CHAPTER 7

Advanced Linear programming

1

$Q = \{x_1, x_2 \mid x_1 + x_2 \le 1, x_1 \ge 0, x_2 \ge 0$

Let $(\bar{x}_1, \bar{x}_2) \ge 0$ and $(\bar{\bar{x}}_1, \bar{\bar{x}}_2) \ge 0$ be two distinct points in Q and define for $0 \le \lambda \le 1$:

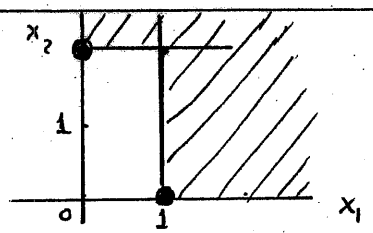
$$(x_1, x_2) = \lambda(\bar{x}_1, \bar{x}_2) + (1-\lambda)(\bar{\bar{x}}_1, \bar{\bar{x}}_2) \ge 0$$

Then,

$$x_1 + x_2 = \lambda \bar{x}_1 + (1-\lambda)\bar{\bar{x}}_1 + \lambda \bar{x}_2 + (1-\lambda)\bar{\bar{x}}_2$$
$$= \lambda(\bar{x}_1 + \bar{x}_2) + (1-\lambda)(\bar{\bar{x}}_1 + \bar{\bar{x}}_2)$$
$$\le \lambda(1) + (1-\lambda)(1) = 1$$

which shows that Q is convex. The result is true even without the nonnegativity restrictions.

2



$Q = \{x_1, x_2 \mid x_1 \ge 1 \text{ or } x_2 \ge 2\}$

Let $(\bar{x}_1, \bar{x}_2) = (1, 0) \in Q$
$(\bar{\bar{x}}_1, \bar{\bar{x}}_2) = (0, 2) \in Q$

Consider

$$(x_1, x_2) = \lambda(1,0) + (1-\lambda)(0,2)$$
$$= (\lambda, 2-2\lambda) \quad 0 \le \lambda \le 1$$

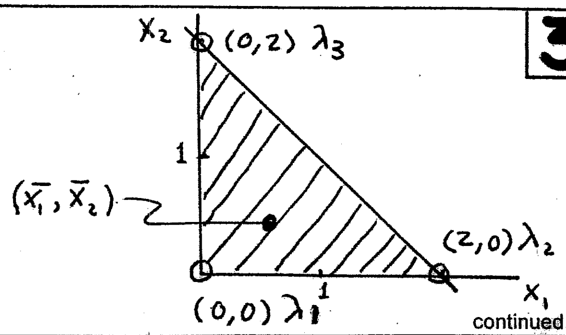
For $0 < \lambda < 1$, we have

$x_1 = \lambda < 1$

$x_2 = 2 - 2\lambda < 2$

Thus, $(x_1, x_2) \notin Q$.

3



continued..

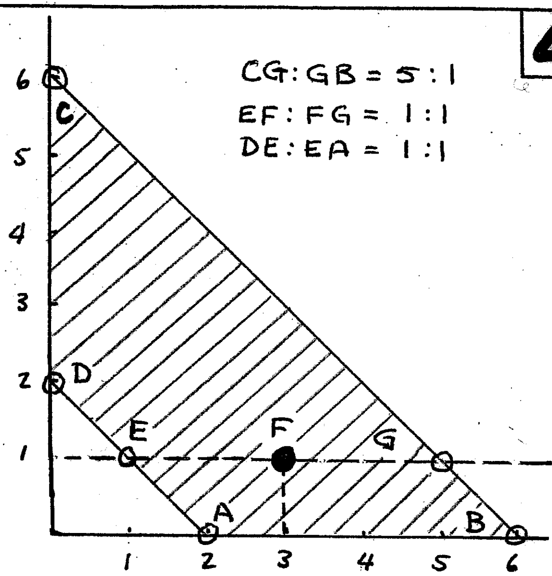
$Q = \{x_1, x_2 \mid x_1 + x_2 \le 2, x_1, x_2 \ge 0\}$

$$(\bar{x}_1, \bar{x}_2) = \lambda_1(0,0) + \lambda_2(2,0) + \lambda_3(0,2)$$
$$= (2\lambda_2, 2\lambda_3)$$

where $\lambda_1, \lambda_2, \lambda_3 \ge 0$

$\lambda_1 + \lambda_2 + \lambda_3 = 1$

4



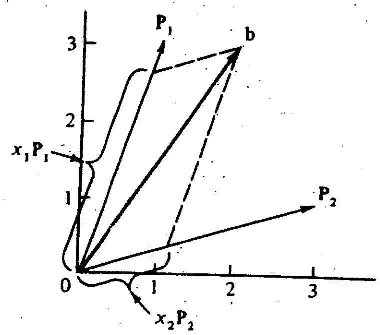
$$E = \frac{1}{2}A + \frac{1}{2}D$$

$$G = \frac{5}{6}B + \frac{1}{6}C$$

$$F = \frac{1}{2}E + \frac{1}{2}G$$
$$= \frac{1}{2}\left(\frac{1}{2}A + \frac{1}{2}D\right) + \frac{1}{2}\left(\frac{5}{6}B + \frac{1}{6}C\right)$$
$$= \frac{1}{4}A + \frac{1}{4}D + \frac{5}{12}B + \frac{1}{12}C$$
$$= \frac{1}{4}(2,0) + \frac{1}{4}(0,2) + \frac{5}{12}(6,0) + \frac{1}{12}(0,6)$$
$$= (3,1)$$

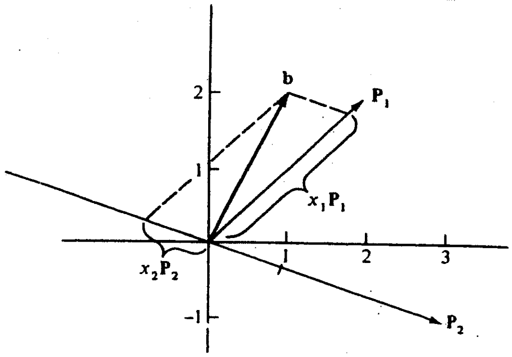
1

(a)



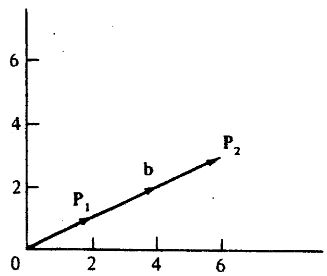
Unique solution:
$(x_1, x_2) = (7/8, 3/8)$,
left-side vectors P_1 and P_2
are independent (basis)

(b)



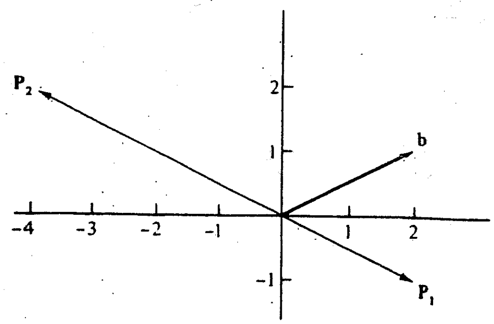
Unique solution:
$(x_1, x_2) = (7/8, -1/4)$,
P_1 and P_2 form a basis

(c)



Infinity of solutions:
P_1 and P_2 are dependent
(no basis); b is also
dependent

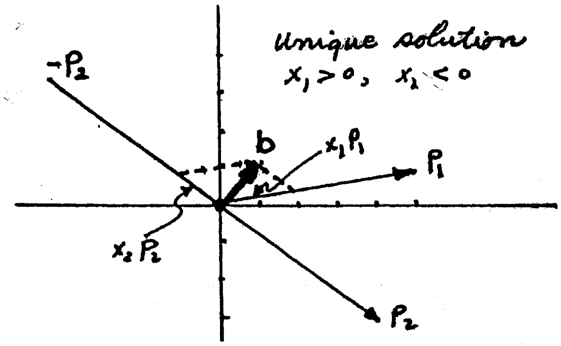
(d)



No solution: P_1 and P_2
are dependent (no basis),
but b is independent

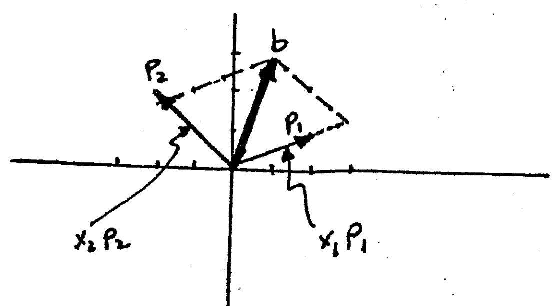
2

(a) $\begin{pmatrix} 5 & 4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



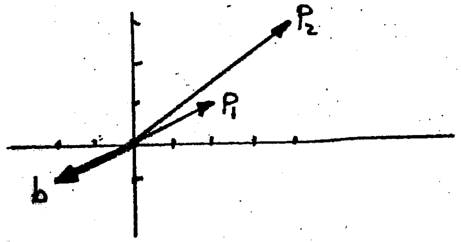
Unique solution
$x_1 > 0$, $x_2 < 0$

b) $\begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$



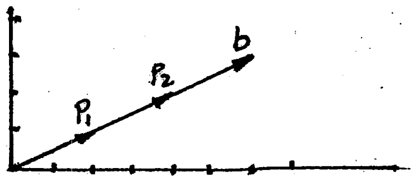
Unique solution: $x_1, x_2 > 0$
$x_1 > 1$, $x_2 < 1$

(c) $\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$



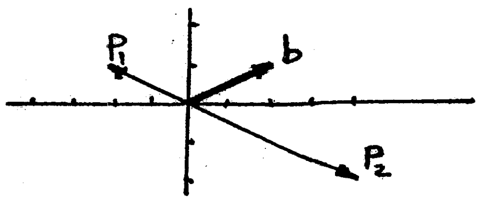
Unique solution: $x_1 < 0$, $x_2 = 0$

(d) $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$



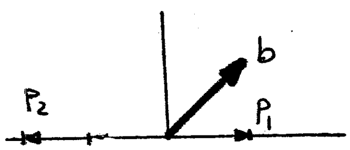
Infinity of solutions

(e) $\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



No solution

(f) $\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



No solution

3

(a) $\det(P_1, P_2, P_3) = \det\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}$

$= -4 \neq 0$, basis

(b) $\det(P_1, P_2, P_4) = \det\begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}$

$= -8 \neq 0$, basis

(c) $\det(P_2, P_3, P_4) = \det\begin{pmatrix} 0 & 1 & 2 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{pmatrix}$

$= 0$, not a basis

(d) In this problem, a basis must include exactly 3 independent vectors.

4

(a) True

(b) True

(c) True

1

$B=(P_3,P_4)=\begin{pmatrix}2&4\\-2&6\end{pmatrix}$

$B^{-1}=\begin{pmatrix}.3&-.2\\.1&.1\end{pmatrix}$, $X_B=\begin{pmatrix}x_3\\x_4\end{pmatrix}$, $C_B=(7,5)$

$X_B=B^{-1}b=\begin{pmatrix}.3&-.2\\.1&.1\end{pmatrix}\begin{pmatrix}10\\5\end{pmatrix}=\begin{pmatrix}2\\1.5\end{pmatrix}\geq\begin{pmatrix}0\\0\end{pmatrix}$

$C_BB^{-1}=(7,5)\begin{pmatrix}.3&-.2\\.1&.1\end{pmatrix}=(2.6,-.9)$

$\{z_j-c_j\}_{j=1,2}=(2.6,-.9)\begin{pmatrix}2&1\\3&-1\end{pmatrix}-(1,4)$

$=(1.5,-.5)$

$B^{-1}(P_1,P_2)=\begin{pmatrix}.3&-.2\\.1&.1\end{pmatrix}\begin{pmatrix}2&1\\3&-1\end{pmatrix}=\begin{pmatrix}0&.5\\.5&0\end{pmatrix}$

X_B is feasible but not optimal.

