But note that the following two functions

$$L_2 = m\dot{x}\dot{y} - mgx \quad \text{and} \quad L_3 = mgy - \frac{1}{2}mgx\dot{y}^2 - m\dot{x}^3\dot{y} - \frac{1}{2}m\dot{y}^2$$

also lead to the same equations of motion, so they are also Lagrangians for this system. But because the action integral of the Lagrangians L_2 and L_3 is not the classical action, we can use this as a guide to reject them and select a correct Lagrangian for the system.

4.6 INTEGRALS OF MOTION AND CONSERVATION LAWS

Lagrange's equations of motion for a system of n degrees of freedom are a set of n second-order differential equations. The solution of each equation has two constants of integration, which usually can be determined from the initial values of the generalized coordinate q and the generalized velocity $\dot{q}$. Sometimes Lagrange's equations can be solved in terms of known functions but not always. In general, most problems either cannot be solved completely or are too tedious to solve. Fortunately, very often, a great deal of information about the system is contained in a number of so-called first integrals of the motion, which are often of greater interest and importance than a complete knowledge of all the q's as a function of time t. The first integrals of motion are functions of the generalized coordinates q_j and the generalized velocities $\dot{q}_j$ of the form $f(q\text{'s}, \dot{q}\text{'s}, t) = \alpha_i$ (constant) whose values remain constant during the motion of the system and are dependent only on the initial conditions of the system. The conservation laws of energy, momentum, and angular momentum that we deduced in Newtonian formalism are of this exact type. These conservation laws can be deduced easily in Lagrangian formalism in a very general and elegant fashion. In the process, they make quite clear the relationship between conservation laws and the symmetry properties of the system. The association goes beyond these conservation laws, beyond classical systems; it finds wide application in modern physics, especially in quantum field theories and particle physics. Hence, the study of symmetry and its uses in learning about the laws of nature is very important.

4.6.1 Cyclic Coordinates and Conservation Theorems

We begin by examining the first integrals of the motion associated with the so-called cyclic coordinates. Coordinates that do not appear explicitly in the Lagrangian of a system are said to be cyclic or ignorable. Be aware that this definition is not universal. Other authors may use them differently. If q_i is a cyclic coordinate, then the Lagrangian L will take the form

$$L = L(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_i, \ldots, \dot{q}_n, t) \tag{4.31}$$

and so

$$\partial L/\partial q_i = 0.$$

Lagrange's equations of motion reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0, i = 1, 2, \ldots, n$$

or

$$\mathrm{d}p_i/\mathrm{d}t = 0,$$

From this, it follows that

$$P_i = \alpha_i,\ i = 1, 2,\ldots, n \tag{4.32}$$

where the α_i are constants evaluated from the initial conditions. They are called the first integrals of motion. We now find a general conservation theorem: The generalized momentum conjugate to a cyclic coordinate is conserved during the motion. This conservation theorem for generalized momentum is more general than the conservation theorems for linear momentum and angular momentum. In fact, the latter two conservation theorems are contained in the general conservation theorem. For example, if q_1 does not appear in L and if a change δq_1 in q_1 corresponds to a translation of the system through a distance along a certain direction, say x, then

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial}{\partial \dot{q}_1}\left(\frac{1}{2}\sum_i m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right) = \sum_i m_i\left(\dot{x}_i\frac{\partial \dot{x}_i}{\partial \dot{q}_1} + \dot{y}_i\frac{\partial \dot{y}_i}{\partial \dot{q}_1} + \dot{z}_i\frac{\partial \dot{z}_i}{\partial \dot{q}_1}\right)$$

$$= \sum_i m_i\left(\dot{x}_i\frac{\partial x_i}{\partial q_1} + \dot{y}_i\frac{\partial y_i}{\partial q_1} + \dot{z}_i\frac{\partial z_i}{\partial q_1}\right) = \sum_i m_i\left(\dot{x}_i\frac{\partial l}{\partial q_1} + 0 + 0\right) = \sum_i m_i\dot{x}_i = \alpha_1$$

or

$$p_i = \sum_i m_i\dot{x}_i = \alpha_i \quad \text{(constant).} \tag{4.33}$$

This is the law of momentum conservation along the x_i axis.

