Calculating the derivatives, we find

$$\ddot{r} - r\dot{\theta}^2 \sin^2 \alpha + g \sin \alpha \cos \alpha = 0 \tag{7.31}$$

which is the equation of motion for the coordinate r.

We shall return to this example in Section 8.10 and examine the motion in more detail.

EXAMPLE 7.5

The point of support of a simple pendulum of length b moves on a massless rim of radius a rotating with constant angular velocity ω. Obtain the expression for the Cartesian components of the velocity and acceleration of the mass m. Obtain also the angular acceleration for the angle θ shown in Figure 7-3.

Solution. We choose the origin of our coordinate system to be at the center of the rotating rim. The Cartesian components of mass m become

$$\left.\begin{aligned} x &= a \cos \omega t + b \sin \theta \\ y &= a \sin \omega t - b \cos \theta \end{aligned}\right\} \tag{7.32}$$

The velocities are

$$\left.\begin{aligned} \dot{x} &= -a\omega \sin \omega t + b\dot{\theta} \cos \theta \\ \dot{y} &= a\omega \cos \omega t + b\dot{\theta} \sin \theta \end{aligned}\right\} \tag{7.33}$$

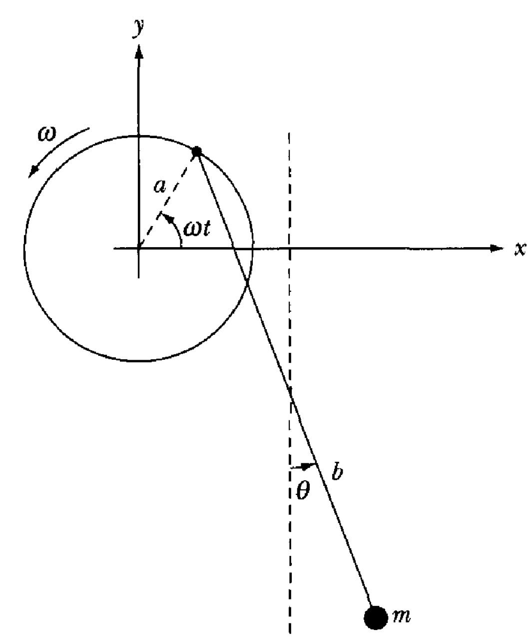

FIGURE 7-3 Example 7.5. A simple pendulum is attached to a rotating rim.

Taking the time derivative once again gives the acceleration:

$$\ddot{x} = -a\omega^2 \cos \omega t + b(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta)$$

$$\ddot{y} = -a\omega^2 \sin \omega t + b(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta)$$

It should now be clear that the single generalized coordinate is θ. The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$U = mgy$$

where $U = 0$ at $y = 0$. The Lagrangian is

$$L = T - U = \frac{m}{2}[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega \sin (\theta - \omega t)] - mg(a \sin \omega t - b \cos \theta) \tag{7.34}$$

The derivatives for the Lagrange equation of motion for θ are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega) \cos(\theta - \omega t)$$

$$\frac{\partial L}{\partial \theta} = mb\dot{\theta}a\omega \cos(\theta - \omega t) - mgb \sin \theta$$

which results in the equation of motion (after solving for $\ddot{\theta}$)

$$\ddot{\theta} = \frac{\omega^2 a}{b} \cos(\theta - \omega t) - \frac{g}{b} \sin \theta \tag{7.35}$$

Notice that this result reduces to the well-known equation of motion for a simple pendulum if $\omega = 0$.

EXAMPLE 7.6

Find the frequency of small oscillations of a simple pendulum placed in a railroad car that has a constant acceleration a in the x-direction.