Tableau:

	x_1	x_2	x_3	x_4	
z	1.5	-.5	0	0	21.5
x_3	0	.5	1	0	2
x_4	.5	0	0	1	1.5

2

Maximize $z=(5,12,4)\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$

Subject to

$$\begin{pmatrix}1&2&1&1\\2&-2&-1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}10\\2\end{pmatrix}$$

$P_1\quad P_2\quad P_3\quad P_4$

$\det(P_1\ P_2)=\det\begin{pmatrix}1&2\\2&-2\end{pmatrix}$

$=-6\neq 0\Rightarrow$ basis

$\det(P_2\ P_3)=\det\begin{pmatrix}2&1\\-2&-1\end{pmatrix}$

$=0\Rightarrow$ not a basis

$\det(P_3\ P_4)=\det\begin{pmatrix}1&1\\-1&0\end{pmatrix}$

$=1\neq 0\Rightarrow$ basis

3

$X_B=(x_1,x_2,x_5)^T$, $C_B=(2,1,0)$

$B^{-1}=\begin{pmatrix}3&1&0\\4&3&0\\1&2&1\end{pmatrix}^{-1}=\begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}$

$C_BB^{-1}=(2,1,0)\begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}$

$=(2/5,\ 1/5,\ 0)$

$(z_3-c_3,\ z_4-c_4)$

$=(2/5,1/5,0)\begin{pmatrix}-1&0\\0&-1\\0&0\end{pmatrix}-(0,0)$

$=(-2/5,-1/5)\Rightarrow$ optimal

$B^{-1}(P_1\ P_2\ P_3\ P_4\ P_5\mid b)$

$=\begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}\left(\begin{array}{ccccc|c}3&1&-1&0&0&3\\4&3&0&-1&0&6\\1&2&0&0&1&3\end{array}\right)$

$=\left(\begin{array}{ccccc|c}1&0&-3/5&1/5&0&3/5\\0&1&4/5&-3/5&0&6/5\\0&0&-1&1&1&0\end{array}\right)$

Feasible

$z=C_B(B^{-1}b)=(2,1,0)\begin{pmatrix}3/5\\6/5\\0\end{pmatrix}=12/5$

	x_1	x_2	x_3	x_4	x_5	Solution
z	0	0	$-2/5$	$-1/5$	0	$12/5$
x_1	1	0	$-3/5$	$1/5$	0	$3/5$
x_2	0	1	$4/5$	$-3/5$	0	$6/5$
x_5	0	0	-1	1	1	0

4

$X_B=(x_3,x_2,x_1)^T$, $C_B=(0,c_2,c_1)$

$C_BB^{-1}=(0,c_2,c_1)\begin{pmatrix}1&1&-1\\0&1&0\\0&-1&1\end{pmatrix}=(0,c_2-c_1,c_1)$

For x_3, x_4, and x_5,

$\{z_j-c_j\}=C_BB^{-1}(P_3,P_4,P_5)-(0,0,0)$

$=C_BB^{-1}\ =C_BB^{-1}=(0,c_2-c_1,c_1)$

From the tableau, we have

$(0,c_2-c_1,c_1)=(0,3,2)$

which gives

$c_1=2$

$c_2=5$

Continued...

Hence,

Optimum $z = c_1x_1 + c_2x_2 + c_3x_3$

$= 2\times 2 + 5\times 6 + 0\times 2 = 34$

To construct the original problem,

$$B^{-1}(P_1\ P_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus,

$$(P_1\ P_2) = B\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Similarly,

$$b = B\begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$$

Original model:

Maximize $z = 2x_1 + 5x_2$

Subject to

$$\begin{aligned} x_1 \quad\quad\ & \le 4 \\ x_2 & \le 6 \\ x_1 + x_2 & \le 8 \\ x_1, x_2 & \ge 0 \end{aligned}$$

5

All that is needed is to show that the computations lead to the column under x_{II}.

For x_{II}, we have,

$$\{z_j - c_j\} = C_B B^{-1} I - C_{II}$$

$$= C_B B^{-1} - C_{II}$$

Constraint coefficients

$$= B^{-1} I = B^{-1}$$

1

(a) current $B = (P_1, P_2)$

P_1 must leave so that b is enclosed between P_2 and P_3, hence yeilding feasible values of x_2 and x_3

(b) $B = (P_2\ P_4)$ is a feasible basis

2

$$z_j - c_j = c_B B^{-1} P_j - c_j$$

Assume for convenience that

$$B = (P_1, P_2, \ldots, P_m)$$

Then, for the basic vectors P_1, $P_2, \ldots,$ and P_m, we have

$$\{z_j - c_j\}_{j=1,2,\ldots,m} = c_B B^{-1}(P_1, \ldots, P_m) - (c_1, \ldots, c_m)$$
$$= c_B B^{-1} B - c_B$$
$$= c_B I - c_B = 0$$

3

Let NB represent the set of nonbasic variables at any iteration. Then

$$z = z^* - \sum_{j \in NB} (z_j - c_j) x_j$$

(a) Since

$$z_j - c_j \begin{cases} > 0 & \text{for max} \\ < 0 & \text{for min} \end{cases}$$

it follows that all $x_j = 0$, $j \in NB$ because if any x_j, $j \in NB$ becomes positive $z < z^*$ for max and $z > z^*$ for min, which is not optimal. Thus, $X_B = B^{-1} b$ and $x_j = 0$, $j \in NB$ shows that the solution is unique.

Continued...

(b) If $z_j - c_j = 0$ for at least one $j \in NB$, then x_j can become basic at a value other than zero without changing the optimum value of z. Thus, alternative optima exist.

4

Starting tableau (max):

	x_1	$x_2 \cdots$	$x_j \cdots$	x_n	
z	$-c_1$	$-c_2 \cdots$	$-c_j \cdots$	$-c_n$	0

At the starting iteration:

$$B = I, \quad c_B = 0$$

Hence

$$z_j - c_j = c_B B^{-1} P_j - c_j$$
$$= 0(B^{-1} P_j) - c_j$$
$$= -c_j$$

5

Starting tableau (assuming max):

	$\cdots\ x_j\ \cdots$	R_1	$R_2 \cdots$	R_m	
	$\cdots\ -c_j\ \cdots$	M	$M \cdots$	M	0
R_1 $\vdots$ R_m	$\cdots\ P_j\ \cdots$		I		b

$$B = B^{-1} = I, \quad c_B = (-M, -M, \ldots, -M)$$

$$c_B B^{-1} = (-M, -M, \ldots, -M)$$

$$\{z_j - c_j\} = (-M, -M, \ldots, -M)(P_1, \ldots, P_n \mid I) - (c_1, c_2, \ldots, c_n, -M, \ldots, -M)$$

$$= \big((-M, -M, \ldots, -M) P_1 - c_1, \ldots, (-M, -M, \ldots, -M) P_n - c_n, 0, \ldots, 0\big)$$

which yields the following tableau

$\cdots$	x_j	$\cdots$	R_1	$\cdots$	R_m	
$\cdots$	$(-M, \ldots, -M) P_j - c_j$	$\cdots$	0	$\cdots$	0	$(-M, \ldots, -M) b$

Continued...

Set 7.2a

6

The vectors

$$\begin{pmatrix} c_k \\ p_k \end{pmatrix} \text{ and } \begin{pmatrix} -c_k \\ -p_k \end{pmatrix}$$

correspond to x_k^- and x_k^+, respectively. Assume that both x_k^- and x_k^+ are nonbsic, and let **B** and $\mathbf{c_B}$ correspond to the current solution. Then

$$z_k^- - c_k^- = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{P}_k - c_k$$
$$z_k^+ - c_k^+ = -\mathbf{c}_B\mathbf{B}^{-1}\mathbf{P}_k + c_k = -(z_k^- - c_k^-)$$

Thus, if x_k^- is a candidate for entering the basic solution, then x_k^+ cannot be an entering candidate, and vice versa.

If $z_k^+ - c_k^+ = (z_k^- - c_k^-) = 0$, then possibly one of the two variables may enter the basic solution to provide an alternative optimum. The two variables cannot be basic simultaneously because a basis **B** cannot include two dependent vectors $\mathbf{P}_k$ and $-\mathbf{P}_k$

To show that the two variables cannot replace one another in alternative optima, assume that x_k^- is basic in the optimum solution. Then

$$\mathbf{B}^{-1}\mathbf{P}_k = (0,\ldots,1,\ldots,0)^T$$
$$\mathbf{B}^{-1}(-\mathbf{P}_k) = (0,\ldots,-1,\ldots,0)^T$$

According to the feasibility condition, x_k^+ cannot replace x_k^- because the corresponding pivot element $\mathbf{B}^{-1}(-\mathbf{P}_k)$ is negative, unless $x_k^- = 0$, which is a trivial case.

7

Number of nonbasic variables = $n - m$. In the case of *nondegeneracy*, each entering nonbasic variable will be associated with a *distinct* adjacent extreme point. In the case of *degeneracy*, an entering nonbasic variable can result in a different basic solution without changing the extreme point itself. In this situation, the number of adjacent extreme points in less than n - m.

8

Let $x_k = d_k$ (≥ 0) represent the current basic solution. Then, the new basic solution after x_j enters and x_r leaves is

$$x_j = \frac{d_r}{(\mathbf{B}^{-1}\mathbf{P}_j)_r} = \frac{0}{(\mathbf{B}^{-1}\mathbf{P}_j)_r} = 0, \text{ provided } (\mathbf{B}^{-1}\mathbf{P}_j)_r \neq 0$$
$$x_k' = d_k - x_j(\mathbf{B}^{-1}\mathbf{P}_j)_k, \text{ all basic } x_k, k \neq j$$

The last equation is independent of $(\mathbf{B}^{-1}\mathbf{P}_j)_k$ for all k, because $x_j = 0$. Hence, x_k' remains feasible for all k.

9

1. If the minimum ratio corresponds to more than one basic variable, the next iteration is degenerate.

2. If x_j is the entering variable and if the basic variable x_j is zero, the next iteration will continue to be degenerate if $(\mathbf{B}^{-1}\mathbf{P}_j)_k > 0$.

3. ~~I~~f for every zero basic variable, *xk*, the pivot element $(\mathbf{B}^{-1}\mathbf{P}j)_k \leq$, then the next iteration will not be degenerate.

