Here we consider only the motion of systems subject to conservative forces. Such forces can always be derived from potential functions, so that condition 1 is satisfied. This is not a necessary restriction on either Hamilton's Principle or Lagrange's equations; the theory can readily be extended to include nonconservative forces. Similarly, we can formulate Hamilton's Principle to include certain types of nonholonomic constraints, but the treatment here is confined to holonomic systems. We return to nonholonomic constraints in Section 7.5.

We now want to work several examples using Lagrange's equations. Experience is the best way to determine a set of generalized coordinates, realize the constraints, and set up the Lagrangian. Once this is done, the remainder of the problem is for the most part mathematical.

EXAMPLE 7.3

Consider the case of projectile motion under gravity in two dimensions as was discussed in Example 2.6. Find the equations of motion in both Cartesian and polar coordinates.

Solution. We use Figure 2-7 to describe the system. In Cartesian coordinates, we use x (horizontal) and y (vertical). In polar coordinates we use r (in radial direction) and θ (elevation angle from horizontal). First, in Cartesian coordinates we have

$$\left.\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \\ U &= mgy \end{aligned}\right\} \tag{7.19}$$

where $U = 0$ at $y = 0$.

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy \tag{7.20}$$

We find the equations of motion by using Equation 7.18:

x:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$0 - \frac{d}{dt}m\dot{x} = 0$$

$$\ddot{x} = 0 \tag{7.21}$$

y:

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$-mg - \frac{d}{dt}(m\dot{y}) = 0$$

$$\ddot{y} = -g \tag{7.22}$$

By using the initial conditions, Equations 7.21 and 7.22 can be integrated to determine the appropriate equations of motion.

In polar coordinates, we have

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2$$

$$U = mgr\sin\theta$$

where $U = 0$ for $\theta = 0$.

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\sin\theta \tag{7.23}$$

r:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\theta}^2 - mg\sin\theta - \frac{d}{dt}(m\dot{r}) = 0$$

$$r\dot{\theta}^2 - g\sin\theta - \ddot{r} = 0 \tag{7.24}$$

θ:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$-mgr\cos\theta - \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$-gr\cos\theta - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} = 0 \tag{7.25}$$

The equations of motion expressed by Equations 7.21 and 7.22 are clearly simpler than those of Equations 7.24 and 7.25. We should choose Cartesian coordinates as the generalized coordinates to solve this problem. The key in recognizing this was that the potential energy of the system only depended on the y coordinate. In polar coordinates, the potential energy depended on both r and θ.

EXAMPLE 7.4

A particle of mass m is constrained to move on the inside surface of a smooth cone of half-angle α (see Figure 7-2). The particle is subject to a gravitational force. Determine a set of generalized coordinates and determine the constraints. Find Lagrange's equations of motion, Equation 7.18.

Solution. Let the axis of the cone correspond to the z-axis and let the apex of the cone be located at the origin. Since the problem possesses cylindrical symmetry, we choose r, θ, and z as the generalized coordinates. We have, however, the equation of constraint

$$z = r\cot\alpha \tag{7.26}$$

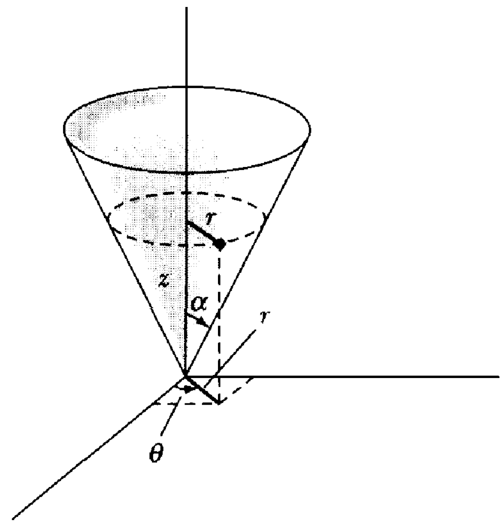

FIGURE 7-2 Example 7.4. A smooth cone of half-angle α. We choose r, θ, and z as the generalized coordinates.

so there are only two degrees of freedom for the system, and therefore only two proper generalized coordinates. We may use Equation 7.26 to eliminate either the coordinate z or r; we choose to do the former. Then the square of the velocity is

$$\begin{aligned} v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 \\ &= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2 \cot^2\alpha \\ &= \dot{r}^2 \csc^2\alpha + r^2\dot{\theta}^2 \end{aligned} \tag{7.27}$$

The potential energy (if we choose $U = 0$ at $z = 0$) is

$$U = mgz = mgr \cot \alpha$$

so the Langrangian is

$$L = \frac{1}{2}m\,(\dot{r}^2 \csc^2\alpha + r^2\dot{\theta}^2) - mgr \cot \alpha \tag{7.28}$$

We note first that L does not explicitly contain θ. Therefore $\partial L/\partial\theta = 0$, and the Lagrange equation for the coordinate θ is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

Hence

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant} \tag{7.29}$$

But $mr^2\dot{\theta} = mr^2\omega$ is just the angular momentum about the z-axis. Therefore, Equation 7.29 expresses the conservation of angular momentum about the axis of symmetry of the system.

The Lagrange equation for r is

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \tag{7.30}$$

Calculating the derivatives, we find

$$\ddot{r} - r\dot{\theta}^2 \sin^2\alpha + g \sin\alpha \cos\alpha = 0 \tag{7.31}$$

which is the equation of motion for the coordinate r.

We shall return to this example in Section 8.10 and examine the motion in more detail.

EXAMPLE 7.5

The point of support of a simple pendulum of length b moves on a massless rim of radius a rotating with constant angular velocity ω. Obtain the expression for the Cartesian components of the velocity and acceleration of the mass m. Obtain also the angular acceleration for the angle θ shown in Figure 7-3.

Solution. We choose the origin of our coordinate system to be at the center of the rotating rim. The Cartesian components of mass m become

$$\left.\begin{aligned} x &= a\cos\omega t + b\sin\theta \\ y &= a\sin\omega t - b\cos\theta \end{aligned}\right\} \tag{7.32}$$

The velocities are

$$\left.\begin{aligned} \dot{x} &= -a\omega\sin\omega t + b\dot{\theta}\cos\theta \\ \dot{y} &= a\omega\cos\omega t + b\dot{\theta}\sin\theta \end{aligned}\right\} \tag{7.33}$$

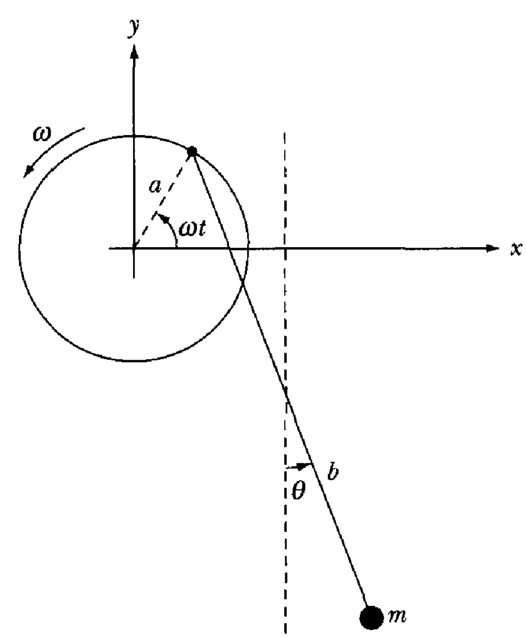

FIGURE 7-3 Example 7.5. A simple pendulum is attached to a rotating rim.