regarding *force*. The equations of motion were obtained only by specifying certain properties associated *with the particle* (the kinetic and potential energies), and without the necessity of explicitly taking into account the fact that there was an external agency acting *on the particle* (the force). Therefore, insofar as *energy* can be defined independently of Newtonian concepts, Hamilton's Principle allows us to calculate the equations of motion of a body completely without recourse to Newtonian theory. We shall return to this important point in Sections 7.5 and 7.7.

7.3 Generalized Coordinates

We now seek to take advantage of the flexibility in specifying coordinates that the two examples of the preceding section have suggested is inherent in Lagrange's equations.

We consider a general mechanical system consisting of a collection of n discrete point particles, some of which may be connected to form rigid bodies. We discuss such systems of particles in Chapter 9 and rigid bodies in Chapter 11. To specify the state of such a system at a given time, it is necessary to use n radius vectors. Because each radius vector consists of three numbers (e.g., the rectangular coordinates), $3n$ quantities must be specified to describe the positions of all the particles. If there exist equations of constraint that relate some of these coordinates to others (as would be the case, for example, if some of the particles were connected to form rigid bodies or if the motion were constrained to lie along some path or on some surface), then not all the $3n$ coordinates are independent. In fact, if there are m equations of constraint, then $3n - m$ coordinates are independent, and the system is said to possess $3n - m$ *degrees of freedom.*

It is important to note that if $s = 3n - m$ coordinates are required in a given case, we need not choose s rectangular coordinates or even s curvilinear coordinates (e.g., spherical, cylindrical). We can choose *any* s independent parameters, as long as they completely specify the state of the system. These s quantities need not even have the dimensions of length. Depending on the problem at hand, it may prove more convenient to choose some of the parameters with dimensions of *energy,* some with dimensions of $(length)^2$, some that are *dimensionless,* and so forth. In Example 6.5, we described a disk rolling down an inclined plane in terms of one coordinate that was a length and one that was an angle. We give the name **generalized coordinates** to any set of quantities that completely specifies the state of a system. The generalized coordinates are customarily written as $q_1, q_2, \ldots$, or simply as the q_j. A set of independent generalized coordinates whose number equals the number s of degrees of freedom of the system and not restricted by the constraints is called a *proper* set of generalized coordinates. In certain instances, it may be advantageous to use generalized coordinates whose number exceeds the number of degrees of freedom and to explicitly take into account the constraint relations through the use of the Lagrange undetermined multipliers. Such would be the case, for example, if we desired to calculate the forces of constraint (see Example 7.9).

The choice of a set of generalized coordinates to describe a system is not unique; there are in general many sets of quantities (in fact, an *infinite* number!) that completely specify the state of a given system. For example, in the problem of the disk rolling down the inclined plane, we might choose as coordinates the height of the center of mass of the disk above some reference level and the distance through which some point on the rim has traveled since the start of the motion. The ultimate test of the "suitability" of a particular set of generalized coordinates is whether the resulting equations of motion are sufficiently simple to allow a straightforward interpretation. Unfortunately, we can state no general rules for selecting the "most suitable" set of generalized coordinates for a given problem. A certain skill must be developed through experience, and we present many examples in this chapter.

In addition to the generalized coordinates, we may define a set of quantities consisting of the time derivatives of $\dot{q}_j$: $\dot{q}_1, \dot{q}_2, \ldots$, or simply $\dot{q}_j$. In analogy with the nomenclature for rectangular coordinates, we call $\dot{q}_j$ the **generalized velocities.**

If we allow for the possibility that the equations connecting $x_{\alpha,i}$ and q_j explicitly contain the time, then the set of transformation equations is given by*

$$\begin{aligned} x_{\alpha,i} &= x_{\alpha,i}(q_1, q_2, \ldots, q_s, t), \qquad \begin{cases} \alpha = 1, 2, \ldots, n \\ i = 1, 2, 3 \end{cases} \\ &= x_{\alpha,i}(q_j, t), \qquad\qquad\qquad\quad j = 1, 2, \ldots, s \end{aligned} \tag{7.5}$$

In general, the rectangular components of the velocities depend on the generalized coordinates, the generalized velocities, and the time:

$$\dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(q_j, \dot{q}_j, t) \tag{7.6}$$

We may also write the inverse transformations as

$$q_j = q_j(x_{\alpha,i}, t) \tag{7.7}$$

$$\dot{q}_j = \dot{q}_j(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) \tag{7.8}$$

Also, there are $m = 3n - s$ equations of constraint of the form

$$f_k(x_{\alpha,i}, t) = 0, \quad k = 1, 2, \ldots, m \tag{7.9}$$

EXAMPLE 7.1

Find a suitable set of generalized coordinates for a point particle moving on the surface of a hemisphere of radius R whose center is at the origin.

Solution. Because the motion always takes place on the surface, we have

$$x^2 + y^2 + z^2 - R^2 = 0, \quad z \geq 0 \tag{7.10}$$

Let us choose as our generalized coordinates the cosines of the angles between the x-, y-, and z-axes and the line connecting the particle with the origin.

*In this chapter, we attempt to simplify the notation by reserving the subscript i to designate rectangular axes; therefore, we always have $i = 1, 2, 3$.

Therefore,

$$q_1 = \frac{x}{R}, \quad q_2 = \frac{y}{R}, \quad q_3 = \frac{z}{R} \tag{7.11}$$

But the sum of the squares of the direction cosines of a line equals unity. Hence,

$$q_1^2 + q_2^2 + q_3^2 = 1 \tag{7.12}$$

This set of q_j does not constitute a proper set of generalized coordinates, because we can write q_3 as a function of q_1 and q_2:

$$q_3 = \sqrt{1 - q_1^2 - q_2^2} \tag{7.13}$$

We may, however, choose $q_1 = x/R$ and $q_2 = y/R$ as proper generalized coordinates, and these quantities, together with the equation of constraint (Equation 7.13)

$$z = \sqrt{R^2 - x^2 - y^2} \tag{7.14}$$

are sufficient to uniquely specify the position of the particle. This should be an obvious result, because only two coordinates (e.g., latitude and longitude) are necessary to specify a point on the surface of a sphere. But the example illustrates the fact that the equations of constraint can always be used to reduce a trial set of coordinates to a proper set of generalized coordinates.

EXAMPLE 7.2

Use the (x, y) coordinate system of Figure 7-1 to find the kinetic energy T, potential energy U, and the Lagrangian L for a simple pendulum (length ℓ, mass bob m) moving in the x, y plane. Determine the transformation equations from the (x, y) rectangular system to the coordinate θ. Find the equation of motion.

Solution. We have already examined this general problem in Sections 4.4 and 7.1. When using the Lagrangian method, it is often useful to begin with

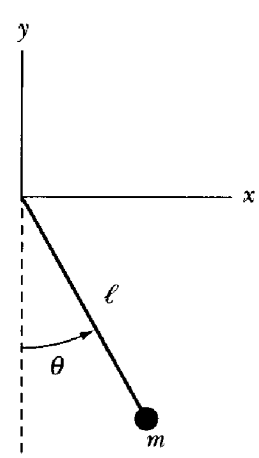

FIGURE 7-1 Example 7.2. A simple pendulum of length ℓ and bob of mass m.