10

Under nondegeneracy:

number of extreme points
= number of basic solutions

Under degeneracy:

number of extreme points
< number of basic solutions

11

(a) $x_j = \theta = \dfrac{x_r}{(B^{-1}P_j)_r}, \; (B^{-1}P_j)_r > 0$

For P_j, we have

$$\frac{\text{new } x_j}{\text{old } x_j} = \frac{\dfrac{x_r}{\alpha(B^{-1}P_j)_r}}{\dfrac{x_r}{(B^{-1}P_j)_r}} = \frac{1}{\alpha}$$

(b)

$$\frac{\text{new } x_j}{\text{old } x_j} = \frac{\dfrac{\beta(B^{-1}b)_r}{\alpha(B^{-1}P_j)_r}}{\dfrac{(B^{-1}b)_r}{(B^{-1}P_j)_r}} = \frac{\beta}{\alpha}$$

12

$$\text{New}(z_j - c_j) = C_B\left(\frac{1}{\beta}B^{-1}P_j\right) - \frac{1}{\beta}c_j$$

$$= \frac{1}{\beta}\left(C_B B^{-1}P_j - c_j\right)$$

$$= \frac{1}{\beta}(\text{old } z_j - c_j), \; \beta > 0$$

Conclusion: x_j remains nonbasic

A variable x_j can be made profitable either by increasing c_j or by decreasing z_j (which is the unit usage of resources by activity j). Of course, a combination of the two changes will work as well.

13

$C_B = (c_1, c_2, \cdots, c_m)$

$B = (P_1, P_2, \cdots, P_m)$

For the basic variables

$$z_j - c_j = C_B B^{-1}(P_1, \cdots, P_m) - (c_1, \cdots, c_m)$$

$$= C_B B^{-1}B - C_B$$

$$= C_B I - C_B = \underline{0}$$

Thus, for the basic variable, $z_j - c_j = 0$ regardless of the specific assignment to the vector C_B (e.g., D_B).

This result implies that changes in C_B cannot affect the optimality of the basic variables since these variables are already basic. It may, however, cause a nonbasic variable to become basic.

Set 7.2b

1

	x_1	x_2	x_3	x_4	x_5	x_6	
z	0	$-2/3$	$5/6$	0	0	0	20
x_1		$2/3$					4
x_4		$4/3$					2
x_5		$5/3$					5
x_6		1					2

2

(a) Starting iteration:

Let x_4 and x_5 be the slacks.

$X_B=(x_4,x_5)^T,\ C_B=(0,0),\ B=B^{-1}=I$

First iteration:

$C_B B^{-1}=(0,0)$

$$(z_j-c_j)_{j=1,2,3}=(0,0)\begin{pmatrix}2&-1&2\\1&0&4\end{pmatrix}-(6,-2,3)$$
$$=(-6,2,-3)\Rightarrow x_1 \text{ enters}$$

$$X_B=B^{-1}b=Ib=b=\begin{pmatrix}2\\4\end{pmatrix}$$

$$\alpha^1=B^{-1}P_1=P_1=\begin{pmatrix}2\\1\end{pmatrix}$$

$$\theta=\min_{k=4,5}\{2/2,\ 4/1\}=1 \Rightarrow x_4 \text{ leaves}$$

$$B^{-1}_{next}=\begin{pmatrix}2&0\\1&1\end{pmatrix}=\begin{pmatrix}1/2&0\\-1/2&1\end{pmatrix}$$

$$X_B=(x_1,x_5)^T=(1,3)^T$$

Second iteration:

$$C_B B^{-1}=(6,0)\begin{pmatrix}1/2&0\\-1/2&1\end{pmatrix}=(3,0)$$

$$(z_j-c_j)_{j=2,3,4}=(3,0)\begin{pmatrix}-1&2&1\\0&4&0\end{pmatrix}-(-2,3,0)$$
$$=(-1,3,3)\Rightarrow x_2 \text{ enters}$$

$$X_B=\begin{pmatrix}x_1\\x_5\end{pmatrix}=\begin{pmatrix}1/2&0\\-1/2&1\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$$

$$\alpha^2=B^{-1}P_2=\begin{pmatrix}1/2&0\\-1/2&1\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}-1/2\\1/2\end{pmatrix}$$

$$\theta=\min_{k=1,5}\left\{-,\ \frac{3}{1/2}\right\}=6\Rightarrow x_6 \text{ leaves}$$

$$B^{-1}_{next}=\begin{pmatrix}2&-1\\1&1\end{pmatrix}^{-1}=\begin{pmatrix}0&1\\-1&2\end{pmatrix}$$

$$X_B=(x_1,x_2)^T=(4,6)^T,\ C_B=(6,-2)$$

continued…

Third iteration:

$$C_B B^{-1}=(6,-2)\begin{pmatrix}0&1\\-1&2\end{pmatrix}=(2,2)$$

$$(z_j-c_j)_{j=3,4,5}=(2,2)\begin{pmatrix}2&1&0\\4&0&1\end{pmatrix}-(3,0,0)$$
$$=(9,2,2)\Rightarrow \text{optimal}$$

Optimal solution:

$$X_B=\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0&1\\-1&2\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}4\\6\end{pmatrix}$$

$$z=C_BX_B=6\times4+(-2)(6)=12$$

(b)

Starting iteration: Let x_4, x_5, and x_6 be the slack variables.

$X_B=(x_4,x_5,x_6)^T,\ C_B=(0,0,0),\ B=B^{-1}=I$

First iteration: $C_B B^{-1}=(0,0,0)$

$$(z_j-c_j)_{j=1,2,3}=(0,0,0)\begin{pmatrix}4&3&8\\4&1&12\\4&-1&3\end{pmatrix}-(2,1,2)$$
$$=(-2,-1,-2)\Rightarrow x_1 \text{ enters}$$

$$X_B=B^{-1}b=Ib=b=(12,8,8)^T$$

$$\alpha^1=B^{-1}P_1=P_1=(4,4,4)^T$$

$$\theta=\min_{k=4,5,6}\left\{\frac{12}{4},\frac{8}{4},\frac{8}{4}\right\}=2\Rightarrow x_5 \text{ leaves}$$

$$B^{-1}_{next}=\begin{pmatrix}1&4&0\\0&4&0\\0&4&1\end{pmatrix}^{-1}=\begin{pmatrix}1&-1&0\\0&1/4&0\\0&-1&1\end{pmatrix}$$

$$X_B=(x_4,x_1,x_6),\ C_B=(0,2,0)$$

Second iteration: $C_B B^{-1}=(0,1/2,0)$

$$(z_j-c_j)_{j=2,3,5}=(0,1/2,0)\begin{pmatrix}1&-1&0\\0&1/4&0\\0&-1&1\end{pmatrix}-(1,2,0)$$
$$=(-1/2,4,1/2)\Rightarrow x_2 \text{ enters}$$

$$X_B=\begin{pmatrix}x_4\\x_1\\x_6\end{pmatrix}=B^{-1}b=\begin{pmatrix}1&-1&0\\0&1/4&0\\0&-1&1\end{pmatrix}\begin{pmatrix}12\\8\\8\end{pmatrix}=\begin{pmatrix}4\\2\\0\end{pmatrix}$$

$$\alpha^2=\begin{pmatrix}1&-1&0\\0&1/4&0\\0&-1&1\end{pmatrix}\begin{pmatrix}3\\1\\-1\end{pmatrix}=\begin{pmatrix}2\\1/4\\-2\end{pmatrix}$$

$$\theta=\min_{k=4,1,6}=\left\{\frac{4}{2},\frac{2}{1/4},-\right\}=2,\ x_4 \text{ leaves}$$

$$B^{-1}_{next}=\begin{pmatrix}3&4&0\\1&4&0\\-1&4&1\end{pmatrix}^{-1}=\begin{pmatrix}1/2&-1/2&0\\-1/8&3/8&0\\1&-2&1\end{pmatrix}$$

$$X_B=(x_2,x_1,x_6)^T,\ C_B=(1,2,0)$$

continued…

Third iteration: $C_B B^{-1} = (1/4, 1/4, 0)$

$$(z_j - c_j)_{j=3,4,5} = (1/4, 1/4, 0)\begin{pmatrix} 8 & 1 & 0 \\ 12 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} - (2, 4, 0)$$
$$= (3, 1/4, 1/4) \Rightarrow \text{optimal.}$$

Optimal solution:

$$X_B = \begin{pmatrix} x_2 \\ x_1 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/8 & 3/8 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 12 \\ 8 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 3/2 \\ 4 \end{pmatrix}$$

$$z = 2 \times 3/2 + 1 \times 2 + 2 \times 0 = 5$$

(c)

Adding artificials, we get

$$\min z = 2x_1 + x_2 + Mx_4 + Mx_5$$

s.t. $\left(\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$

where x_3 and x_6 are slacks, and x_4 and x_5 are artificials.

Starting solution:

$X_B = (x_4, x_5, x_6)$, $C_B = (M, M, 0)$

$B = B^{-1} = I$

First iteration: $C_B B^{-1} = (M, M, 0)$

$$(z_j - c_j)_{j=1,2,3} = (M, M, 0)\begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & -1 \\ 1 & 2 & 0 \end{pmatrix} - (2, 1, 0)$$
$$= (-2 + 7M, -1 + 4M, -M)$$

Thus, x_1 enters.

$$\theta = \min_{k=4,5,6}\left\{\frac{3}{3}, \frac{6}{4}, \frac{3}{1}\right\} = 1 \Rightarrow x_4 \text{ leaves}$$

$$B_{next}^{-1} = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

$X_B = (x_1, x_5, x_6)^T$, $C_B = (2, M, 0)$

Second iteration: $C_B B^{-1} = \left(\frac{2-4M}{3}, M, 0\right)$

$$(z_j - c_j)_{j=2,3,4} = \left(\frac{2-4M}{3}, M, 0\right)\begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix} - (1, 0, 0)$$
$$= \left(\frac{5M-1}{3}, -M, \frac{2-4M}{3}\right) \Rightarrow x_2 \text{ enters}$$

$$X_B = \begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 5/3 \\ 5/3 \end{pmatrix}$$

$$\theta = \min_{k=1,5,6}\left\{\frac{1}{1/3}, \frac{2}{5/3}, \frac{2}{5/3}\right\} \Rightarrow x_5 \text{ leaves}$$

Continued...

$$B_{next}^{-1} = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$X_B = (x_1, x_2, x_6)^T$, $C_B = (2, 1, 0)$

Third iteration: $C_B B^{-1} = (2/5, 1/5, 0)$

$$(z_j - c_j)_{j=3,4,5} = (2/5, 1/5, 0)\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - (0, M, M)$$
$$= (-1/5, 2/5 - M, 1/5 - M)$$
$$\Rightarrow \text{optimal solution.}$$

Optimal solution:

$$X_B = \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}$$

$$z = 2 \times \frac{3}{5} + 1 \times \frac{6}{5} = 12/5$$

(d)

Minimize $z = 5x_1 - 4x_2 + 6x_3 + 8x_4 + Mx_8$

subject to

$$\begin{aligned} x_1 + 7x_2 + 3x_3 + 7x_4 \quad\quad + x_6 \quad\quad &= 46 \\ 3x_1 - x_2 + x_3 + 2x_4 \quad\quad + x_7 \quad &= 20 \\ 2x_1 + 3x_2 - x_3 + x_4 - x_5 \quad\quad + x_8 &= 18 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0$$

Iteration 0:

$X_B = (x_6, x_7, x_8)$, $C_B = (0, 0, M)$, $B_0 = B_0^{-1} = I$

$\{z_j - c_j\}_{j=1,2,3,4,5}$

$$= (0, 0, M)\begin{pmatrix} 1 & 7 & 3 & 7 & 0 \\ 3 & -1 & 1 & 2 & 0 \\ 2 & 3 & -1 & 1 & -1 \end{pmatrix} - (5, -4, 6, 8, 0)$$
$$= (2M - 5, \boxed{3M + 4}, -M - 6, M - 8, -M)$$

x_2 enters

$$B^{-1}P_2 = \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}, \quad B^{-1}b = \begin{pmatrix} 46 \\ 20 \\ 18 \end{pmatrix}, \quad \theta = \min\left\{\frac{46}{7}, -, \frac{18}{3}\right\}$$

x_8 leaves

$$B_1 = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}, \quad B_1^{-1} = \begin{pmatrix} 1 & 0 & -7/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1/3 \end{pmatrix}$$

$$X_{B_1} = \begin{pmatrix} x_6 \\ x_7 \\ x_2 \end{pmatrix} = B_1^{-1} b = \begin{pmatrix} 4 \\ 26 \\ 6 \end{pmatrix}$$

Continued...

