Chapter 22

Probabilistic Dynamic Programming

Set 22.1a

1

$$f_6(j) = 2j$$

$$f_i(j) = \max \begin{cases} \text{end: } 2j \\ \text{spin: } \frac{1}{8}\sum_{k=1}^{8} f_{i+1}(k) \end{cases} \quad i = 2, 3, 4, 5$$

$$f_1(0) = \frac{1}{8}\sum_{k=1}^{8} f_2(k)$$

<u>Stage 6:</u>

Spin 5 outcome	Opt. Soln. $f_6(j)$	Decision
1	2	End
2	4	End
3	6	End
4	8	End
5	10	End
6	12	End
7	14	End
8	16	

<u>Stage 5:</u> $f_5(j) = \max\{2j, \frac{1}{8}(f_6(1)+f_6(2)+\cdots+f_6(8))\}$

$$= \max\{2j, \tfrac{72}{8}\}$$

$$= \max\{2j, 9\}$$

Spin 4 outcome		Spin	Opt. Sol. $f_5(j)$	Decision
1	2	9	9	Spin
2	4	9	9	Spin
3	6	9	9	Spin
4	8	9	9	Spin
5	10	9	10	End
6	12	9	12	End
7	14	9	14	End
8	16	9	16	End

<u>Stage 4:</u>

$$f_4(j) = \max\{2j, \tfrac{1}{8}(9+9+9+9+10+12+14+16)\}$$

$$= \max\{2j, 11\}$$

Spin 3 outcome	End	Spin	Opt. Sol. $f_4(j)$	Decision
1	2	11	11	Spin
2	4	11	11	Spin
3	6	11	11	Spin
4	8	11	11	Spin
5	10	11	11	Spin
6	12	11	12	End
7	14	11	14	End
8	16	11	16	End

<u>Stage 3:</u> $f_3(j) = \max\{2j, 12.125\}$

Spin 2 outcome	End	Spin	Opt. Sol. $f_3(j)$	Decision
$1 \le j \le 6$	$2j$	12.125	12.125	Spin
$j = 7, 8$	$2j$	12.125	$2j$	End

<u>Stage 2:</u> $f_2(j) = \max\{2j, 12.84375\}$

Spin 1 outcome	End	Spin	Opt. Sol. $f_2(j)$	Decision
$1 \le j \le 6$	$2j$	12.84375	12.84375	Spin
$j = 7, 8$	$2j$	12.84375	$2j$	End

<u>Stage 1:</u> $f_1(0) = \frac{1}{8}(6 \times 12.84375 + 14 + 16) = 13.38$

<u>Solution:</u>

Spin #	Strategy
1	Continue to spin
2	Continue if #1 produces 1→6, else end
3	Continue if #2 produces 1→6, else end
4	Continue if #3 produces 1→5, else end.
5	Continue if #4 produces 1→4, else end
6	End

Expected return = \$13.38

continued...

2

Let O_j represent the best offer at the end of day i, where

$$j = \begin{cases} 1, \text{ high offer} \\ 2, \text{ medium offer} \\ 3, \text{ low offer} \end{cases}$$

$i = 1, 2, 3$

$f_4(j) = O_j$

$$f_i(j) = \max \begin{cases} \text{accept}: O_j \\ \text{continue}: \frac{1}{3}\left(f_{i+1}(1) + f_{i+1}(2) + f_{i+1}(3)\right) \end{cases}$$

$f_1(0) = \frac{1}{3}\{f_2(1) + f_2(2) + f_3(2)\}$

Stage 4:

Day 3 best offer j	Opt. Sol. $f_4(j)$	Decision
1	1050	Accept
2	1900	Accept
3	2500	Accept

Stage 3:

$f_3(j) = \max\{O_j, \frac{1}{3}\left(f_4(1) + f_4(2) + f_4(3)\right)\}$

$= \max\{O_j, 1816.67\}$

Day 2 best offer j	Accept	Continue	Opt. Sol. $f_3(j)$	Decision
1	1050	1816.67	1816.67	Continue
2	1900	1816.67	1900	Accept
3	2500	1816.67	2500	Accept

Stage 2:

$f_2(j) = \max\{O_j, \frac{1}{3}\left(f_3(1) + f_3(2) + f_3(3)\right)\}$

$= \max\{O_j, 2072.33\}$

Day 1 best offer j	Accept	Continue	Opt. Sol. $f_2(j)$	Decision
1	1050	2072.33	2072.33	Continue
2	1900	2072.33	2072.33	Continue
3	2500	2072.33	2072.33	Accept

Stage 1:

$f_1(0) = \frac{1}{3}(2 \times 2072.33 + 2500)$

$= \$2214.82$

Solution:

Day 1: Accept if offer is high

Day 2: Accept if offer is medium or high

Day 3: Accept any offer.