In a similar fashion, it can be shown that if cyclic coordinate q_j is such that δq_j corresponds to a rotation of the system around some axis, then the conservation of its conjugate momentum corresponds to the conservation of an angular momentum. We do not plan to give a general proof here because of its length and tediousness and refer interested students to the book by Goldstein (1980). The two-dimensional harmonic oscillator described in plane polar coordinates gives a simple illustrative example. The Lagrange L is

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}kr^2$$

where θ is a cyclic coordinate. Hence,

$$\partial L/\partial\dot{\theta} = mr^2\dot{\theta} = \beta = \text{constant}.$$

$mr^2\dot{\theta}$ is seen to be the angular momentum about the origin.

Summarizing these results, we see that if a translation coordinate is cyclic, the system is invariant under translation along a given direction, and the corresponding linear momentum is conserved. Similarly, when a rotation coordinate is cyclic, the system is invariant under rotation about the given axis, and the conjugate angular momentum is conserved. That is, when arbitrary changes δq_i of a coordinate q_i make no difference to the description of the motion by the Lagrangian, q_i need not appear in the Lagrangian and the description possesses symmetry expressible as an invariance of the system to q_i. The consequent conservation property testifies to a close association between conservation laws and invariances or symmetries. In the following, we shall explore the purely geometric types of symmetry that reflect the general properties of the homogeneity of space and time and the isotropy of space in an inertial reference frame.

4.6.2 Symmetries and Conservation Laws

The homogeneity of space and time means that there are no fixed reference points in space and that there is no preferred instant in time. In other words, the displacement in space of the system as a whole or a shift in time will not change the mechanical properties of a closed system (i.e., one that does not interact with other systems). The isotropy of space means that all directions in space are equivalent; hence, rotation in space does not change the properties of a closed system.

In Lagrangian formalism, the laws of motion of a system are given by Lagrange's equations of motion and so are uniquely determined by the Lagrange of the system. In other words, the effects of a symmetry operation on a system's equations of motion can be determined from its effect on the Lagrangian L. That is what we shall proceed to do, and we begin with the law of energy conservation.

4.6.2.1 Homogeneity of Time and Conservation of Energy

Assume that a system of particles is in unchanging external conditions; this occurs if the system is closed or in a stationary force field (a time-independent constant external force field). In this case, the time, because of its homogeneity, cannot enter the Lagrangian explicitly, and so we have $\partial L/\partial t = 0$. Then the total derivative of the Lagrangian becomes

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right) = \sum_i \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right) = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i \right)$$

or

$$\sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \right) = 0.$$

Thus, we see that, for a closed system or one in a stationary external force field, the quantity in the parentheses, a function of the generalized coordinates and the generalized velocities, remains constant during the motion:

$$\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \alpha \quad \text{(a constant)}.$$

Functions of the quantities q_i and $\dot{q}_i$ that remain, during the motion, a constant value determined by the initial conditions are called integrals of motion. Accordingly, α is an integral of constant.