Iteration 1:

$X_B = (x_6, x_7, x_2)^T$, $C_B = (0, 0, -4)$

$C_B B^{-1} = (0, 0, -4/3)$

$\{z_j - c_j\}_{1,3,4,5}$

$$= (0, 0, -4/3)\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{pmatrix} - (5, 6, 8, 0)$$

$$= (-23/3, -30/3, -28/3, \boxed{4/3})$$

x_5 enters

$$B_1^{-1}P_5 = \begin{pmatrix} \boxed{7/3} \\ -1/3 \\ -1/3 \end{pmatrix}, \quad B_1^{-1}b = \begin{pmatrix} 4 \\ 26 \\ 6 \end{pmatrix}$$

x_6 leaves

Iteration 2:

$X_B = (x_5, x_7, x_2)^T$, $C_B = (0, 0, -4)$

$$B_2 = \begin{pmatrix} 0 & 0 & 7 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{pmatrix}, \quad B_2^{-1} = \begin{pmatrix} 3/7 & 0 & 0 \\ 1/7 & 1 & 0 \\ 1/7 & 0 & 1 \end{pmatrix}$$

$$X_{B_2} = \begin{pmatrix} x_5 \\ x_7 \\ x_2 \end{pmatrix} = B^{-1}b = \begin{pmatrix} 12/7 \\ 186/7 \\ 46/7 \end{pmatrix}$$

$C_B B^{-1} = (-4/7, 0, 0)$

$\{z_j - c_j\}_{1,3,4,6}$

$$= (-4/7, 0, 0)\begin{pmatrix} 1 & 3 & 7 & 1 \\ 3 & 1 & 2 & 0 \\ 2 & -1 & 1 & 0 \end{pmatrix} - (5, 6, 8, 0)$$

$$= (-39/7, -54/7, -12, -4/7) \text{ optimum}$$

$X_{B_2} = (x_5, x_7, x_2)^T = (12/7, 186/7, 46/7)$

$z = -184/7$

3

Iteration 0:

$X_{B_0} = (x_2, x_4, x_5)^T$, $C_B = (7, -10, 0)$

$$B_0 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad B_0^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

continued...

$$X_B = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = B_0^{-1}b = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$$

$$C_B B_0^{-1} = (7, -10, 0)\begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} = (17, 7, -17)$$

$\{z_j - c_j\}_{j=1,3,6}$

$$= (17, 7, -17)\begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 3 \\ 1 & -3 & 0 \end{pmatrix} - (0, 11, 26)$$

$$= (-17, \boxed{16}, 12) \quad x_3 \text{ enters}$$

$$B_0^{-1}b = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}, \quad B_0^{-1}P_3 = \begin{pmatrix} \boxed{1} \\ -2 \\ -2 \end{pmatrix} \; x_2 \text{ leaves}$$

Iteration 1:

$X_{B_1} = (x_3, x_4, x_5)^T$, $C_B = (11, -10, 0)$

$$B_1 = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix}, \quad B_1^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$X_B = B_1^{-1}b = \begin{pmatrix} 2 \\ 10 \\ 8 \end{pmatrix}$$

$$C_B B_1^{-1} = (11, -10, 0)\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} = (1, -9, -1)$$

$\{z_j - c_j\}_{1,2,6}$

$$= (1, -9, -1)\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} - (0, 7, 26)$$

$$= (-1, -16, -52) \Rightarrow \text{optimum}$$

$X_{B_1} = (x_3, x_4, x_5)^T = (2, 10, 8)^T$

$z = -78$

4

(a) Minimize $z = 2x_1 + x_2 + M(x_4 + x_5)$

subject to

$$\begin{aligned} 3x_1 + x_2 \qquad + x_4 \qquad &= 3 \\ 4x_1 + 3x_2 - x_3 \qquad + x_5 &= 6 \\ x_1 + 2x_2 \qquad\qquad + x_6 &= 3 \end{aligned}$$

$x_1, \ldots, x_6 \geq 0$

Phase I:

Iteration 0:

$X_B = (x_4, x_5, x_6)^T$, $C_B = (1, 1, 0)$

$B_0^{-1} = I$, $C_B B^{-1} = (1, 1, 0)$

continued...

$\{z_j - c_j\}_{1,2,3}$

$= (1,1,0)\begin{pmatrix}3 & 1 & 0\\ 4 & 3 & -1\\ 1 & 2 & 0\end{pmatrix} - (0,0,0)$

$= (\boxed{7}, 4, -1)$, x_1 enters

$B_0^{-1}P_1 = B_0^{-1}\begin{pmatrix}3\\4\\1\end{pmatrix} = \begin{pmatrix}3\\4\\1\end{pmatrix}$, $B_0^{-1}b = \begin{pmatrix}3\\6\\3\end{pmatrix}$

$\theta = \min\left\{\boxed{\frac{3}{3}}, \frac{6}{4}, \frac{3}{1}\right\} \Rightarrow x_4$ leaves

Iteration 1:

$X_B = (x_1, x_5, x_6)^T$, $C_B = (0, 1, 0)$

$B_1 = \begin{pmatrix}3 & 0 & 0\\ 4 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}$, $B_1^{-1} = \begin{pmatrix}1/3 & 0 & 0\\ -4/3 & 1 & 0\\ -1/3 & 0 & 1\end{pmatrix}$

$X_B = \begin{pmatrix}x_1\\x_5\\x_6\end{pmatrix} = \begin{pmatrix}1\\2\\2\end{pmatrix}$

$C_B B_1^{-1} = (-4/3, 1, 0)$

$\{z_j - c_j\}_{2,3,4}$

$= (-4/3, 1, 0)\begin{pmatrix}1 & 0 & 1\\ 3 & -1 & 0\\ 2 & 0 & 0\end{pmatrix} - (0,0,1)$

$= (\boxed{5/3}, -1, -7/3)$ x_2 enters

$B_1^{-1}P_2 = B_1^{-1}\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}1/3\\5/3\\5/3\end{pmatrix}$,

$B_1^{-1}b = \begin{pmatrix}1\\2\\2\end{pmatrix}$

$\theta = \min\left\{\frac{1}{1/3}, \frac{2}{5/3}, \frac{2}{5/3}\right\}$, x_5 leaves

Iteration 2:

$X_B = (x_1, x_2, x_6)^T$, $C_B = (0, 0, 0)$

$B_2 = \begin{pmatrix}3 & 1 & 0\\ 4 & 3 & 0\\ 1 & 2 & 1\end{pmatrix}$, $B_2^{-1} = \begin{pmatrix}3/5 & -1/5 & 0\\ -4/5 & 3/5 & 0\\ 1 & -1 & 1\end{pmatrix}$

Since X_B does not include the artificials x_4 and x_5, we can use to start Phase II.

Continued...

Phase II: objective max $z = 2x_1 + x_2$

Iteration 0:

$X_B = (x_1, x_2, x_6)$, $C_B = (2, 1, 0)$

$B_0^{-1} = \begin{pmatrix}3/5 & -1/5 & 0\\ -4/5 & 3/5 & 0\\ 1 & -1 & 1\end{pmatrix}$

$X_B = \begin{pmatrix}x_1\\x_2\\x_6\end{pmatrix} = B_2^{-1}b = \begin{pmatrix}3/5 & -1/5 & 0\\ -4/5 & 3/5 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}3\\6\\3\end{pmatrix} = \begin{pmatrix}3/5\\6/5\\0\end{pmatrix}$

$C_B B^{-1} = (2,1,0)\begin{pmatrix}3/5 & -1/5 & 0\\ -4/5 & 3/5 & 0\\ 1 & 1 & 1\end{pmatrix} = (2/5, 1/5, 0)$

$\{z_j - c_j\}_{j=3} = (2/5, 1/5, 0)\begin{pmatrix}0\\-1\\0\end{pmatrix} - 0 = -1/5$

x_3 enters

$B_0^{-1}P_3 = \begin{pmatrix}1/5\\-3/5\\1\end{pmatrix}$, $B_0^{-1}b = \begin{pmatrix}3/5\\6/5\\0\end{pmatrix}$ x_6 leaves

Iteration 1:

$X_B = (x_1, x_2, x_3)$, $C_B = (2, 1, 0)$

$B_1 = \begin{pmatrix}3 & 1 & 0\\ 4 & 3 & -1\\ 1 & 2 & 0\end{pmatrix}$, $B_1^{-1} = \begin{pmatrix}2/5 & 0 & -1/5\\ -1/5 & 0 & 3/5\\ 1 & -1 & 1\end{pmatrix}$

$\{z_j - c_j\}_{j=6}$

$= (3/5, 0, 1/5)\begin{pmatrix}0\\0\\1\end{pmatrix} - 0 = 1/5 > 0$

optimum!

$X_B = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2/5 & 0 & -1/5\\ -1/5 & 0 & 3/5\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}3\\6\\3\end{pmatrix} = \begin{pmatrix}3/5\\6/5\\0\end{pmatrix}$

$z = 12/5$

5

Minimize $z = 3x_1 + 2x_2$

subject to

$$\begin{aligned} -3x_1 - x_2 + x_3 &= -3\\ -4x_1 - 3x_2 + x_4 &= -6\\ x_1 + x_2 + x_5 &= 3 \end{aligned}$$

Iteration 0:

$X_B = \begin{pmatrix}x_3\\x_4\\x_5\end{pmatrix}$, $B_0 = B_0^{-1} = I$

Continued...

$$X_B = \begin{pmatrix} -3 \\ -6 \\ 3 \end{pmatrix} \Rightarrow x_4 \text{ leaves}$$

$$C_B = (0,0,0),\ C_B B^{-1} = (0,0,0)$$

$$\{z_j - c_j\}_{1,2} = (0,0,0)\begin{pmatrix} -3 & -1 \\ -4 & -3 \\ 1 & 1 \end{pmatrix} - (3,2) = (-3,\ -2)$$

$$(\text{row 2 of } B_0^{-1})(P_1\ P_2) = (0,\ 1,\ 0)\begin{pmatrix} -3 & -1 \\ -4 & -3 \\ 1 & 1 \end{pmatrix} = (-4, -3)$$

$$\theta = \min_{j=1,2}\left\{\left|\frac{-3}{-4}\right|, \left|\frac{-2}{-3}\right|\right\} = 2/3 \Rightarrow x_2 \text{ enters}$$

