developed. All such approaches are in essence *a posteriori*, because we know beforehand that a result equivalent to the Newtonian equations must be obtained. Thus, to effect a simplification we need not formulate a *new* theory of mechanics—the Newtonian theory is quite correct—but only devise an alternate method of dealing with complicated problems in a general manner. Such a method is contained in **Hamilton's Principle,** and the equations of motion resulting from the application of this principle are called **Lagrange's equations.**

If Lagrange's equations are to constitute a proper description of the dynamics of particles, they must be equivalent to Newton's equations. On the other hand, Hamilton's Principle can be applied to a wide range of physical phenomena (particularly those involving *fields*) not usually associated with Newton's equations. To be sure, each of the results that can be obtained from Hamilton's Principle was *first* obtained, as were Newton's equations, by the correlation of experimental facts. Hamilton's Principle has not provided us with any new physical theories, but it has allowed a satisfying unification of many individual theories by a single basic postulate. This is not an idle exercise in hindsight, because it is the goal of physical theory not only to give precise mathematical formulation to observed phenomena but also to describe these effects with an economy of fundamental postulates and in the most unified manner possible. Indeed, Hamilton's Principle is one of the most elegant and far-reaching principles of physical theory.

In view of its wide range of applicability (even though this is an after-the-fact discovery), it is not unreasonable to assert that Hamilton's Principle is more "fundamental" than Newton's equations. Therefore, we proceed by first postulating Hamilton's Principle; we then obtain Lagrange's equations and show that these are equivalent to Newton's equations.

Because we have already discussed (in Chapters 2, 3, and 4) dissipative phenomena at some length, we henceforth confine our attention to *conservative* systems. Consequently, we do not discuss the more general set of Lagrange's equations, which take into account the effects of nonconservative forces. The reader is referred to the literature for these details.*

7.2 Hamilton's Principle

Minimal principles in physics have a long and interesting history. The search for such principles is predicated on the notion that nature always minimizes certain important quantities when a physical process takes place. The first such minimum principles were developed in the field of optics. Hero of Alexandria, in the second century B.C., found that the law governing the reflection of light could be obtained by asserting that a light ray, traveling from one point to another by a reflection from a plane mirror, always takes the shortest possible path. A simple geometric construction verifies that this minimum principle does indeed lead to

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*See, for example, Goldstein (Go80, Chapter 2) or, for a comprehensive discussion, Whittaker (Wh37, Chapter 8).

the equality of the angles of incidence and reflection for a light ray reflected from a plane mirror. Hero's principle of the *shortest path* cannot, however, yield a correct law for *refraction*. In 1657, Fermat reformulated the principle by postulating that a light ray always travels from one point to another in a medium by a path that requires the least time.* Fermat's principle of *least time* leads immediately, not only to the correct law of reflection, but also to Snell's law of refraction (see Problem 6-7).†

Minimum principles continued to be sought, and in the latter part of the seventeenth century the beginnings of the calculus of variations were developed by Newton, Leibniz, and the Bernoullis when such problems as the brachistochrone (see Example 6.2) and the shape of a hanging chain (a catenary) were solved.

The first application of a general minimum principle in mechanics was made in 1747 by Maupertuis, who asserted that dynamical motion takes place with minimum action.‡ Maupertuis's **principle of least action** was based on theological grounds (action is minimized through the "wisdom of God"), and his concept of "action" was rather vague. (Recall that *action* is a quantity with the dimensions of *length* × *momentum* or *energy* × *time*.) Only later was a firm mathematic foundation of the principle given by Lagrange (1760). Although it is a useful form from which to make the transition from classical mechanics to optics and to quantum mechanics, the principle of least action is less general than Hamilton's Principle and, indeed, can be derived from it. We forego a detailed discussion here.§

In 1828, Gauss developed a method of treating mechanics by his principle of **least constraint;** a modification was later made by Hertz and embodied in his principle of **least curvature.** These principles‖ are closely related to Hamilton's Principle and add nothing to the content of Hamilton's more general formulation; their mention only emphasizes the continual concern with minimal principles in physics.

In two papers published in 1834 and 1835, Hamilton¶ announced the dynamical principle on which it is possible to base all of mechanics and, indeed, most of classical physics. Hamilton's Principle may be stated as follows**:

> *Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.*

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*Pierre de Fermat (1601–1665), a French lawyer, linguist, and amateur mathematician.

†In 1661, Fermat correctly deduced the law of refraction, which had been discovered experimentally in about 1621 by Willebrord Snell (1591–1626), a Dutch mathematical prodigy.

‡Pierre-Louise-Moreau de Maupertuis (1698–1759), French mathematician and astronomer. The first use to which Maupertuis put the principle of least action was to restate Fermat's derivation of the law of refraction (1744).

§See, for example, Goldstein (Go80, pp. 365–371) or Sommerfeld (So50, pp. 204–209).

‖See, for example, Lindsay and Margenau (Li36, pp. 112–120) or Sommerfeld (So50, pp. 210–214).

¶Sir William Rowan Hamilton (1805–1865), Irish mathematician and astronomer, and later, Irish Astronomer Royal.

**The general meaning of "the path of a system" is made clear in Section 7.3.

In terms of the calculus of variations, Hamilton's Principle becomes

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0 \tag{7.1}$$

where the symbol δ is a shorthand notation to describe the variation discussed in Sections 6.3 and 6.7. This variational statement of the principle requires only that the integral of $T - U$ be an *extremum*, not necessarily a *minimum*. But in almost all important applications in dynamics, the minimum condition occurs.

The kinetic energy of a particle expressed in fixed, rectangular coordinates is a function only of the $\dot{x}_i$, and if the particle moves in a conservative force field, the potential energy is a function only of the x_i:

$$T = T(\dot{x}_i), \quad U = U(x_i)$$

If we define the difference of these quantities to be

$$L \equiv T - U = L(x_i, \dot{x}_i) \tag{7.2}$$

then Equation 7.1 becomes

$$\boxed{\delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i)\, dt = 0} \tag{7.3}$$

The function L appearing in this expression may be identified with the function f of the variational integral (see Section 6.5),

$$\delta \int_{x_1}^{x_2} f\{y_i(x), y_i'(x); x\}\, dx$$

if we make the transformations

$$\begin{aligned} x &\rightarrow t \\ y_i(x) &\rightarrow x_i(t) \\ y_i'(x) &\rightarrow \dot{x}_i(t) \\ f\{y_i(x), y_i'(x); x\} &\rightarrow L(x_i, \dot{x}_i) \end{aligned}$$

The Euler-Lagrange equations (Equation 6.57) corresponding to Equation 7.3 are therefore

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1, 2, 3} \quad \text{Lagrange equations of motion} \tag{7.4}$$

These are the **Lagrange equations of motion** for the particle, and the quantity L is called the **Lagrange function** or **Lagrangian** for the particle.