CHAPTER 4

Duality and Post-Optimal Analysis

Set 4.1a

1

Primal:

Minimize $z = 5x_1 + 12x_2 + 4x_3$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 + S_1 &= 10 \\ 2x_1 - x_2 + 3x_3 &= 8 \\ x_1, x_2, x_3, S_1 &\geq 0 \end{aligned}$$

Dual:

Maximize $w = 10y_1 + 8y_2$

subject to

$$\begin{aligned} y_1 + 2y_2 &\leq 5 \\ 2y_1 - y_2 &\leq 12 \\ y_1 + 3y_2 &\leq 4 \\ y_1 &\leq 0 \\ y_2 &\text{ unrestricted} \end{aligned}$$

2

Primal:

Minimize $z = 15x_1 + 12x_2$

subject to

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ 2x_1 - 4x_2 + x_4 &= 5 \\ 3x_1 + x_2 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Dual:

Maximize $z = 3y_1 + 5y_2 + 4y_3$

subject to

$$\begin{aligned} y_1 + 2y_2 + 3y_3 &\leq 15 \\ 2y_1 - 4y_2 + y_3 &\leq 12 \\ -y_1 &\leq 0 \Rightarrow y_1 \geq 0 \\ y_2 &\leq 0 \\ y_3 &\text{ unrestricted} \end{aligned}$$

3

Primal:

Minimize $z = 5x_1^+ - 5x_1^- + 6x_2$

subject to

$$\begin{aligned} x_1^+ - x_1^- + 2x_2 &= 5 \\ -x_1^+ + x_1^- + 5x_2 - x_3 &= 3 \\ 4x_1^+ - 4x_1^- + 7x_2 + x_4 &= 8 \\ x_1^+, x_1^-, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Dual:

Maximize $z = 5y_1 + 3y_2 + 8y_3$

subject to

$$\left.\begin{aligned} y_1 - y_2 + 4y_3 &\leq 5 \\ -y_1 + y_2 - 4y_3 &\leq -5 \end{aligned}\right\} \Rightarrow y_1 - y_2 + 4y_3 = 5$$

$$\begin{aligned} 2y_1 + 5y_2 + 7y_3 &\leq 6 \\ -y_2 &\leq 0 \Rightarrow y_2 \geq 0 \\ y_3 &\leq 0 \\ y_1 &\text{ unrestricted} \end{aligned}$$

4

(a) Primal:

Maximize $z = -5x_1 + 2x_2$

s.t.

$$\begin{aligned} x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Dual:

Minimize $w = 2y_1 + 5y_2$

Subject to

$$\begin{aligned} y_1 + 2y_2 &\geq -5 \\ -y_1 + 3y_2 &\geq 2 \\ -y_1 &\geq 0 \Rightarrow y_1 \leq 0 \\ y_2 &\geq 0 \end{aligned}$$

(b) Primal:

Minimize $z = 6x_1 + 3x_2$

Subject to

$$\begin{aligned} 6x_1 - 3x_2 + x_3 - x_4 &= 2 \\ 3x_1 + 4x_2 + x_3 - x_5 &= 5 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Dual:

Maximize $w = 2y_1 + 5y_2$

Subject to

$$\begin{aligned} 6y_1 + 3y_2 &\leq 6 \\ -3y_1 + 4y_2 &\leq 3 \\ y_1 + y_2 &\leq 0 \end{aligned}$$

$$\left.\begin{aligned} -y_1 &\leq 0 \\ -y_2 &\leq 0 \end{aligned}\right\} \Rightarrow y_1, y_2 \geq 0$$

(c) Primal:

Maximize $z = x_1 + x_2$

Subject to

$$\begin{aligned} 2x_1 + x_2 &= 5 \\ 3x_1 - x_2 &= 6 \\ x_1, x_2 &\text{ unrestricted} \end{aligned}$$

Dual:

Minimize $w = 5y_1 + 6y_2$

Subject to

$$\begin{aligned} 2y_1 + 3y_2 &= 1 \\ y_1 - y_2 &= 1 \\ y_1, y_2 &\text{ unrestricted} \end{aligned}$$

5

Primal:

Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR_2$

$$x_1 + 2x_2 + x_3 + S_1 = 10$$
$$2x_1 - x_2 + 3x_3 + R_2 = 8$$
$$x_1, x_2, x_3, S_1, R_2 \geq 0$$