<u>Iteration 1:</u>

$$X_B = \begin{pmatrix} x_3 \\ x_2 \\ x_5 \end{pmatrix},\ B_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & 1 \end{pmatrix},\ B_1^{-1} = \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & -1/3 & 0 \\ 0 & 1/3 & 1 \end{pmatrix}$$

$$X_B = \begin{pmatrix} x_3 \\ x_2 \\ x_5 \end{pmatrix} = B_1^{-1} b = \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & -1/3 & 0 \\ 0 & 1/3 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} x_3 \text{ leaves}$$

$$C_B = (0, 2, 0)$$

$$C_B B^{-1} = (0, -2/3, 0)$$

$$\{z_j - c_j\}_{j=1,4} = (0, -2/3, 0)\begin{pmatrix} -3 & 0 \\ -4 & 1 \\ 1 & 0 \end{pmatrix} - (3, 0) = (-1/3, -2/3)$$

$$(\text{row 1 of } B_1^{-1})(P_1\ P_4) = (1, -1/3, 0)\begin{pmatrix} -3 & 0 \\ -4 & 1 \\ 1 & 0 \end{pmatrix} = (-5/3, -1/3)$$

$$\theta = \min_{j=1,4}\left\{\left|\frac{-1/3}{-5/3}\right|, \left|\frac{-2/3}{-1/3}\right|\right\} = 1/5$$

x_1 enters

<u>Iteration 2:</u>

$$X_B = \begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} -3 & -1 & 0 \\ -4 & -3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$B_2^{-1} = \begin{pmatrix} -3/5 & 1/5 & 0 \\ 4/5 & -3/5 & 0 \\ -1/5 & 2/5 & 1 \end{pmatrix}$$

$$X_B = B_2^{-1} b = \begin{pmatrix} -3/5 & 1/5 & 0 \\ 4/5 & -3/5 & 0 \\ -1/5 & 2/5 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 6/5 \end{pmatrix}$$

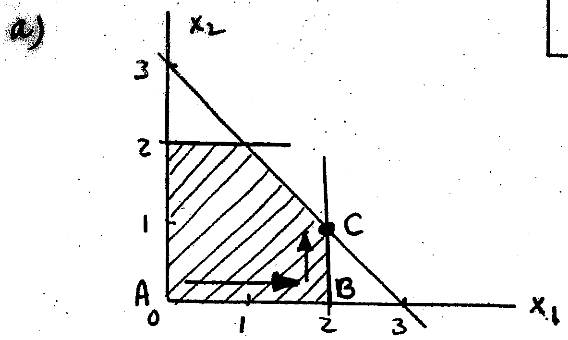
Feasible!

$$Z = 3 \times 3/5 + 2 \times 6/5 = 21/5$$

continued...

1

a)



b)

<u>Iteration 1:</u> x_1 enters

	x_1	x_2	x_3	solution
z	-2	-1	0	0
x_3	1	1	1	3

$\theta = \min\{3/1, -, 2\} = 2$

Substitute x_1 at its upper bound: $x_1 = 2 - x_1'$

	x_1'	x_2	x_3	solution
z	2	-1	0	2
x_3	-1	1	1	1

This solution $(x_1 = 2,\ x_2 = 0)$ coincides with point B in the solution space above. The solution now has $x_1' = 0$, which implies that $x_1 = 2$, thus reducing the solution space to line segment BC.

<u>Iteration 2:</u> x_2 enters

$\theta = \min\{1/1, -, 2\} = 1$

	x_1'	x_2	x_3	solution
z	1	0	1	3
x_2	-1	1	1	1

<u>Optimum:</u> $x_1' = 0 \Rightarrow x_1 = 2,\ x_2 = 1$ which is the same as point C.

c) As shown in (b) above, the substitution of the upper bounding method recognizes the extreme point implicitly by using the substitution

$$x_j = \mu_j - x_j'$$

2

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
z	-6	-2	-8	-4	-2	-10	0	0
x_7	8	1	8	2	2	4	1	13

x_6 enters: $\theta = \min\{13/4, -, 1\} = 1$

$x_6 = 1 - x_6'$

	x_1	x_2	x_3	x_4	x_5	x_6'	x_7	
z	-6	-2	-8	-4	-2	10	0	10
x_7	8	1	8	2	2	-4	1	9

x_3 enters: $\theta = \min\{9/8, -, 1\} = 1$

$x_3 = 1 - x_3'$

	x_1	x_2	x_3'	x_4	x_5	x_6'	x_7	
z	-6	-2	8	-4	-2	10	0	18
x_7	8	1	-8	2	2	-4	1	1

x_1 enters: $\theta = \min\{1/8, -, 1\} = 1/8$, x_7 leaves

	x_1	x_2	x_3'	x_4	x_5	x_6'	x_7	
z	0	$-5/4$	2	$-5/2$	$-1/2$	7	3/4	18 3/4
x_1	1	1/8	-1	1/4	1/4	$-1/2$	1/8	1/8

x_4 enters: $\theta = \min\left\{\frac{1/8}{1/4}, -, 1\right\} = 1/2$, x_1 leaves

	x_1	x_2	x_3'	x_4	x_5	x_6'	x_7	
z	10	0	-8	0	2	2	2	20
x_4	4	1/2	-4	1	1	-2	1/2	1/2

x_3' enters: $\theta = \min\left\{-, \frac{1/2 - 1}{(-4)}, 1\right\} = 1/8$

x_4 leaves, $x_4 = 1 - x_4'$

	x_1	x_2	x_3'	x_4'	x_5	x_6'	x_7	
z	2	-1	0	2	0	6	1	21
x_3'	-1	$-1/8$	1	1/4	$-1/4$	1/2	$-1/8$	1/8

x_2 enters: $\theta = \min\left\{-, \frac{1/8 - 1}{-1/8}, 1\right\} = 1$

$x_2 = 1 - x_2'$

	x_1	x_2'	x_3'	x_4'	x_5	x_6'	x_7	
z	2	1	0	2	0	6	1	22
x_3'	-1	1/8	1	1/4	$-1/4$	1/2	$-1/8$	1/4

Optimum solution:

$x_1 = 0$
$x_2 = 1$
$x_3 = 3/4$
$x_4 = 1$
$x_5 = 0$
$x_6 = 1$

$z = 22$

3

(a) Minimize

	x_1	x_2	x_3	x_4	x_5	
z	−6	2	3	0	0	0
x_4	2	4	2	1	0	8
x_5	1	−2	3	0	1	7

x_3 enters: $\theta = \min\{\frac{7}{3}, -, 1\} = 1$; $x_3 = 1 - x_3'$

	x_1	x_2	x_3'	x_4	x_5	
z	−6	2	−3	0	0	−3
x_4	2	4	−2	1	0	6
x_5	1	−2	−3	0	1	4

x_2 enters: $\theta = \min\{\frac{6}{4}, -, 2\} = 3/2$; x_4 leaves

	x_1	x_2	x_3'	x_4	x_5	
z	−7	0	−2	−1/2	0	−6
x_2	1/2	1	−1/2	1/4	0	3/2
x_5	2	0	−4	1/2	1	7

Optimum: $x_1 = 0$, $x_2 = 3/2$, $x_3 = 1$, $z = -6$

b) Maximize

	x_1	x_2	x_3	x_4	x_5	
z	−3	−5	−2	0	0	0
x_4	1	2	2	1	0	10
x_5	2	4	3	0	1	15

x_2 enters: $\theta = \min\{\frac{15}{4}, -, 3\} = 3$; $x_2 = 3 - x_2'$

	x_1	x_2'	x_3	x_4	x_5	
z	−3	5	−2	0	0	15
x_4	1	−2	2	1	0	4
x_5	2	−4	3	0	1	3

x_1 enters: $\theta = \min\{\frac{3}{2}, -, 4\}$; x_5 leaves

	x_1	x_2'	x_3	x_4	x_5	
z	0	−1	5/2	0	3/2	39/2
x_4	0	0	1/2	1	−1/2	5/2
x_1	1	−2	3/2	0	1/2	3/2

x_2' enters: $\theta = \min\{-, \frac{3/2-2}{-2}, 2\} = 1/4$

x_1 leaves, $x_1 = 4 - x_1'$

	x_1'	x_2'	x_3	x_4	x_5	
z	1/2	0	7/4	0	5/4	83/4
x_4	0	0	1/2	1	−1/2	5/2
x_2'	1/2	1	−3/4	0	−1/4	5/4

Optimum: $x_1 = 4$, $x_2 = 7/4$, $x_3 = 0$, $z = 83/4$

4

(a) Substitute $x_1 = 1 + y_1$, $x_3 = y_3 + 2$

Phase 1: $0 \le y_1 \le 2$, $0 \le x_2 \le 3$, $y_3 \ge 0$

	y_1	x_2	y_3	x_4	x_5	R	
z	1	2	−1	−1	0	0	4
x_5	2	1	1	0	1	0	4
R_1	1	2	−1	−1	0	1	4
z	0	0	0	0	0	−1	0
x_5	3/2	0	3/2	1/2	1	0	2
x_2	1/2	1	−1/2	−1/2	0	1	2

Phase 2:

	y_1	x_2	y_3	x_4	x_5	
z	−2	0	1	−1	0	3
x_5	3/2	0	3/2	1/2	1	2
x_2	1/2	1	−1/2	−1/2	0	2

y_1 enters: $\theta = \min\{\frac{2}{3/2}, -, 2\} = 4/3$; x_5 leaves

	y_1	x_2	y_3	x_4	x_5	
z	0	0	3	−1/3	4/3	17/6
y_1	1	0	1	1/3	2/3	4/3
x_2	0	1	−1	−2/3	−1/3	4/3

x_4 enters: $\theta = \min\{\frac{4/3}{1/3}, \frac{4/3-3}{-2/3}, -\} = 5/2$

x_2 leaves, $x_2 = 1 - x_2'$

	y_1	x_2'	y_3	x_4	x_5	
z	0	1/2	7/2	0	3/2	13/2
y_1	1	−1/2	1/2	0	1/2	1/2
x_4	0	3/2	3/2	1	1/2	5/2

Optimum: $x_1 = 3/2$, $x_2 = 3$, $x_3 = 2$, $z = 13/2$

b) Set $x_1 = 1 + y_1$, $0 \le y_1 \le 2$, $0 \le x_2 \le 1$

Phase 1:

	y_1	x_2	x_3	R	x_4	x_5	
z	−1	2	0	0	0	0	1
R	−1	2	−1	1	0	0	1
x_4	3	2	0	0	1	0	7
x_5	−1	1	0	0	0	1	2
z	−2	0	−1	1	0	0	0
x_2	−1/2	1	−1/2	1/2	0	0	1/2
x_4	4	0	1	−1	1	0	6
x_5	−1/2	0	1/2	−1/2	0	1	3/2

Phase 2:

	y_1	x_2'	x_3	x_4	x_5	
z	0	4	1	0	0	4
y_1	1	2	1	0	0	1
x_4	0	−8	−3	1	0	2
x_5	0	1	1	0	1	2