The quantity $\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$ is denoted by the symbol H, called the Hamiltonian of the system:

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \sum_i p_i\dot{q}_i - L. \tag{4.34}$$

If the potential energy V is velocity-independent, and if the equations of transformation (Equation 4.1) do not depend on time explicitly, then H is equal to the total energy of the system. It is easy to show this explicitly. By the first condition, we have $V = V(x_{i\alpha})$, where $i = 1, 2, \ldots, n$ (number of particles) and $\alpha = 1, 2, 3$ (coordinate axes). By making use of the second condition, namely, the equations of transformation connecting the rectangular and generalized coordinates do not depend on time explicitly [$x_{i,\alpha} = x_{i,\alpha}(q_j)$ or $q_j = q_j(x_{i\alpha})$], we can express V in terms of q_j as $V = V(q_j)$, and so $\partial V(q_j)/\partial \dot{q}_j = 0$. Furthermore, under the second condition, the kinetic energy T is a homogeneous, quadratic function of the q's, and Euler's theorem gives

$$\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 2T.$$

Substituting these into the expression for the Hamiltonian H, we obtain

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \sum_i \frac{\partial (T-V)}{\partial \dot{q}_i}\dot{q}_i - (T-V)$$
$$= \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - (T-V) \ = 2T - (T-V) = T + V = E.$$

Hence, the homogeneity of time leads to the following law: The energy of the system of a closed conservative system of particles (or a system in a stationary external force field) remains constant.

When the equations of transformation connecting the rectangular and generalized coordinates depend on time explicitly, the Hamiltonian H is no longer the total energy of the system, but it is still conserved.

4.6.2.2 Spatial Homogeneity and Momentum Conservation

Consider a closed system. Because of the homogeneity of space, the displacement of the system by a small amount δr must not change the mechanical properties of the system, and so the Lagrangian must retain its previous value. This would not be true for an unclosed system because such a displacement would cause a change in the arrangement of the particles relative to the bodies interacting with them, and so the mechanical properties of the system would be affected. As the displacement δr is very small, we can write

$$\delta L = \sum_j \frac{\partial L}{\partial \vec{r}_j}\cdot \delta \vec{r}_j = \delta \vec{r} \cdot \sum_j \frac{\partial L}{\partial \vec{r}_j} \ = 0 \tag{4.35}$$

where j is the number of particles, and we have made use of the fact that each particle in the system is displaced by the same amount, and so $\delta \vec{r}_j = \delta \vec{r}$. Before we continue further, we ought to digress for a moment to explain a mathematical notation here: the derivative of a scalar function with respect to a vector quantity. By the derivative of the scalar ϕ with respect to the vector λ, it is understood as a vector having the components $\partial\phi/\partial\lambda_x$, $\partial\phi/\partial\lambda_y$, and $\partial\phi/\partial\lambda_z$. Consequently, the symbol $\partial\phi/\partial r$ stands for a vector with the components $\partial\phi/\partial x$, $\partial\phi/\partial y$, and $\partial\phi/\partial z$, and

$$\frac{\partial \varphi}{\partial \vec{r}}\cdot d\vec{r} = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz.$$

We now return to Equation 4.35. Because $\delta r \neq 0$, we have

$$\sum_j \frac{\partial L}{\partial \vec{r}_j} = 0. \tag{4.36}$$

Lagrange's equations allow us to write

$$\frac{\partial L}{\partial x_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j}, \quad \frac{\partial L}{\partial y_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_j}, \quad \frac{\partial L}{\partial z_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}_j}.$$

Multiplying the first, second, and third of these equations by the unit vectors $\hat{e}_x, \hat{e}_y, \hat{e}_z$, respectively, and summating them, we obtain the expression

$$\frac{\partial L}{\partial \vec{r}_j} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vec{r}}_j}.$$

Equation 4.36 can thus be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_j \frac{\partial L}{\partial \dot{\vec{r}}_j} = 0. \tag{4.37}$$

The quantity $\partial L/\partial \dot{\vec{r}}_j$ is a vector with the components $\partial L/\partial \dot{x}_j$, $\partial L/\partial \dot{y}_j$, and $\partial L/\partial \dot{z}_j$. These are the projections of the conventional (not generalized) momentum p_j of the jth particle onto the coordinate axes. Hence,

$$\partial L/\partial \dot{\vec{r}}_j = \vec{p}_j.$$

Accordingly, Equation 4.37 can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_j \vec{p}_j = 0.$$

Hence, it follows that

$$\vec{P} = \sum_j \vec{p}_j = \text{constant}. \tag{4.38}$$

Thus, the homogeneity of space leads to the momentum conservation: The total momentum of a closed system of particles remains constant, that is, it is also an integral of motion.