Set 22.2a

1

Stage 4:

$$f_4(x_4) = x_4(1+.8\times.6+.4\times.2+.2\times.2) = 1.6x_4$$

State	Opt. Sol. $f_4(x_4)$	y_4
x_4	$1.6x_4$	x_4 (invest all)

Stage 3:

$$f_3(x_3) = \max_{0\le y_3\le x_3}\{.2\times1.6(x_3+4y_3) + .4\times1.6(x_3-y_3) + .4\times1.6(x_3-y_3)\}$$
$$= \max_{0\le y_3\le x_3}\{1.6x_3\}$$

State	Optimum $f_3(x_3)$	y_3
x_3	$1.6x_3$	$0\le y_3\le x_3$

Stage 2:

$$f_2(x_2) = \max_{0\le y_2\le x_2}\{.4\times1.6(x_2+y_2) + .4\times1.6x_2 + .2\times1.6(x_2-y_2)\}$$
$$= \max_{0\le y_2\le x_2}\{1.6x_2+.32y_2\}$$

State	Opt. Sol. $f_2(x_2)$	y_2
x_2	$1.92x_2$	x_2

Stage 1: $f_1(x_1) = \max_{0\le y_1\le x_1}\{.1\times1.92(x_1+2y_1) + .4\times1.92(x_1+y_1) + .5\times1.92(x_1+.5y_1)\}$

$$= \max_{0\le y_1\le x_1}\{1.92x_1+1.632y_1\}$$

State	Opt. Sol. $f_1(x_1)$	y_1
x_1	$3.552x_1$	x_1

Solution: accumulation = \$35,520

Invest \$10,000 in year 1, all in year 2, none in year 3, and all in year 4.

2

c_i = penalty cost/shortage unit of item i

z_i = number of units of item i

v_i = volume per unit of item i

x_i = m^3 assigned to items $i,\dots,n$

p_{ij} = probability of j demand units of item i

$f_i(x_i)$ = minimum expected penalty cost for items $i, i+1,\dots,$ and n, given x_i

$$f_i(x_i) = \min_{0\le z_i\le\left[\frac{x_i}{v_i}\right]}\Big\{c_i\sum_{j>z_i}(j-z_i)p_{ij} + f_{i+1}(x_i - z_iv_i)\Big\}$$
$$i = 1,2,\dots,n$$

$$f_{n+1}(\cdot)\equiv 0$$

Table for exp. shortage cost:

item i	$z_i=1$	$z_i=2$	$z_i=3$
1	$8(1\times.5)=4$	0	0
2	$10(1\times.4+2\times.2+3\times.1)=11$	$10(1\times.2+2\times.1)=4$	$10(1\times.1)=1$
3	$15(1\times.2+2\times.5)=18$	$15(1\times.5)=7.5$	0

Stage 3: $f_3(x_3) = \min_{z_3\le\left[\frac{x_3}{v_3}\right]}\Big\{15\sum_{j>z_3}(j-z_3)p_{3j}\Big\}$

x_3	$v_3=3$: $z_3=1$	$z_3=2$	$z_3=3$	Opt. Sol. $f_3(x_3)$	z_3
3	18	—	—	18	1
4	18	—	—	18	1
5	18	—	—	18	1
6	18	7.5	—	7.5	2
7	18	7.5	—	7.5	2
8	18	7.5	—	7.5	2
9	18	7.5	0	0	3
10	18	7.5	0	0	3

Stage 2: $v_2=1$

$$f_2(x_2) = \min_{z_2\le\left[\frac{x_2}{v_2}\right]}\Big\{10\sum_{j>z_2}(j-z_2)p_{2j} + f_3(x_2-z_2)\Big\}$$

2 continued

x_2	$z_2=1$	2	3	4	5	6	7	f_3	z_3
4	11+18=29	—	—	—	—	—	—	29	1
5	11+18=29	4+18=22	—	—	—	—	—	22	2
6	11+18=29	4+18=22	1+18=19	—	—	—	—	19	3
7	11+7.5=18.5	4+18=22	1+18=19	0+18=18	—	—	—	18	4
8	11+7.5=18.5	4+7.5=11.5	1+18=19	0+18=18	0+18=18	—	—	11.5	2
9	11+7.5=18.5	4+7.5=11.5	1+7.5=8.5	0+18=18	0+18=18	0+18=18	—	8.5	3
10	11+0=11	4+7.5=11.5	1+7.5=8.5	0+7.5=7.5	0+18=18	0+18=18	0+18=18	7.5	4

