canonical transformations that preserve the canonical form of Hamilton's equations of motion; that is, given that the q's and p's satisfy Hamilton's equations

$$\dot{q}_j=\frac{\partial H}{\partial p_j},\quad \dot{p}_j=-\frac{\partial H}{\partial q_j}$$

for some H, then the transformation

$$Q_j=Q_j(q_k,p_k,t),\quad P_j=P_j(q_k,p_k,t) \tag{5.21}$$

is canonical if and only if there exists a function K such that the time evolutions of the Q's and P's are still governed by Hamilton's equations

$$\dot{Q}_j=\frac{\partial K}{\partial P_j},\quad \dot{P}_j=-\frac{\partial K}{\partial Q_j}. \tag{5.22}$$

Here, $K(Q, P, t)$ is the new Hamiltonian that may be different from the old Hamiltonian $H(q, p, t)$.