m_3:

$$v_3 = \frac{d}{dt}(l_1 - x + l_2 - y) = -\dot{x} - \dot{y} \tag{7.53}$$

$$\begin{aligned} T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 \\ &= \frac{1}{2}m_1 \dot{x}^2 + \frac{1}{2}m_2(\dot{y} - \dot{x})^2 + \frac{1}{2}m_3(-\dot{x} - \dot{y})^2 \end{aligned} \tag{7.54}$$

Let the potential energy $U = 0$ at $x = 0$.

$$\begin{aligned} U &= U_1 + U_2 + U_3 \\ &= -m_1 g x - m_2 g(l_1 - x + y) - m_3 g(l_1 - x + l_2 - y) \end{aligned} \tag{7.55}$$

Because T and U have been determined, the equations of motion can be obtained using Equation 7.18. The results are

$$m_1\ddot{x} + m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_1 - m_2 - m_3)g \tag{7.56}$$

$$-m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_2 - m_3)g \tag{7.57}$$

Equations 7.56 and 7.57 can be solved for $\ddot{x}$ and $\ddot{y}$.

Examples 7.2–7.8 indicate the ease and usefulness of using Lagrange's equations. It has been said, probably unfairly, that Lagrangian techniques are simply recipes to follow. The argument is that we lose track of the "physics" by their use. Lagrangian methods, on the contrary, are extremely powerful and allow us to solve problems that otherwise would lead to severe complications using Newtonian methods. Simple problems can perhaps be solved just as easily using Newtonian methods, but the Lagrangian techniques can be used to attack a wide range of complex physical situations (including those occurring in quantum mechanics*).

7.5 Lagrange's Equations with Undetermined Multipliers

Constraints that can be expressed as algebraic relations among the coordinates are holonomic constraints. If a system is subject only to such constraints, we can always find a proper set of generalized coordinates in terms of which the equations of motion are free from explicit reference to the constraints.

Any constraints that must be expressed in terms of the *velocities* of the particles in the system are of the form

$$f(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) = 0 \tag{7.58}$$

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*See Feynman and Hibbs (Fe65).

and constitute nonholonomic constraints *unless* the equations can be integrated to yield relations among the coordinates.*

Consider a constraint relation of the form

$$\sum_i A_i \dot{x}_i + B = 0, \quad i = 1, 2, 3 \tag{7.59}$$

In general, this equation is nonintegrable, and therefore the constraint is nonholonomic. But if A_i and B have the forms

$$A_i = \frac{\partial f}{\partial x_i}, \quad B = \frac{\partial f}{\partial t}, \quad f = f(x_i, t) \tag{7.60}$$

then Equation 7.59 may be written as

$$\sum_i \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial f}{\partial t} = 0 \tag{7.61}$$

But this is just

$$\frac{df}{dt} = 0$$

which can be integrated to yield

$$f(x_i, t) - \text{constant} = 0 \tag{7.62}$$

so the constraint is actually holonomic.

From the preceding discussion, we conclude that constraints expressible in differential form as

$$\sum_j \frac{\partial f_k}{\partial q_j} dq_j + \frac{\partial f_k}{\partial t} dt = 0 \tag{7.63}$$

are equivalent to those having the form of Equation 7.9.

If the constraint relations for a problem are given in differential form rather than as algebraic expressions, we can incorporate them directly into Lagrange's equations by using the Lagrange undetermined multipliers (see Section 6.6) without first performing the integrations; that is, for constraints expressible as in Equation 6.71,

$$\sum_j \frac{\partial f_k}{\partial q_j} dq_j = 0 \quad \begin{cases} j = 1, 2, \ldots, s \\ k = 1, 2, \ldots, m \end{cases} \tag{7.64}$$

the Lagrange equations (Equation 6.69) are

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda_k(t)\frac{\partial f_k}{\partial q_j} = 0} \tag{7.65}$$

In fact, because the variation process involved in Hamilton's Principle holds the time constant at the endpoints, we could add to Equation 7.64 a term $(\partial f_k/\partial t)dt$

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*Such constraints are sometimes called "semiholonomic."

without affecting the equations of motion. Thus constraints expressed by Equation 7.63 also lead to the Lagrange equations given in Equation 7.65.