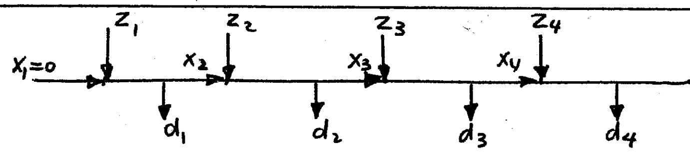
Stage 1: $f_1(x_1) = \min\limits_{z_1 \le \left[\frac{x_1}{v_1}\right]} \left\{ 8 \sum\limits_{j>z_1} (j-z_1) P_{1j} + f_2(x_1 - 2z_1) \right\}$

x_1	$z_1=1$	$z_1=2$	$z_1=3$	Opt. Sol. f_1	z_1
10	4+11.5=15.5	0+19=19	0+29=29	15.5	1

Solution:

$(x_1=10) \to z_1=1 \to (x_2=8) \to z_2=2 \to (x_3=6) \to z_3=2$

3



x = inventory
z = production
d = demand

$$f_n(x_n) = \min_{z_n} \{ C(x_n) \}$$

$$f_i(x_i) = \min_{z_i} \left\{ C(x_i) + \sum_{d=0}^{3} f_{i+1}(x_i + z_i - d_i)\, p(d_i) \right\}, \quad i = 1, 2, \dots, n-1 \qquad C(x_i) = \begin{cases} x_i, & x_i \ge 0 \\ -2x_i, & x_i \le 0 \end{cases}$$

Stage 4:

x_4	$z_4=0$	1	2	3	Opt. Sol. f_4	z_4
−3	—	—	—	6	6	3
−2	—	—	4	—	4	2
−1	—	2	—	—	2	1
0	0	—	—	—	0	0
1	1	—	—	—	1	0
2	2	—	—	—	2	0
3	3	—	—	—	3	0

Notice that negative x_4 allows for the possibility of backordering by producing for year 3 in period 4.

continued...

Set 22.2a

3 continued

Stage 3: $f_3(x_3) = \min_{z_3} \{C(x_3) + .5 f_4(x_3+z_3-1) + .3 f_4(x_3+z_3-2) + .2 f_4(x_3+z_3-3)\}$

x_3	$z_3=0$	1	2	3	Opt. Sol. f_3	z_3
-3	—	—	—	6+.5x2+.3x4+.2x6=9.4	9.4	3
-2	—	—	4+.5x2+.3x4+.2x6=7.4	4+.5x0+.3x2+.2x4=5.4	5.4	3
-1	—	2+.5x2+.3x4+.2x6=5.4	2+.5x0+.3x2+.2x4=3.4	2+.5x1+.3x0+.2x2=2.9	2.9	3
0	0+.5x2+.3x4+.2x6=3.4	0+.5x0+.3x2+.2x4=1.4	0+.5x1+.3x0+.2x2=.9	0+.5x2+.3x1+.2x0=1.3	.9	2
1	1+.5x0+.3x2+.2x4=2.4	1+.5x1+.3x0+.2x2=1.9	1+.5x2+.3x1+.2x0=2.3	—	1.9	1
2	2+.5x1+.3x0+.2x2=2.9	2+.5x2+.3x1+.2x0=3.3	—	—	2.9	0
3	3+.5x2+.3x1+.2x0=4.3	—	—	—	4.3	0

Stage 2: $f_2(x_2) = \min_{z_2} \{C(x_2) + .5 f_3(x_2+z_2-1) + .3 f_3(x_2+z_2-2) + .2 f_3(x_2+z_2-3)\}$