Optimum: $x_1 = 2$, $x_2 = 1$, $z = 4$

c) Let $x_1 = 1 + y_1$

$0 \le y_1 \le 2,\ 0 \le x_2 \le 5,\ 0 \le x_3 \le 2$

	x_1	x_2	x_3	x_4	x_5	x_6	
z	-4	-2	-6	0	0	0	4
x_4	4	-1	0	1	0	0	5
x_5	-1	1	2	0	1	0	9
x_6	-3	1	4	0	0	1	15

x_3 enters: $\theta = \min\{15/4, -, 2\} = 2;\ x_3 = 2 - x_3'$

	y_1	x_2	x_3'	x_4	x_5	x_6	
z	-4	-2	6	0	0	0	16
x_4	4	-1	0	1	0	0	5
x_5	-1	1	-2	0	1	0	5
x_6	-3	1	-4	0	0	1	7

y_1 enters: $\theta = \min\{\frac{5}{4}, -, 2\} = 5/4;\ x_4$ leaves

	y_1	x_2	x_3'	x_4	x_5	x_6	
z	0	-3	6	1	0	0	21
y_1	1	-1/4	0	1/4	0	0	5/4
x_5	0	3/4	-2	1/4	1	0	25/4
x_6	0	1/4	-4	3/4	0	1	43/4

x_2 enters: $\theta = \min\{\frac{25}{3}, \frac{5/4-2}{-1/4}, 5\} = 3$

y_1 leaves, $y_1 = 2 - y_1'$

	y_1'	x_2	x_3'	x_4	x_5	x_6	
	12	0	6	-2	0	0	30
x_2	4	1	0	-1	0	0	3
x_5	-3	0	-2	1	1	0	4
x_6	-1	0	-4	1	0	1	10

x_4 enters: $\theta = \min\{4, \frac{3-5}{-1}, -\} = 2$

x_2 leaves, $x_2 = 5 - x_2'$

	y_1'	x_2'	x_3'	x_4	x_5	x_6	
z	4	2	6	0	0	0	34
x_4	-4	1	0	1	0	0	2
x_5	3	-1	-2	0	1	0	2
x_6	1	-1	-4	0	0	1	8

Optimum Solution:

$x_1 = 3$

$x_2 = 5$

$x_3 = 2$

$z = 34$

5

Let X_z represent the basic *and* nonbasic variables in X that have been substituted at their upper bound. Also, let X_z be the remaining basic *and* nonbasic variables. Suppose that the order of the vectors of (A, I) corresponding to X_z and X_u are given by the matrices D_z and D_u, and let the vector C of the objective function be partitioned correspondingly to give (C_z, C_u). The equations of the linear programming problem at any iteration then become

$$\begin{pmatrix} 1 & -C_z & -C_u \\ 0 & D_z & D_u \end{pmatrix}\begin{pmatrix} z \\ X_z \\ X_u \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Instead of dealing with two types of variables, X_z and X_u, X_u is put at zero level by using the substitution

$$X_u = U_u - X_u'$$

where U_u is a subset of U representing the upper bounds for the variables in X_u. This gives

$$\begin{pmatrix} 1 & -C_z & C_u \\ 0 & D_z & -D_u \end{pmatrix}\begin{pmatrix} z \\ X_z \\ X_u' \end{pmatrix} = \begin{pmatrix} C_u U_u \\ b - D_u U_u \end{pmatrix}$$

The optimality and the feasibility conditions can be developed more easily now, since all nonbasic variables are at zero level. However, it is still necessary to check that no basic or nonbasic variable will exceed its upper bound.

Define X_B as the basic variables of the current iteration, and let C_B represent the elements corresponding to X_B in C. Also, let B be the basic matrix corresponding to X_B. The current solution is determined from

$$\begin{pmatrix} 1 & -C_B \\ 0 & B \end{pmatrix}\begin{pmatrix} z \\ X_B \end{pmatrix} = \begin{pmatrix} C_u U_u \\ b - D_u U_u \end{pmatrix}$$

By inverting the partitioned matrix as in Section 4.1.3, the current basic solution is given by

$$\begin{pmatrix} z \\ X_B \end{pmatrix} = \begin{pmatrix} 1 & C_B B^{-1} \\ 0 & B^{-1} \end{pmatrix}\begin{pmatrix} C_u U_u \\ b - D_u U_u \end{pmatrix} = \begin{pmatrix} C_u U_u + C_B B^{-1}(b - D_u U_u) \\ B^{-1}(b - D_u U_u) \end{pmatrix}$$

By using

$$b' = b - D_u U_u$$

the complete simplex tableau corresponding to any iteration is

Basic	X_z^T	$X_u'^T$	Solution
z	$C_B B^{-1} D_z - C_z$	$-C_B B^{-1} D_u + C_u$	$C_B B^{-1} b' + C_u U_u$
X_B	$B^{-1} D_z$	$-B^{-1} D_u$	$B^{-1} b'$

6

(a) $b' = b - D_u U_u$

$$= \begin{pmatrix} 7 \\ 15 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix}(3) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_1 \end{pmatrix} = B^{-1} b' = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}$$

(b) $X_B = \begin{pmatrix} x_4 \\ x_2' \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 0 & -4 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & -1/4 \\ 0 & -1/4 \end{pmatrix}$

$b' = b - D_u U_u$

$$= \begin{pmatrix} 7 \\ 15 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$$X_B = \begin{pmatrix} x_4 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -1/4 \\ 0 & -1/4 \end{pmatrix}\begin{pmatrix} 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 5/4 \\ 5/4 \end{pmatrix}$$

7

Minimize $z = 6x_1 - 2x_2 - 3x_3$

Subject to

$$2x_1 + 4x_2 + 2x_3 + x_4 = 8$$
$$x_1 - 2x_2 + 3x_3 + x_5 = 7$$
$$0 \le x_1 \le 2,\quad 0 \le x_2 \le 2,\quad 0 \le x_3 \le 1$$

We use the tableau developed in Problem 5 above.

Iteration 0:

$$X_B = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix},\quad B = B^{-1} = I$$

$$C_B = (0,0),\quad C_B B^{-1} = (0,0)$$

$$\{z_j - c_j\}_{1,2,3} = (0,0)\begin{pmatrix} 2 & 4 & 2 \\ 1 & -2 & 3 \end{pmatrix} - (6,-2,-3)$$

$$= (-6, 2, 3),\quad x_3 \text{ enters}$$

$$B^{-1}P_3 = B^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = B^{-1}b = \begin{pmatrix} 8 \\ 7 \end{pmatrix} \Longrightarrow \theta_1 = 7/3$$

Since $B^{-1}P_3 > 0$, $\theta_2 = \infty$

$$\theta = \min\{7/3, \infty, 1\} = 1$$

Thus, x_3 becomes nonbasic at its upper bound.

New Solution: $X_Z = (x_1, x_2)$, $X_u = x_3$

$U_u = 1$, $C_u = -3$

$$D_Z = \begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix},\quad D_u = \begin{pmatrix} 2 \\ 3 \end{pmatrix},\quad C_Z = (6,-2)$$

$$b' = \begin{pmatrix} 8 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}(1) = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = B^{-1}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix},\quad z = -3$$

Iteration 1: $C_Z = (6,-2)$, $C_u = C_3' = 3$

$$P_3' = \begin{pmatrix} -2 \\ -3 \end{pmatrix},\quad B = B^{-1} = I,\ C_B = (0,0),\ C_B B^{-1} = (0,0)$$

$$\{z_j - c_j\}_{Z(j=1,2)} = (0,0)\begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} - (6,-2) = (-6, 2)$$

continued...

$$\{z_j - c_j\}_{u(j=3)} = (0,0)\begin{pmatrix} -2 \\ -3 \end{pmatrix} - (3) = -3$$

x_2 enters

$$B^{-1}P_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix},\quad X_B = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\theta_1 = \frac{6}{4} = 3/2,\quad \theta_2 = \infty \text{ (because } U_5 = \infty)$$

$$\theta = \min\{3/2, \infty, 2\} = 3/2$$

x_4 leaves

Iteration 2: $C_Z = (x_1, x_4)$, $X_u = x_3$

$$X_B = \begin{pmatrix} x_2 \\ x_5 \end{pmatrix},\quad P_3' = \begin{pmatrix} -2 \\ -3 \end{pmatrix},\quad b' = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix},\quad B^{-1} = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 1 \end{pmatrix}$$

$$C_B = (-2, 0),\quad C_B B^{-1} = (-1/2, 0)$$

$$\{z_j - c_j\}_{Z(j=1,4)} = (-1/2, 0)\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} - (6,0) = (-7, 0)$$

$$\{z_j - c_j\}_{u(j=3)} = (-1/2, 0)\begin{pmatrix} -2 \\ -3 \end{pmatrix} - 3 = -2$$

Optimum!

$$X_B = \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 7 \end{pmatrix}$$

$$x_3 = 1 - 0 = 1$$

$$z = -6$$

8

(a)

To convert the problem into a dual feasible solution, we use the following substitutions:

$x_1 = 2 - x_1'$, $x_2 = 3 - x_2'$

Thus,

minimize $z = 3x_1' + 2x_2' + 2x_3 - 12$

Subject to

$-2x_1' - x_2' + x_3 \le 1$

$-x_1' + 2x_2' - x_3 \le -9$

$0 \le x_1' \le 2,\ 0 \le x_2' \le 3,\ 0 \le x_3 \le 1$

	x_1'	x_2'	x_3	x_4	x_5	
z	-3	-2	-2	0	0	-12
x_4	-2	-1	1	1	0	1
x_5	-1	2	$\boxed{-1}$	0	1	-9

x_5 leaves and x_3 enters

	x_1'	x_2'	x_3	x_4	x_5	
z	-1	-6	0	0	-2	6
x_4	-2	1	0	1	1	-8
x_3	1	-2	1	0	-1	9

x_3 above its upper bound, substitute $x_3 = 1 - x_3'$, then multiply the second row by -1.

	x_1'	x_2'	x_3'	x_4	x_5	
z	-1	-6	0	0	-2	6
x_4	-2	1	0	1	1	-8
x_3'	-1	2	1	0	1	-8

x_3' leaves and x_1' enters

	x_1'	x_2'	x_3'	x_4	x_5	
z	0	-8	-1	0	-3	14
x_4	0	-3	-2	1	-1	8
x_1'	1	-2	-1	0	-1	8

Substitute $x_1' = 2 - x_1$ and multiply second row by -1

Continued...

	x_1	x_1'	x_3'	x_4	x_5	
z	0	-8	-1	0	-3	14
x_4	0	-3	-2	1	-1	8
x_1	1	2	1	0	1	-8

x_1-row shows that the problem has no feasible solution

(b) Let $x_1 = 2 - x_1'$

$x_2 = 3 - x_2'$

This substitution will result in a dual feasible starting solution

	x_1'	x_2'	x_3	x_4	x_5	
z	1	5	2	0	0	17
x_4	-4	-2	2	1	0	12
x_5	1	3	$\boxed{-4}$	0	1	-6
z	3/2	13/2	0	0	1/2	14
x_4	-7/2	-1/2	0	1	1/2	9
x_3	-1/4	-3/4	1	0	-1/4	3/2

Optimum!