The great advantage of the Lagrangian formulation of mechanics is that the explicit inclusion of the forces of constraint is not necessary; that is, the emphasis is placed on the dynamics of the system rather than the calculation of the forces acting on each component of the system. In certain instances, however, it might be desirable to know the forces of constraint. For example, from an engineering standpoint, it would be useful to know the constraint forces for design purposes. It is therefore worth pointing out that in Lagrange's equations expressed as in Equation 7.65, **the undetermined multipliers $\lambda_k(t)$ are closely related to the forces of constraint.*** The generalized forces of constraint Q_j are given by

$$Q_j = \sum_k \lambda_k \frac{\partial f_k}{\partial q_j} \tag{7.66}$$

EXAMPLE 7.9

Let us consider again the case of the disk rolling down an inclined plane (see Example 6.5 and Figure 6-7). Find the equations of motion, the force of constraint, and the angular acceleration.

Solution. The kinetic energy may be separated into translational and rotational terms†

$$T = \frac{1}{2} M\dot{y}^2 + \frac{1}{2} I\dot{\theta}^2$$

$$= \frac{1}{2} M\dot{y}^2 + \frac{1}{4} MR^2 \dot{\theta}^2$$

where M is the mass of the disk and R is the radius; $I = \frac{1}{2} MR^2$ is the moment of inertia of the disk about a central axis. The potential energy is

$$U = Mg(l - y) \sin \alpha \tag{7.67}$$

where l is the length of the inclined surface of the plane and where the disk is assumed to have zero potential energy at the bottom of the plane. The Lagrangian is therefore

$$L = T - U$$

$$= \frac{1}{2}M\dot{y}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mg(y - l) \sin \alpha \tag{7.68}$$

*See, for example, Goldstein (Go80, p. 47). Explicit calculations of the forces of constraint in some specific problems are carried out by Becker (Be54, Chapters 11 and 13) and by Symon (Sy71, p. 372ff).

†We anticipate here a well-known result from rigid-body dynamics discussed in Chapter 11.

The equation of constraint is

$$f(y, \theta) = y - R\theta = 0 \tag{7.69}$$

The system has only one degree of freedom if we insist that the rolling takes place without slipping. We may therefore choose either y or θ as the proper coordinate and use Equation 7.69 to eliminate the other. Alternatively, we may continue to consider *both* y and θ as generalized coordinates and use the method of undetermined multipliers. The Lagrange equations in this case are

$$\left.\begin{aligned} \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda\frac{\partial f}{\partial y} = 0 \\ \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \end{aligned}\right\} \tag{7.70}$$

Performing the differentiations, we obtain, for the equations of motion,

$$Mg\sin\alpha - M\ddot{y} + \lambda = 0 \tag{7.71a}$$

$$-\frac{1}{2}MR^2\ddot{\theta} - \lambda R = 0 \tag{7.71b}$$

Also, from the constraint equation, we have

$$y = R\theta \tag{7.72}$$

These equations (Equations 7.71 and 7.72) constitute a soluble system for the three unknowns y, θ, λ. Differentiating the equation of constraint (Equation 7.72), we obtain

$$\ddot{\theta} = \frac{\ddot{y}}{R} \tag{7.73}$$

Combining Equations 7.71b and 7.73, we find

$$\lambda = -\frac{1}{2}M\ddot{y} \tag{7.74}$$

and then using this expression in Equation 7.71a there results

$$\ddot{y} = \frac{2g\sin\alpha}{3} \tag{7.75}$$

with

$$\lambda = -\frac{Mg\sin\alpha}{3} \tag{7.76}$$

so that Equation 7.71b yields

$$\ddot{\theta} = \frac{2g\sin\alpha}{3R} \tag{7.77}$$

Thus, we have three equations for the quantities $\ddot{y}$, $\ddot{\theta}$, and λ that can be immediately integrated.

We note that if the disk were to slide without friction down the plane, we would have $\ddot{y} = g \sin \alpha$. Therefore, the rolling constraint reduces the acceleration to $\frac{2}{3}$ of the value of frictionless sliding. The magnitude of the force of friction producing the constraint is just λ—that is, $(Mg/3) \sin \alpha$.

The generalized forces of constraint, Equation 7.66, are

$$Q_y = \lambda \frac{\partial f}{\partial y} = \lambda = -\frac{Mg \sin \alpha}{3}$$

$$Q_\theta = \lambda \frac{\partial f}{\partial \theta} = -\lambda R = \frac{MgR \sin \alpha}{3}$$

Note that Q_y and Q_θ are a force and a torque, respectively, and they are the generalized forces of constraint required to keep the disk rolling down the plane without slipping.

Note that we may eliminate $\dot{\theta}$ from the Lagrangian by substituting $\dot{\theta} = \dot{y}/R$ from the equation of constraint:

$$L = \frac{3}{4} M\dot{y}^2 + Mg(y - l) \sin \alpha \tag{7.78}$$

The Lagrangian is then expressed in terms of only one proper coordinate, and the single equation of motion is immediately obtained from Equation 7.18:

$$Mg \sin \alpha - \frac{3}{2} M\ddot{y} = 0 \tag{7.79}$$

which is the same as Equation 7.75. Although this procedure is simpler, it cannot be used to obtain the force of constraint.