x_2	$z_2=0$	1	2	3	Opt. Sol. f_2	z_2
-3	—	—	—	6+.5x2.9+.3x5.4+.2x9.4=10.94	10.94	3
-2	—	—	4+.5x2.9+.3x5.4+.2x9.4=8.94	4+.5x.9+.3x2.9+.2x5.4=6.4	6.4	3
-1	—	2+.5x2.9+.3x5.4+.2x9.4=6.94	2+.5x.9+.3x2.9+.2x5.4=4.4	2+.5x1.9+.3x.9+.2x2.9=3.8	3.8	3
0	0+.5x2.9+.3x5.4+.2x9.4=4.94	0+.5x.9+.3x2.9+.2x5.4=2.4	0+.5x1.9+.3x.9+.2x2.9=1.8	0+.5x2.9+.3x1.9+.2x.9=2.2	1.8	2
1	1+.5x.9+.3x2.9+.2x5.4=3.4	1+.5x1.9+.3x.9+.2x2.9=2.8	1+.5x2.9+.3x1.9+.2x.9=3.2	—	2.8	1
2	2+.5x1.9+.3x.9+.2x2.9=3.8	2+.5x2.9+.3x1.9+.2x.9=4.2	2+.5x4.3+.3x2.9+.2x1.9=5.4	—	3.8	0
3	3+.5x2.9+.3x1.9+.2x.9=5.2	3+.5x4.3+.3x2.9+.2x1.9=6.4	—	—	5.2	0

Stage 1: $f_1(x_1) = \min_{z_1} \{C(x_1) + .5 f_2(x_1+z_1-1) + .3 f_2(x_1+z_1-2) + .2 f_2(x_2+z_2-3)\}$

x_1	$z_1=0$	1	2	3	Opt. Sol. f_1	z_1
0	0+.5x3.8+.3x6.4+.2x10.94=6.008	0+.5x1.8+.3x3.8+.2x6.4=3.32	0+.5x2.8+.3x1.8+.2x3.8=2.7	0+.5x3.8+.3x2.8+.2x1.8=3.1	2.7	2

Solution:

$(x_1=0) \rightarrow z_1=2$

- $(x_2=1) \rightarrow z_2=1$
- $(x_2=2) \rightarrow z_2=2$
- $(x_2=-1) \rightarrow z_2=3$

- $(x_3=1) \rightarrow z_3=1$
- $(x_3=0) \rightarrow z_3=2$
- $(x_3=-1) \rightarrow z_3=3$

- $(x_4=1) \rightarrow z_4=0$
- $(x_4=0) \rightarrow z_4=0$
- $(x_4=-1) \rightarrow z_4=1$

continued...

Stage i = center i

alternative y_i = number of bikes assigned to center i

State x_i = number of bikes assigned to centers $i, i+1, \ldots,$ and n

d_i = demand in center i

$f_i(x_i)$ = maximum expected revenue for stages $i, i+1, \ldots,$ and n given x_i.

$$f_n(x_n) = \max_{y_n \le x_n} \{ C_n E\{d_n \mid y_n\} \}$$

$$f_i(x_i) = \max_{y_i \le x_i} \{ C_i E\{d_i \mid y_i\} + f_{i+1}(x_i - y_i) \}, \quad i = 1, 2, \ldots, n-1$$

where

$E\{d_i \mid y_i\}$ = Average demand at center i given y_i bikes are allocated to center i

$$= 0P_0 + 1P_1 + \cdots + y_{i-1} P_{y_{i-1}} + y_i (P_{y_i} + P_{y_{i+1}} + \cdots + P_8)$$

Example calculations:

$$E\{d_1 \mid y_1 = 2\} = 0P_1 + 1P_1 + 2(P_2 + P_3 + \cdots + P_8)$$
$$= 0 + 1 \times .15 + 2 \times .85 = 1.85$$

Table for $C_i E\{d_i \mid y_i\}$:

i	C_i	$y_i = 0$	1	2	3	4	5	6	7	8
1	5	0	5.40	9.60	12.00	13.20	13.80	13.80	13.80	13.8
2	7	0	6.86	13.51	19.46	23.66	25.76	26.81	27.51	27.86
3	6	0	5.00	9.25	18.25	19.75	20.50	20.75	20.875	20.875

Stage 3: $f_3(x_3) = \max_{y_3 \le x_3} \{ C_3 E\{d_3 \mid y_3\} \}$

x_3	$y_3 = 0$	1	2	3	4	5	6	7	8	Opt. Sol. f_3	y_3
0	0	—								0	0
1	0	5	—							5.00	1
2	0	5	9.25	—						9.25	2
3	0	5	9.25	18.25	—					18.25	3
4	0	5	9.25	18.25	19.75	—				19.75	4
5	0	5	9.25	18.25	19.75	20.50	—			20.50	5
6	0	5	9.25	18.25	19.75	20.50	20.75	—		20.75	6
7	0	5	9.25	18.25	19.75	20.50	20.75	20.875	—	20.875	7
8	0	5	9.25	18.25	19.75	20.50	20.75	20.875	20.875	20.85	8

Continued...