Solution. A schematic diagram is shown in Figure 7-4a for the pendulum of length ℓ, mass m, and displacement angle θ. We choose a fixed cartesian coordinate system with $x = 0$ and $\dot{x} = v_0$ at $t = 0$. The position and velocity of m become

$$x = v_0 t + \frac{1}{2}at^2 + \ell \sin \theta$$

$$y = -\ell \cos \theta$$

$$\dot{x} = v_0 + at + \ell\dot{\theta} \cos \theta$$

$$\dot{y} = \ell\dot{\theta} \sin \theta$$

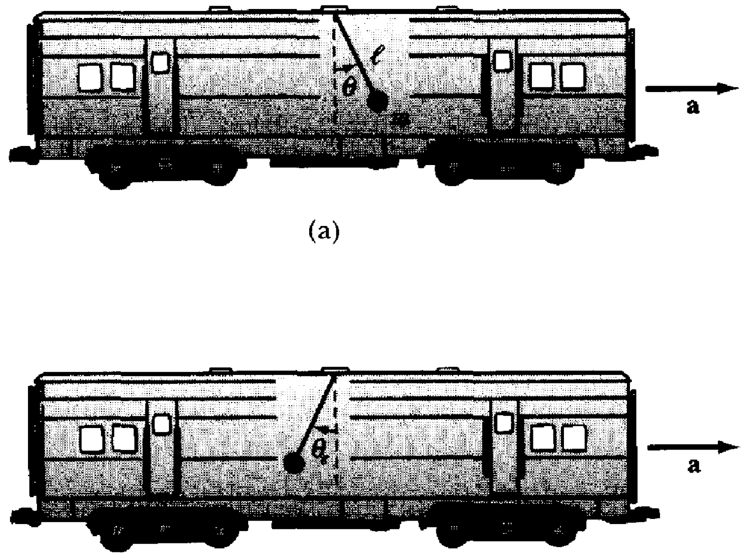

FIGURE 7-4 Example 7.6. (a) A simple pendulum swings in an accelerating railroad car. (b) The angle θ_e is the equilibrium angle due to the car's acceleration a and acceleration of gravity g.

The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad U = -mg\ell \cos\theta$$

and the Lagrangian is

$$L = T - U = \frac{1}{2}m(v_0 + at + \ell\dot{\theta}\cos\theta)^2 + \frac{1}{2}m(\ell\dot{\theta}\sin\theta)^2 + mg\ell\cos\theta$$

The angle θ is the only generalized coordinate, and after taking the derivatives for Lagrange's equations and suitable collection of terms, the equation of motion becomes (Problem 7-2)

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta - \frac{a}{\ell}\cos\theta \tag{7.36}$$

We determine the equilibrium angle $\theta = \theta_e$ by setting $\ddot{\theta} = 0$,

$$0 = g\sin\theta_e + \text{a}\cos\theta_e \tag{7.37}$$

The equilibrium angle θ_e, shown in Figure 7-4b, is obtained by

$$\tan\theta_e = -\frac{a}{g} \tag{7.38}$$

Because the oscillations are small and are about the equilibrium angle, let $\theta = \theta_e + \eta$, where η is a small angle.

$$\ddot{\theta} = \ddot{\eta} = -\frac{g}{\ell}\sin(\theta_e + \eta) - \frac{a}{\ell}\cos(\theta_e + \eta) \tag{7.39}$$

We expand the sine and cosine terms and use the small angle approximation for $\sin \eta$ and $\cos \eta$, keeping only the first terms in the Taylor series expansions.

$$\begin{aligned}\ddot{\eta} &= -\frac{g}{\ell}(\sin\theta_e \cos\eta + \cos\theta_e \sin\eta) - \frac{a}{\ell}(\cos\theta_e \cos\eta - \sin\theta_e \sin\eta) \\ &= -\frac{g}{\ell}(\sin\theta_e + \eta\cos\theta_e) - \frac{a}{\ell}(\cos\theta_e - \eta\sin\theta_e) \\ &= -\frac{1}{\ell}[(g\sin\theta_e + a\cos\theta_e) + \eta(g\cos\theta_e - a\sin\theta_e)]\end{aligned}$$

The first term in the brackets is zero because of Equation 7.37, which leaves

$$\ddot{\eta} = -\frac{1}{\ell}(g\cos\theta_e - a\sin\theta_e)\eta \tag{7.40}$$

We use Equation 7.38 to determine $\sin\theta_e$ and $\cos\theta_e$ and after a little manipulation (Problem 7-2), Equation 7.40 becomes

$$\ddot{\eta} = -\frac{\sqrt{a^2+g^2}}{\ell}\eta \tag{7.41}$$

Because this equation now represents simple harmonic motion, the frequency ω is determined to be

$$\omega^2 = \frac{\sqrt{a^2+g^2}}{\ell} \tag{7.42}$$

This result seems plausible, because $\omega \to \sqrt{g/\ell}$ for $a = 0$ when the railroad car is at rest.