4.6.2.3 Isotropy of Space and Angular Momentum Conservation

Because of the isotropy of space, the Lagrangian of a closed system should not be affected by an infinitesimal rotation of the system as a whole in space. Accordingly, the Lagrangian should be unchanged, $\delta L = 0$. We are now interested in the increment of the Lagrangian δL in an arbitrary very small rotation of a system through an angle $\delta\theta$. All the vectors characterizing the system will rotate together with it. As a result, they will receive certain increments that will be of the same order as $\delta\theta$. According to Equation 4.36, we have

$$\delta\vec{r}_\alpha = \delta\vec{\theta}\times\vec{r}_\alpha, \quad \text{and} \quad \delta\dot{\vec{r}}_\alpha = \delta\vec{v}_\alpha = \delta\vec{\theta}\times\vec{v}_\alpha. \tag{4.39}$$

Because of the smallness of the quantities $\delta\vec{r}_\alpha$ and $\delta\vec{v}_\alpha$, we have

$$\delta L(\vec{r}_\alpha, \vec{v}_\alpha) = \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha}\cdot\delta\vec{r}_\alpha + \sum_\alpha \frac{\partial L}{\partial \vec{v}_\alpha}\cdot\delta\vec{v}_\alpha \tag{4.40}$$

which becomes, in view of Equation 4.39,

$$\delta L(\vec{r}_\alpha, \vec{v}_\alpha) = \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} \cdot (\delta\vec{\theta} \times \vec{r}_\alpha) + \sum_\alpha \frac{\partial L}{\partial \vec{v}_\alpha} \cdot (\delta\vec{\theta} \times \vec{v}_\alpha). \tag{4.41}$$

Now, a cyclic transposition of the multipliers may be performed in a scalar triple product, $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$. Such a transposition in Equation 4.41 yields

$$\delta L = \sum_\alpha \delta\vec{\theta} \cdot \left(\vec{r}_\alpha \times \frac{\partial L}{\partial \vec{r}_\alpha} \right) + \sum_\alpha \delta\vec{\theta} \cdot \left(\vec{v}_\alpha \times \frac{\partial L}{\partial \vec{v}_\alpha} \right) = \delta\vec{\theta} \cdot \sum_\alpha \left(\vec{r}_\alpha \times \frac{\partial L}{\partial \vec{r}_\alpha} + \vec{v}_\alpha \times \frac{\partial L}{\partial \vec{v}_\alpha} \right).$$

δL can be further simplified with the help of Lagrange's equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \vec{v}_\alpha} - \frac{\partial L}{\partial \vec{r}_\alpha} = 0 \quad \text{or} \quad \frac{\partial L}{\partial \vec{r}_\alpha} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \vec{v}_\alpha}$$

$$\begin{aligned} \delta L &= \delta\vec{\theta} \cdot \sum_\alpha \left(\vec{r}_\alpha \times \frac{\partial L}{\partial \vec{r}_\alpha} + \vec{v}_\alpha \times \frac{\partial L}{\partial \vec{v}_\alpha} \right) \\ &= \delta\vec{\theta} \cdot \sum_\alpha \left(\vec{r}_\alpha \times \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \vec{v}_\alpha} + \vec{v}_\alpha \times \frac{\partial L}{\partial \vec{v}_\alpha} \right) = \delta\vec{\theta} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha \left[\vec{r}_\alpha \times \frac{\partial L}{\partial \vec{v}_\alpha} \right]. \end{aligned} \tag{4.42}$$

Because $\delta\vec{\theta} \neq 0$, the condition $\delta L = 0$ is equivalent to the condition

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha \left[\vec{r}_\alpha \times \frac{\partial L}{\partial \vec{v}_\alpha} \right] = 0 \tag{4.43}$$

from which it follows that

$$\vec{L} = \sum_\alpha \left[\vec{r}_\alpha \times \frac{\partial L}{\partial \vec{v}_\alpha} \right] = \sum_\alpha \left[\vec{r}_\alpha \times \vec{p}_\alpha \right] = \text{constant}. \tag{4.44}$$

The symbol $\vec{L}$ for angular momentum should not be confused with the Lagrangian L.