Set 22.2a

Stage 2: $f_2(x_2) = \max\limits_{y_2 \le x_2} \{ C_2 E\{d_2 | y_2\} + f_3(x_2 - y_2) \}$

x_2	$y_2 = 0$	1	2	3	4	5	6	7	8	Optm f_2	Optm y_2
0	0 + 0 = 0	—								0	0
1	0 + 5 = 5	6.86 + 0 = 6.86	—							6.86	1
2	0 + 9.25 = 9.25	6.86 + 5 = 11.86	13.51 + 0 = 13.51	—						13.51	2
3	0 + 18.25 = 18.25	686 + 9.25 = 16.11	13.51 + 5 = 18.51	19.46 + 0 = 19.46	—					19.46	3
4	0 + 19.75 = 19.75	6.86 + 18.25 = 25.11	13.51 + 9.25 = 22.76	19.46 + 5 = 24.46	23.66 + 0 = 23.66	—				25.11	1
5	0 + 20.50 = 20.50	6.86 + 19.75 = 26.61	13.51 + 18.25 = 31.76	19.46 + 9.25 = 28.71	23.66 + 5 = 28.66	25.76 + 0 = 25.76	—			31.76	2
6	0 + 20.75 = 20.75	6.86 + 20.5 = 27.36	13.51 + 19.75 = 33.26	19.46 + 18.25 = 37.71	23.66 + 9.25 = 32.91	25.76 + 5 = 30.76	26.81 + 0 = 26.81	—		37.71	3
7	0 + 20.875 = 20.875	6.86 + 20.75 = 27.61	13.51 + 20.5 34.01	19.46 + 19.75 = 39.21	23.66 + 18.25 = 41.91	25.76 + 9.25 = 35.01	26.81 + 5 = 31.81	27.51 + 0 = 27.51	—	41.91	4
8	0 + 20.875 = 20.875	6.86 + 20.875 = 27.735	13.51 + 20.75 = 34.20	19.46 + 20.5 39.96	23.66 + 19.75 = 43.41	25.76 + 18.25 = 44.01	26.81 + 9.25 = 36.06	27.51 + 5 = 32.51	27 + .86 27.86	44.01	5

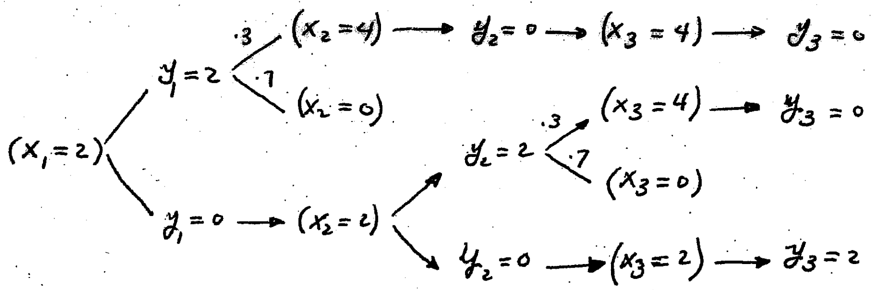
Stage 1: $f_1(x_1) = \max\limits_{y_1 \le x_1} \{ C_1 E\{ d_1 | y_1 \} + f_2(x_1 - y_1) \}$

x_1	$y_1 = 0$	1	2	3	4	5	6	7	8	Opt f_1	Opt y_1
8	0 + 44.01 = 44.01	5.4 + 41.9 = 47.3	9.6 + 37.71 = 47.31	12 + 31.76 = 43.76	13.2 + 25.11 = 38.31	13.8 + 19.46 = 33.26	13.8 + 13.5 = 27.3	13.8 + 6.86 = 20.66	13.8 + 0 13.8	47.31	2

Optimum solution:

$(x_1 = 8) \rightarrow y_1 = 2 \rightarrow (x_2 = 6) \rightarrow y_2 = 3 \rightarrow (x_3 = 3) \rightarrow y_3 = 3$

1



2

Stage 3:

$.6\,P\{x_3+y_3 \geq 3\} + .4\,P\{x_3-y_3 \geq 3\}$

x_3	$y_3=0$	1	2	3	4	Opt. Sol. f_3	y_3
0	$.6\times0+.4\times0=0$	—	—	—	—	0	0
1	$.6\times0+.4\times0=0$	$.6\times0+.4\times0=0$	—	—	—	0	0
2	$.6\times0+.4\times0=0$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	—	—	.6	1
3	$.6\times1+.4\times1=1$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	—	1	0
4	$.6\times1+.4\times1=1$	$.6\times1+.4\times1=1$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	1	0