$x_1 = 2 - 0 = 2$

$x_2 = 3 - 0 = 3$

$x_3 = 3/2$

$z = 14$

1

<u>Primal:</u>

Maximize $z = CX$

Subject to

$$AX = b \quad \leftarrow Y$$
$$X \geq 0$$

<u>Dual:</u>

Minimize $w = Yb$

Subject to

$$YA \geq C$$
$$Y \text{ unrestricted}$$

<u>Dual in equation form:</u>

Minimize $w = Yb$

Subject to

$$YA - IS = C \quad \leftarrow X$$
$$Y \text{ unrestricted}$$
$$S \geq 0$$

<u>Dual of dual:</u>

Maximize $z = CX$

Subject to

$$AX = b$$
$$-X \leq 0 \Rightarrow X \geq 0$$

The first set of constraints is equation because Y is unrestricted

The last problem shows that the dual of the dual is the primal

2

<u>Primal in equation form:</u>

Minimize $z = CX$

Subject to

$$AX - IS = b \quad \leftarrow Y$$
$$X \geq 0$$
$$S \geq 0$$

<u>Dual:</u>

Maximize $w = Yb$

Subject to

$$YA \leq C$$
$$-Y \leq 0 \Rightarrow Y \geq 0$$

Set 7.4b

1

Primal in equation form:

Maximize $z = x_1 + x_2$

Subject to

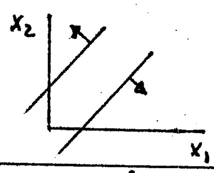
$$x_1 - x_2 + S_1 = -1 \quad \leftarrow y_1$$
$$-x_1 + x_2 + S_2 = -1 \quad \leftarrow y_2$$

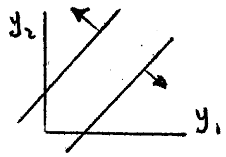
Dual:

Minimize $w = -y_1 - y_2$

Subject to

$$y_1 - y_2 \geq 1$$
$$-y_1 + y_2 \geq 1$$
$$y_1, y_2 \geq 0$$





2

(a) Dual:

Minimize $w = y_1 - 5y_2 + 6y_3$

Subject to

$$2y_1 + 4y_3 \geq 50$$
$$y_1 + 2y_2 \geq 30$$
$$y_3 \geq 10$$
$$y_1, y_2, y_3 \text{ unrestricted.}$$

(b) $2x_1 = -5 \Rightarrow x_1 < 0$, infeasible

(c) Inspection of the second dual constraint shows that y_2 can be increased indefinitely without violating any of the dual constraints. Thus, $w = y_1 - 5y_2 + 6y_3$ is unbounded.

(d)

$$\text{Primal infeasible} \Rightarrow \begin{cases} \text{dual infeasible} \\ \text{or} \\ \text{dual unbounded} \end{cases}$$

Primal unbounded $\Rightarrow$ dual infeasible

3

(a) Minimize $w = 2y_1 + 5y_2$

Subject to

$$2y_1 + y_2 \geq 5$$
$$-y_1 + 2y_2 \geq 12$$
$$3y_1 + y_2 \geq 4$$
$$y_2 \geq 0$$
$$y_1 \text{ unrestricted}$$

(b)

(i) $B = (P_4\ P_3) = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} -1/3 & 1 \\ 1/3 & 0 \end{pmatrix}$

$X_B = \begin{pmatrix} -1/3 & 1 \\ 1/3 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 13/3 \\ 2/3 \end{pmatrix}$ feasible

$C_B = (0, 4)$

$Y = C_B B^{-1} = (0, 4)\begin{pmatrix} -1/3 & 1 \\ 1/3 & 0 \end{pmatrix} = (4/3, 0)$

Dual feasibility:

$2y_1 + y_2 = 2 \times 4/3 + 1 \times 0 = 8/3 \not\geq 5$

Dual infeasible $\Rightarrow$ primal nonoptimal.

(ii) $B = (P_2 P_3) = \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{pmatrix}$

$X_B = \begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 13/7 \\ 9/7 \end{pmatrix}$ feasible

Dual feasibility:

$Y = C_B B^{-1} = (12, 4)\begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{pmatrix}$

$= (-4/7, 40/7)$

$2y_1 + y_2 = 2(-4/7) + 40/7 = \frac{32}{7} \not\geq 5$

X_B is not optimal

(iii) $B = (P_1\ P_2) = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix}$

$X_B = \begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 9/5 \\ 8/5 \end{pmatrix}$ feasible

Dual feasibility:

$Y = C_B B^{-1} = (5, 12)\begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix}$

$= (-2/5, 29/5)$

Y satisfies all dual constraints. Thus X_B is optimal.

continued...

(iv) $B = (P_1 P_4) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

$B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}$

$X_B = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ feasible

Dual feasibility:

$Y = C_B B^{-1} = (5, 0)\begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} = (5/2, 0)$

Y does not satisfy second dual constraint. X_B is not optimum

4

(a) Dual:

Minimize $w = 4y_1 + 8y_2$

Subject to

$$\left.\begin{aligned} y_1 + y_2 &\ge 2 \\ y_1 + 4y_2 &\ge 4 \\ y_1 &\ge 4 \\ y_2 &\ge -3 \end{aligned}\right\} \text{all } y \text{ unrestr.}$$

(b) $X_B = (x_2, x_3)^T$

$B = \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 0 & 1/4 \\ 1 & -1/4 \end{pmatrix}$

$C_B = (4, 4)$, $C_B B^{-1} = (4, 0)$

$z_1 - c_1 = C_B B^{-1} P_1 - c_1$

$= (4,0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2 = 2 > 0$

$z_4 - c_4 = (4, 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - (-3) = 3 > 0$

X_B optimal

(c) x_3 basic $\Rightarrow z_3 - c_3 = 0$, or

$YP_3 - c_3 = (y_1, y_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 4 = 0$, or

$y_1 - 4 = 0 \Rightarrow y_1 = 4$ (1)

x_2 basic $\Rightarrow z_2 - c_2 = 0$, or

$YP_2 - c_2 = (y_1, y_2)\begin{pmatrix} 1 \\ 4 \end{pmatrix} - 4 = 0$, or

$y_1 + 4y_2 = 4$. Given (1), we get $y_2 = 0$.

5

$B^{-1} b = X_B$

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} \Rightarrow \begin{aligned} b_1 &= 4 \\ b_2 &= 6 \\ b_3 &= 8 \end{aligned}$$

Dual objective value is

$w = Yb = (0, 3, 2)\begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} = 34$

From the dual:

$C_B B^{-1} = Y$

$(c_1, c_2, 0)\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} = (0, 3, 2)$

or $\left.\begin{aligned} c_2 - c_1 &= 3 \\ c_1 &= 2 \end{aligned}\right\} \Rightarrow c_1 = 2,\ c_2 = 5$

Primal objective value is

$z = C_B X_B = (2, 5, 0)\begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = 34$

6

$$\sum_{i=1}^{m} c_i (B^{-1}P_k)_i = (C_B B^{-1})P_k = YP_k = \sum_{i=1}^{m} y_i a_{ik}$$

7

Minimize $w = Yb$

Subject to $YA = C$

Y unrestricted

8

Dual: Minimize $Y_1 b - Y_2 L + Y_3 U$

subject to

$Y_1 A - Y_2 + Y_3 \ge C$

$Y_1, Y_2, Y_3 \ge 0$

Let $Y = Y_3 - Y_2 \Rightarrow Y$ unrestricted. Hence $Y_1 A + (Y_3 - Y_2) \ge C$ can be written as $Y_1 A + Y \ge C$. Since Y is unrestricted, its value can always be selected such that $Y_1 A + Y \ge C$ is satisfied

1

For X_{B_0}:

$\{z_j - c_j\}_{j=1,4,5}$

$= (4+14t, 1-t, 2+3t) \geq (0,0,0)$

The inequalities are satisfied for

$-2/7 \leq t \leq 1$

2

(a) $C_B(t)B_0^{-1} = (2, 5-6t, 0)\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$

$= (1, 2-3t, 0)$

$X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$

$\{z_j - c_j\}_{j=1,4,5}$

$= (1, 2-3t, 0)\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3+3t, 0, 0)$

$= (4-12t, 1, 2-3t) \geq (0,0,0)$

X_{B_0} remains optimal for $t \leq 1/3$

at $t = 1/3$, x_1 enters solution

$B_0^{-1} P_1 = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 3/2 \\ 2 \end{pmatrix}$

x_6 leaves.

$X_{B_1} = (x_2, x_3, x_1)^T$

$B_1 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \\ 4 & 0 & 1 \end{pmatrix}$

$B_1^{-1} = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}$

$X_{B_1} = B_1^{-1} b = (25/4, 90/4, 5)^T$

$C_B(t) B_1^{-1} = (2, 5-6t, 3+3t)\begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}$

$= (5-12t, 3t, -2+6t)$

$\{z_j - c_j\}_{j=4,5,6}$

$= (5-12t, 3t, -2+6t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (0,0,0)$

$= (5-12t, 3t, -2+6t)$

X_{B_1} remains optimal for $1/3 \leq t \leq 5/12$

continued...

at $t = 5/12$, x_4 enters

$B_1^{-1} P_4 = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 3/2 \\ -1 \end{pmatrix}$

x_3 leaves

$X_{B_2} = (x_2, x_4, x_1)^T$

$B_2 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 3 \\ 4 & 0 & 1 \end{pmatrix}$

$B_2^{-1} = \begin{pmatrix} 0 & -1/12 & 1/4 \\ 1 & -1/6 & -1/2 \\ 0 & 1/3 & 0 \end{pmatrix}$

$X_{B_2} = B_2^{-1} b = (5/2, 15, 20)^T$

$C_B(t) B_2^{-1} = (2, 0, 3+3t)\, B_2^{-1}$

$= (0, 5/6 + t, 1/2)$

$\{z_j - c_j\}_{j=3,5,6}$

$= (0, 5/6 + t, 1/2)\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (5-6t, 0, 0)$

$= (-10/3 + 8t, 5/6 + t, 1/2)$

X_{B_2} remains optimal for $5/12 \leq t < \infty$

(b) $X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$

$C_B(t) B_0^{-1} = (2+t, 5+2t, 0)\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$

$= (1 + t/2, 2 + 3t/4, 0)$

$\{z_j - c_j\}_{j=1,4,5}$

$= (1 + t/2, 2 + 3/4\,t, 0)\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3-2t, 0, 0)$

$= (4 + 19t/4, 1 + t/2, 2 + 3t/4) \geq (0,0,0)$

X_{B_0} is optimal for all $t \geq 0$

(c) $X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$

$C_B(t) B_0^{-1} = (2+2t, 5-t, 0)\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$

$= (1+t, 2-t, 0)$

$\{z_j - c_j\}_{j=1,4,5}$

$= (1+t, 2-t, 0)\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3+t, 0, 0)$

$= (4-3t, 1+t, 2-t) \geq (0,0,0)$

continued...