Thus, the isotropy of space leads to the angular momentum conservation law: The resultant angular momentum of a closed system of particles remains constant. The angular momentum of a closed system, like its energy and momentum, is also an integral of motion.

Although the conservation law for angular momentum is valid only for a closed system, the conservation law may hold in a more restricted form for a system in an external force field that possesses an axis of symmetry. In such a field, the Lagrangian of the system is invariant about the symmetry axis; hence, the angular momentum of the system about the axis of symmetry is constant in time, that is, it is conserved. The most important such case is that of a central force field that will be examined in Chapter 6. We consider here a simple illustrative example: the motion of a particle on the inner surface of a cone.

Example 4.9

A particle of mass m is constrained to move under the influence of gravity on the smooth inner surface of the paraboloid of revolution $x^2 + y^2 = az^2$, where a is a constant. Show that the angular momentum of the particle about the axis of symmetry of the system is conserved.

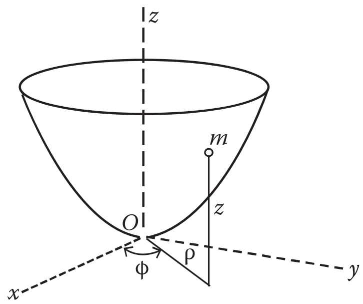

FIGURE 4.11 Particle constrained to move under gravity on the smooth inner surface of the paraboloid.

Solution:

The problem possesses cylindrical symmetry, so we choose ρ, ϕ, and z as the generalized coordinates, and we let the axis of the paraboloid correspond to the z-axis and the vertex of the paraboloid be located at the origin (Figure 4.11). The Lagrangian of the system is

$$L = T - V = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - mgz$$

where the reference level for the potential energy is set at the vertex of the paraboloid.

As ϕ is a cyclic coordinate, $\partial L/\partial\phi = 0$. Then, Lagrange's equation for coordinate ϕ reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\varphi}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\rho^2\dot{\varphi}) = 0$$

from which we obtain

$$m\rho^2\dot{\varphi} = \text{constant}. \tag{4.45}$$

Note that $m\rho^2\dot{\varphi} = m\rho^2\omega$ is just the angular momentum of the particle about the system's axis of symmetry, the z-axis. Thus, Equation 4.45 simply expresses the fact that the angular momentum of the particle about the axis of symmetry is conserved.

We have learned that the laws of conservation of energy, linear momentum, and angular momentum are an immediate consequence of the general symmetry properties of space and time. We should also note that these laws also explain, from their derivations, why the following pairs of variables are associated with each other:

$$(\vec{r},\ \vec{p}), (\vec{\theta},\ \vec{L}), \text{ and } (t, E).$$

4.6.2.4 Noether's Theorem

By now, we have all learned that symmetries of the Lagrangian gave rise to constants of the motion. But the constants of the motion do not always come from the obvious symmetries of the Lagrangian, nor do they always have a simple form. Mathematician Emmy Noether took a general approach to this problem in 1919 and found a theorem that states essentially that if corresponding to a variable α in the Lagrangian, and the Lagrangian remains unchanged for a change of α to α + ε where ε is infinitesimal, we will have a conservation principle. The conservation laws of energy, momentum, and angular momentum are just an example of Noether's theorem. A straightforward and simple exposition of Noether's work is given in Appendix 3.