Stage 2:

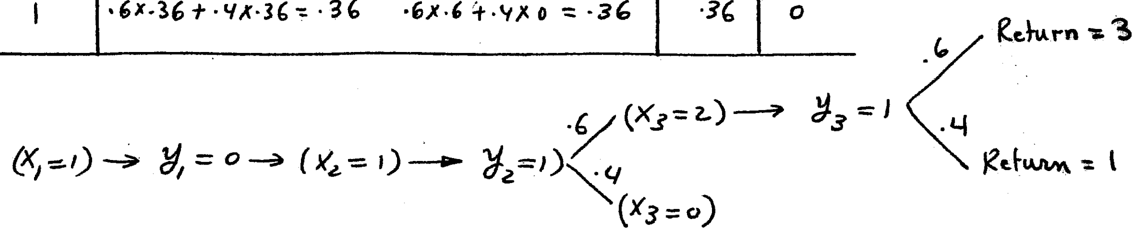
$.6\,f_3(x_2+y_2) + .4\,f_3(x_2-y_2)$

x_2	$y_2=0$	1	2	Opt. Sol. f_2	y_2
0	$.6\times0+.4\times0=0$	—		0	0
1	$.6\times0+.4\times0=0$	$.6\times.6+.4\times0=.36$	—	.36	1
2	$.6\times0+.4\times0=0$	$.6\times1+.4\times0=.6$	$.6\times1+.4\times0=.6$	.6	1

Stage 1:

$.6\,f_2(x_1+y_1) + .4\,f_1(x_1-y_1)$

x_1	$y_1=0$	$y_1=1$	Opt. Sol. f_1	y_1
1	$.6\times.36+.4\times.36=.36$	$.6\times.6+.4\times0=.36$	.36	0



Set 22.3a

3

<u>Stage 3</u>: $f_3(x_3) = \max\limits_{y_3 \le x_3} \{.25\,P\{x_3 + 2y_3 \ge 4\} + .75\,P\{x_3 - y_3 \ge 4\}\}$

x_3	$y_3=0$	1	2	3	4	5	6	7	8	9	Opt. Sol. f_3	y_3
0	0	—									0	0
1	0	0	—								0	0
2	0	.25	.25	—							.25	1,2
3	0	.25	.25	.25	—						.25	1,2,3
4	1	.25	.25	.25	.25	—					1	0
5	1	1	.25	.25	.25	.25	—				1	0,1
6	1	1	1	.25	.25	.25	.25	—			1	0,1,2
7	1	1	1	1	.25	.25	.25	.25	—		1	0,1,2,3
8	1	1	1	1	1	.25	.25	.25	.25	—	1	0→4
9	1	1	1	1	1	1	.25	.25	.25	.25	1	0→5

<u>Stage 2</u>: $f_2(x_2) = \max\limits_{y_2 \le x_2} \{.25 f_3(x_2 + 2y_2) + .75 f_3(x_2 - y_2)\}$

x_2	$y_2=0$	1	2	3	Opt. Sol. f_2	y_2
0	0	—			0	0
1	0	.25×.25 = .0625	—		.0625	1
2	.25	.25×1 + .75×0 = .25	.25×1 + .75×0 = .25	—	.25	0→2
3	.25	.25×1 + .75×.25 = .4375	.25×1 + .75×0 = .25	.25×1 + .75×0 = .25	.4375	1

<u>Stage 1</u>: $f_1(x_1) = \max\limits_{y_1 \le x_1} \{.25 f_2(x_1 + 2y_1) + .75 f_2(x_1 - y_1)\}$

x_1	$y_1 = 0$	1	Opt. Sol. f_1	y_1
1	.25×.0625 + .75×.0625 = .0625	.25×.4375 + .75×0 =	.109375	1

<u>Solution</u>: max probability = .109375

$(x_1 = 1) \to y_1 = 1 \xrightarrow{.25} (x_2 = 3) \to y_2 = 1$

- $\xrightarrow{.25} (x_3 = 5) \to y_3 = 1$: $\xrightarrow{.25}$ Return = 7; $\xrightarrow{.75}$ Return = 4
- $\xrightarrow{.25} (x_3 = 5) \to y_3 = 0 \to$ Return = 5
- $\xrightarrow{.75} (x_3 = 2) \to y_3 = 1 \xrightarrow{.25}$ Return = 4

Bet \$1 in game 1, \$1 in game 2, and \$1 or none in game 3

alternative optima