X_{B_0} remains optimal for the range $t \le 4/3$. At $t = 4/3$, x_1 enters solution. As in Part (a) above, x_6 leaves

$$B_1^{-1} = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}, \quad X_{B_1} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

$$X_{B_1} = B_1^{-1} b = (25/4,\ 90/4,\ 5)^T$$

$$C_B(t) B_1^{-1} = (2+2t,\ 5-t,\ 3+t)\ B_1^{-1} = (5-2t,\ t/2,\ -2+3/2\,t)$$

$$\{z_j - c_j\}_{j=4,5,6} = (5-2t,\ t/2,\ -2+3/2\,t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (0,0,0)$$

$$= (5-2t,\ t/2,\ -2+3/2\,t) \ge (0,0,0)$$

X_{B_1} remains optimal for

$$4/3 \le t \le 5/2$$

At $t = 5/2$, x_4 enters solution. As in Part (a), we have x_3 leaving and

$$B_2^{-1} = \begin{pmatrix} 0 & -1/12 & 1/4 \\ 1 & -1/6 & -1/2 \\ 0 & 1/3 & 0 \end{pmatrix}, \quad X_{B_2} = \begin{pmatrix} x_2 \\ x_4 \\ x_1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 15 \\ 20 \end{pmatrix}$$

$$C_B(t) B_2^{-1} = (2+2t,\ 0,\ 3+t)\begin{pmatrix} 0 & -1/12 & 1/4 \\ 1 & -1/6 & -1/2 \\ 0 & 1/3 & 0 \end{pmatrix} = (0,\ 5/6 + t/6,\ 1/2 + t/2)$$

$$\{z_j - c_j\}_{j=3,5,6} = (0,\ 5/6 + t/6,\ 1/2 + t/2)\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (5-t,\ 0,\ 0)$$

$$= (-10/3 + 4t/3,\ 5/6 + t/6,\ 1/2 + t/2) \ge (0,0,0)$$

X_{B_2} remains optimal for $\frac{5}{2} \le t < \infty$

3

Minimize $z = (4-t)x_1 + (1-3t)x_2 + (2-2t)x_3$

Subject to

$$\begin{aligned} 3x_1 + x_2 + 2x_3 \quad\quad\quad &= 3 \\ 4x_1 + 3x_2 + 2x_3 - x_4 \quad &= 6 \\ x_1 + 2x_2 + 5x_3 \quad + x_5 &= 4 \end{aligned}$$

$$x_1, x_2, \ldots, x_5 \ge 0$$

$$X_{B_0} = (x_1, x_2, x_4)^T = (2/5,\ 9/5,\ 1)$$

$$B_0^{-1} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$C_B(t) B_0^{-1} = (4-t,\ 1-3t,\ 0)\ B_0^{-1} = \left(\frac{7+t}{5},\ 0,\ -\frac{1+8t}{5}\right)$$

$$\{z_j - c_j\}_{j=3,5} = \left(\frac{7+t}{5},\ 0,\ -\frac{1+8t}{5}\right)\begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 5 & 1 \end{pmatrix} - (2-2t,\ 0)$$

$$= \left(-\frac{1+28t}{5},\ -\frac{1+8t}{5}\right) \le (0,0)$$

B_0 remains optimal for all $t \ge 0$.

4

The dual simplex method requires that the LP problem be put in the form:

Minimize $z = CX$

Subject to

$$-AX \le -b, \quad X \ge 0$$

Let B_i be the basis associated with critical value t_i in the parametric analysis. To obtain t_{i+1}, we consider

$$\{z_j - c_j\}_{\text{nonbasic } x_j} = C_B(t) B_i^{-1}(-P_j) - c_j(t) \le 0$$

where P_j is the jth column vector of A.

In the present problem, the first two constraints are of the type $\ge$. Hence, only the first two constraints are multiplied by -1.

$$X_{B_0} = (x_3, x_2, x_6)^T = (3/2,\ 3/2,\ 0)^T$$

$$B_0^{-1} = \begin{pmatrix} -3/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad C_{B_0}(t) = (1,\ 2+4t,\ 0)$$

Continued...

$C_{B_0}(t)B_0^{-1} = (-1/2+2t, -1/2-2t, 0)$

$\{z_j - c_j\}_{1,4,5} = C_{B_0}B_0^{-1}P_j' - c_j(t)$

$= (-1/2+2t, -1/2-2t, 0)\begin{pmatrix}-3 & 1 & 0\\ 3 & 0 & 0\\ 1 & 0 & 1\end{pmatrix} - (3+t, 0, 0)$

$= (-13t-3, -1/2+2t, 0) \le (0,0,0)$

Thus, $t_1 = 1/4 \Rightarrow x_{B_0}$ *remains optimal for* $0 \le t \le 1/4$.

At $t = 1/4$, x_4 *enters and* x_6 *leaves.*

$X_{B_1} = (x_3, x_2, x_4)^T = (3/2, 3/2, 0)^T$

$B_1^{-1} = \begin{pmatrix}0 & 1/2 & 3/2\\ 0 & -1/2 & -1/2\\ 1 & 0 & 1\end{pmatrix}, C_{B_1}(t) = (1, 2+4t, 0)$

$C_{B_1}(t)B_1^{-1} = (0, -1/2-2t, 1/2-2t)$

$\{z_j - c_j\}_{1,5,6} = (0, -1/2-2t, 1/2-2t)\begin{pmatrix}-3 & 0 & 0\\ 3 & 1 & 0\\ 1 & 0 & 1\end{pmatrix} - (3+t, 0, 0)$

$= (-4-9t, -1/2-2t, 1/2-2t) \le (0,0,0)$

Conditions are satisfied for $t \ge 1/4$. *Thus,* X_{B_1} *is optimal for all* $t \ge 1/4$.

Summary:

$X_{B_0} = (x_3, x_2, x_6) = (3/2, 3/2, 0)$ *is optimal for* $0 \le t \le 1/4$

$X_{B_1} = (x_3, x_2, x_4) = (3/2, 3/2, 0)$ *is optimal for* $t \ge 1/4$

OR

$\left.\begin{array}{l} x_1 = 0\\ x_2 = 3/2\\ x_3 = 3/2\end{array}\right\}$ *for all* $t \ge 0$

5

$X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$

$C_{B_0}(t) = (2-2t^2, 5-t, 0)$

$B_0^{-1} = \begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix}$

$C_B(t)\, B_0^{-1} = (2-2t^2, 5-t, 0)\begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix}$

$= (1-t^2, t^2/2 - t/2 + 2, 0)$

continued...

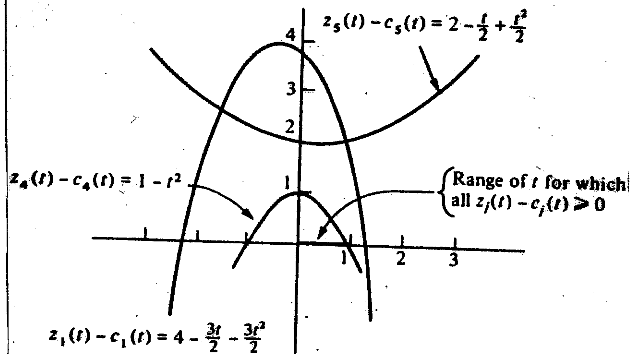
$\{z_j - c_j\}_{j=1,4,5}$

$= (1-t^2, t^2/2 - t/2 + 2, 0)\begin{pmatrix}1 & 1 & 0\\ 3 & 0 & 1\\ 1 & 0 & 0\end{pmatrix} - (3+2t^2, 0, 0)$

$= \left(4 - \frac{3t}{2} - \frac{3t^2}{2}, 1-t^2, 2 - \frac{t}{2} + \frac{t^2}{2}\right)$

$\ge (0,0,0)$

The graph below summarizes the optimality conditions.



X_{B_0} *remains optimal for* $0 \le t \le 1$.

1

(a) $X_{B_0} = (x_2, x_3, x_6)^T$

$$= \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 40+2t \\ 60-3t \\ 30+6t \end{pmatrix}$$

$$= \begin{pmatrix} 5+t/4 \\ 30-3t/2 \\ 10-t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-20 \leq t \leq 10, \quad t_1 = 10$

x_6 leaves at $t=10$.

(row of B_0^{-1} associated with x_6)$(P_1\ P_4\ P_5)$

$$= (-2, 1, 1)\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (2, -2, 1)$$

$\{z_j - c_j\}_{j=1,4,5}$

$$= (2,5,0)\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3,0,0)$$

$= (4, 1, 2)$

	x_1	x_4	x_5
$z_j - c_j$	4	1	2
x_6	2	-2	1

x_4 enters.

$$\text{new } B_1 = (P_2\ P_3\ P_4) = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

(b) $X_{B_0} = (x_2, x_3, x_6)^T$

$$= \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 40-t \\ 60+2t \\ 30-5t \end{pmatrix}$$

$$= \begin{pmatrix} 5-t \\ 30+t \\ 10-t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-30 \leq t \leq 5 \quad t_1 = 5$

x_2 leaves when $t=5$.

(row of B_0^{-1} associated with x_2)$(P_1 P_4 P_5) =$

$$= (1/2, -1/4, 0)\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$= (-1/4, 1/2, -1/4)$

$\{z_j - c_j\}_{j=1,4,5}$

$$= (2,5,0)\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3,0,0)$$

$= (4, 1, 2)$

	x_1	x_4	x_5
$z_j - c_j$	4	1	2
x_6	$-1/4$	$1/2$	$-1/4$

x_5 enters

$$\text{new } B_1 = (P_5\ P_3\ P_6) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2

$X_{B_0} = (x_1, x_2, x_4)^T = (2/5, 9/5, 1)$

x_4 = surplus in constraint 2

x_5 = slack in constraint 3

$$B_0^{-1} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$X_{B_0}(t) = B_0^{-1}\begin{pmatrix} 3+3t \\ 6+2t \\ 4-t \end{pmatrix} = \begin{pmatrix} 2/5+7/5\,t \\ 9/5-6/5\,t \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus, $0 \leq t \leq 3/2$, $t_1 = 3/2$

At $t = 3/2$, x_2 leaves the solution. To determine the entering variable, we use the dual simplex computations.

(row of B_0^{-1} associated with x_2) (P_3, P_5)

$$= (-1/5, 0, 3/5)\begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 5 & 1 \end{pmatrix} = (13/5, 3/5)$$

Because $(13/5, 3/5) \geq 0$, the problem has no feasible solution for $t > 3/2$ (per dual simplex conditions).

Summary:

$x_1 = 2/5, x_2 = 9/5, x_3 = 0$, for $0 \leq t \leq 3/2$

No feasible solution for $t > 3/2$

Continued...

For the dual simplex, the feasibility condition is

$$B^{-1}b'(t) \geq 0$$

where $b'(t)$ is modified such that the element $b_i(t)$ associated with $\geq$ constraint is replaced with $-b_i(t)$.

$$X_{B_0} = (x_3, x_2, x_6)^T = (3/2, 3/2, 0)$$

$$B_0^{-1} = \begin{pmatrix} -3/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$b_0'(t) = \begin{pmatrix} -3-2t \\ -6+t \\ 3-4t \end{pmatrix}$$

The top two elements appear with an opposite sign because the first two constraints are of the type ≥ 0, hence reversing their signs in the dual simplex method.

$$B_0^{-1} b_0'(t) = \begin{pmatrix} -3/2 & -1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3-2t \\ -6+t \\ 3-4t \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 + 5/2\,t \\ 3/2 - 3/2\,t \\ -6t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus,

$x_3 = 3/2 + 5/2\,t \geq 0$ gives $t \geq -\frac{3}{5}$

$x_2 = 3/2 - 3/2\,t \geq 0$ gives $t \leq 1$

$x_6 = -6t$ gives $t \leq 0$

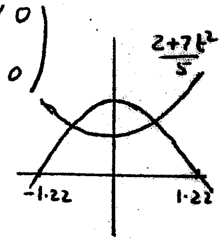
Thus, for $t \geq 0$, the solution X_{B_0} is feasible for $t = 0$ only. Else, the problem has no feasible solution for $t > 0$

4

$$X_{B_0} = (x_1, x_2, x_3)^T$$

$$X_{B_t} = B_0^{-1} b(t) = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3+3t^2 \\ 6+2t^2 \\ 4-t^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2/5 + 7/5\,t^2 \\ 9/5 - 6/5\,t^2 \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



$-1.22 \leq t \leq 1.22$

x_2 leaves at $t = 1.22$

$$(\text{Row 2 of } B_0^{-1})(P_4\ P_5)$$

$$= (-1/5, 0, 3/5) \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} = (0, 3/5)$$

$\Rightarrow$ no feasible solution exists for $t > 1.22$

continued...