EXAMPLE 7.10

A particle of mass m starts at rest on top of a smooth fixed hemisphere of radius a. Find the force of constraint, and determine the angle at which the particle leaves the hemisphere.

Solution. See Figure 7-7. Because we are considering the possibility of the particle leaving the hemisphere, we choose the generalized coordinates to be r and θ. The constraint equation is

$$f(r, \theta) = r - a = 0 \tag{7.80}$$

The Lagrangian is determined from the kinetic and potential energies:

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U = mgr \cos \theta$$

$$L = T - U$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr \cos \theta \tag{7.81}$$

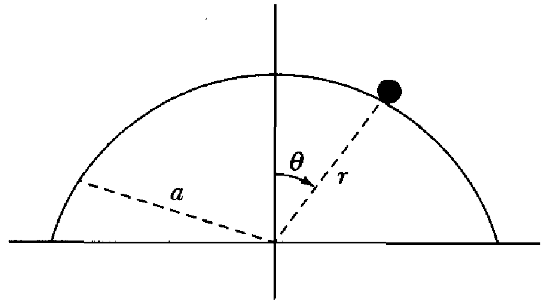

FIGURE 7-7 Example 7.10. A particle of mass m moves on the surface of a fixed smooth hemisphere.

where the potential energy is zero at the bottom of the hemisphere. The Lagrange equations, Equation 7.65, are

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0 \tag{7.82}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \tag{7.83}$$

Performing the differentiations on Equation 7.80 gives

$$\frac{\partial f}{\partial r} = 1, \qquad \frac{\partial f}{\partial \theta} = 0 \tag{7.84}$$

Equations 7.82 and 7.83 become

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0 \tag{7.85}$$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0 \tag{7.86}$$

Next, we apply the constraint $r = a$ to these equations of motion:

$$r = a, \quad \dot{r} = 0 = \ddot{r}$$

Equations 7.85 and 7.86 then become

$$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0 \tag{7.87}$$

$$mga\sin\theta - ma^2\ddot{\theta} = 0 \tag{7.88}$$

From Equation 7.88, we have

$$\ddot{\theta} = \frac{g}{a}\sin\theta \tag{7.89}$$

We can integrate Equation 7.89 to determine $\dot{\theta}^2$.

$$\ddot{\theta} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \tag{7.90}$$

We integrate Equation 7.89,

$$\int \dot{\theta}\, d\dot{\theta} = \frac{g}{a}\int \sin\theta\, d\theta \tag{7.91}$$

which results in

$$\frac{\dot{\theta}^2}{2} = \frac{-g}{a}\cos\theta + \frac{g}{a} \tag{7.92}$$

where the integration constant is g/a, because $\dot{\theta} = 0$ at $t = 0$ when $\theta = 0$. Substituting $\dot{\theta}^2$ from Equation 7.92 into Equation 7.87 gives, after solving for λ,

$$\lambda = mg(3\cos\theta - 2) \tag{7.93}$$

which is the force of constraint. The particle falls off the hemisphere at angle θ_0 when $\lambda = 0$.

$$\lambda = 0 = mg(3\cos\theta_0 - 2) \tag{7.94}$$

$$\theta_0 = \cos^{-1}\left(\frac{2}{3}\right) \tag{7.95}$$

As a quick check, notice that the constraint force is $\lambda = mg$ at $\theta = 0$ when the particle is perched on top of the hemisphere.

The usefulness of the method of undetermined multipliers is twofold:

1. The Lagrange multipliers are closely related to the forces of constraint that are often needed.
2. When a proper set of generalized coordinates is not desired or too difficult to obtain, the method may be used to increase the number of generalized coordinates by including constraint relations between the coordinates.

7.6 Equivalence of Lagrange's and Newton's Equations

As we have emphasized from the outset, the Lagrangian and Newtonian formulations of mechanics are equivalent: The viewpoint is different, but the content is the same. We now explicitly demonstrate this equivalence by showing that the two sets of equations of motion are in fact the same.

In Equation 7.18, let us choose the generalized coordinates to be the rectangular coordinates. Lagrange's equations (for a single particle) then become

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \qquad i = 1, 2, 3 \tag{7.96}$$

or

$$\frac{\partial(T - U)}{\partial x_i} - \frac{d}{dt}\frac{\partial(T - U)}{\partial \dot{x}_i} = 0$$