EXAMPLE 7.7

A bead slides along a smooth wire bent in the shape of a parabola $z = cr^2$ (Figure 7-5). The bead rotates in a circle of radius R when the wire is rotating about its vertical symmetry axis with angular velocity ω. Find the value of c.

Solution. Because the problem has cylindrical symmetry, we choose r, θ, and z as the generalized coordinates. The kinetic energy of the bead is

$$T = \frac{m}{2}[\dot{r}^2 + \dot{z}^2 + (r\dot{\theta}^2)] \tag{7.43}$$

If we choose $U = 0$ at $z = 0$, the potential energy term is

$$U = mgz \tag{7.44}$$

But r, z, and θ are not independent. The equation of constraint for the parabola is

$$z = cr^2 \tag{7.45}$$

$$\dot{z} = 2c\dot{r}r \tag{7.46}$$

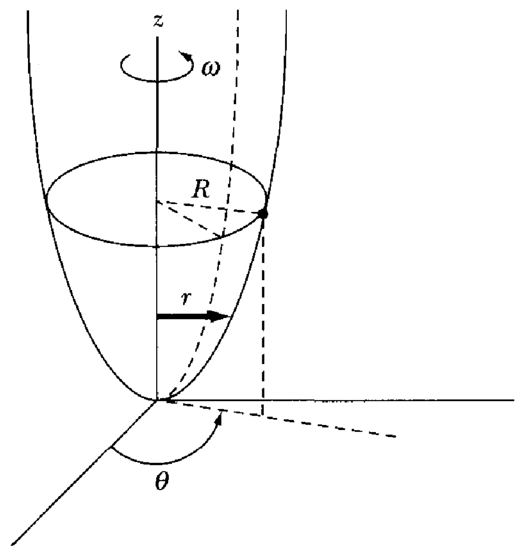

FIGURE 7-5 Example 7.7. A bead slides along a smooth wire that rotates about the z-axis.

We also have an explicit time dependence of the angular rotation

$$\theta = \omega t$$
$$\dot{\theta} = \omega \tag{7.47}$$

We can now construct the Lagrangian as being dependent only on r, because there is no direct θ dependence.

$$\begin{aligned} L &= T - U \\ &= \frac{m}{2}(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2) - mgcr^2 \end{aligned} \tag{7.48}$$

The problem stated that the bead moved in a circle of radius R. The reader might be tempted at this point to let $r = R =$ const. and $\dot{r} = 0$. It would be a mistake to do this now in the Lagrangian. First, we should find the equation of motion for the variable r and then let $r = R$ as a condition of the particular motion. This determines the particular value of c needed for $r = R$.

$$\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}(2\dot{r} + 8c^2r^2\dot{r})$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r})$$

$$\frac{\partial L}{\partial r} = m(4c^2r\dot{r}^2 + r\omega^2 - 2gcr)$$

Lagrange's equation of motion becomes

$$\ddot{r}(1 + 4c^2r^2) + \dot{r}^2(4c^2r) + r(2gc - \omega^2) = 0 \tag{7.49}$$

which is a complicated result. If, however, the bead rotates with $r = R =$ constant, then $\dot{r} = \ddot{r} = 0$, and Equation 7.49 becomes

$$R(2gc - \omega^2) = 0$$

and

$$c = \frac{\omega^2}{2g} \tag{7.50}$$

is the result we wanted.

EXAMPLE 7.8

Consider the double pulley system shown in Figure 7-6. Use the coordinates indicated, and determine the equations of motion.

Solution. Consider the pulleys to be massless, and let l_1 and l_2 be the lengths of rope hanging freely from each of the two pulleys. The distances x and y are measured from the center of the two pulleys.

m_1:

$$v_1 = \dot{x} \tag{7.51}$$

m_2:

$$v_2 = \frac{d}{dt}(l_1 - x + y) = -\dot{x} + \dot{y} \tag{7.52}$$

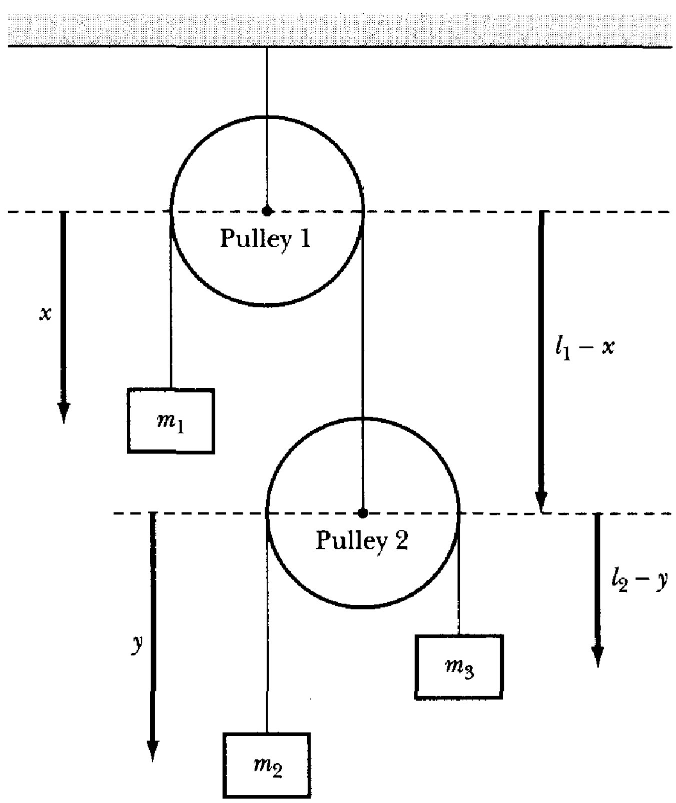

FIGURE 7-6 Example 7.8. The double pulley system.

m_3:

$$v_3 = \frac{d}{dt}(l_1 - x + l_2 - y) = -\dot{x} - \dot{y} \tag{7.53}$$

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{y} - \dot{x})^2 + \frac{1}{2}m_3(-\dot{x} - \dot{y})^2 \tag{7.54}$$

Let the potential energy $U = 0$ at $x = 0$.

$$U = U_1 + U_2 + U_3$$

$$= -m_1 gx - m_2 g(l_1 - x + y) - m_3 g(l_1 - x + l_2 - y) \tag{7.55}$$

Because T and U have been determined, the equations of motion can be obtained using Equation 7.18. The results are

$$m_1\ddot{x} + m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_1 - m_2 - m_3)g \tag{7.56}$$

$$-m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_2 - m_3)g \tag{7.57}$$

Equations 7.56 and 7.57 can be solved for $\ddot{x}$ and $\ddot{y}$.

Examples 7.2–7.8 indicate the ease and usefulness of using Lagrange's equations. It has been said, probably unfairly, that Lagrangian techniques are simply recipes to follow. The argument is that we lose track of the "physics" by their use. Lagrangian methods, on the contrary, are extremely powerful and allow us to solve problems that otherwise would lead to severe complications using Newtonian methods. Simple problems can perhaps be solved just as easily using Newtonian methods, but the Lagrangian techniques can be used to attack a wide range of complex physical situations (including those occurring in quantum mechanics*).

7.5 Lagrange's Equations with Undetermined Multipliers

Constraints that can be expressed as algebraic relations among the coordinates are holonomic constraints. If a system is subject only to such constraints, we can always find a proper set of generalized coordinates in terms of which the equations of motion are free from explicit reference to the constraints.

Any constraints that must be expressed in terms of the *velocities* of the particles in the system are of the form

$$f(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) = 0 \tag{7.58}$$

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*See Feynman and Hibbs (Fe65).