Dual

Minimize $w = 10y_1 + 8y_2$

Subject to

$$y_1 + 2y_2 \geq 5$$
$$2y_1 - y_2 \geq 12$$
$$y_1 + 3y_2 \geq 4$$
$$y_1 \geq 0$$
$$\left.\begin{array}{l} y_2 \geq -M \\ y_2 \text{ unrestricted} \end{array}\right\} \text{same}$$

6

All parts, (a) through (e), are true

7

(1) max + ($\geq$ constraints):

$$\sum a_{ij} x_j \boxed{-S_i} = b_i \Rightarrow -y_i \geq 0 \Rightarrow y_i \leq 0$$

(2) min + ($\geq$ constraints):

$$\sum a_{ij} x_j \boxed{-S_i} = b_i \Rightarrow -y_i \leq 0 \Rightarrow y_i \geq 0$$

(3) max + ($\leq$ constraints):

$$\sum a_{ij} x_j \boxed{+S_i} = b_i \Rightarrow y_i \geq 0$$

(4) min + ($\leq$ constraints):

$$\sum a_{ij} x_j + S_i = b_i \Rightarrow y_i \leq 0$$

(5) max or min + (= constraint)

$$\sum a_{ij} x_j = b_i \Rightarrow y_i \text{ unrestricted}$$

(6) max + ($x_j \geq 0$):

$$\boxed{\begin{array}{c} c_j x_j \\ a_{ij} x_j \end{array}} \Rightarrow \sum_{i=1}^{m} a_{ij} y_i \geq c_j$$

(7) max + ($x_j \leq 0$):

Let $x_j = -x_j'$, $x_j' \geq 0$

$$\boxed{\begin{array}{c} -c_j x_j' \\ -a_{ij} x_j' \end{array}} \Rightarrow -\sum_{i=1}^{m} a_{ij} y_i \geq -c_j$$
$$\Rightarrow \sum_{i=1}^{m} a_{ij} y_i \leq c_j$$

(8) min + ($x_j \geq 0$):

$$\boxed{\begin{array}{c} c_j x_j \\ a_{ij} x_j \end{array}} \Rightarrow \sum_{i=1}^{m} a_{ij} y_i \leq c_j$$

(9) min + ($x_j \leq 0$):

Let $x_j = -x_j'$, $x_j' \geq 0$

$$\boxed{\begin{array}{c} -c_j x_j' \\ -a_{ij} x_j' \end{array}} \Rightarrow -\sum_{i=1}^{m} a_{ij} y_i \leq -c_j$$
$$\Rightarrow \sum_{i=1}^{m} a_{ij} y_i \geq c_j$$

(10) max or min + (x_j unrestricted)

$$\boxed{\begin{array}{c} c_j x_j \\ a_{ij} x_j \end{array}} \Rightarrow \sum_{i=1}^{m} a_{ij} y_i = c_j$$

Set 4.2a

1

(a) $\underset{3\times\underline{2}}{A}\ \underset{\underline{1}\times 2}{V1}$ undefined

(b) $\underset{3\times\underline{2}}{A}\underset{\underline{2}\times 1}{P1} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \\ 15 \end{pmatrix}_{3\times 1}$

(c) $\underset{3\times\underline{2}}{A}\underset{\underline{3}\times 1}{P2}$ undefined

(d) $\underset{1\times\underline{2}}{V1}\ \underset{\underline{3}\times 2}{A}$ undefined

(e) $\underset{1\times\underline{3}}{V2}\ \underset{\underline{3}\times 2}{A} = (-1,-2,-3)\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

$= (-14, -32)_{1\times 2}$

(f) $\underset{2\times\underline{1}}{P1}\ \underset{\underline{3}\times 1}{P2}$ undefined

(g) $\underset{1\times\underline{2}}{V1}\ \underset{2\times 1}{P1} = (11, 22)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$= 55_{1\times 1}$

1

(a)

$$\text{inverse} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3/2 \\ 5/2 \\ 1/2 \end{pmatrix}$$

2

(a)

$$\text{inverse} = \begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Set 4.2c

1

Dual: Maximize $w = 50y$

s.t.

$5y_1 \le 10,\ -7y_1 \le 4,\ 3y_1 \le 5,\ y \ge 0$

The constraints simplify to

$$0 \le y_1 \le 5/3$$

Thus, $\max w = 50 \times \frac{5}{3} = \frac{250}{3} = \min z$

2

Dual:

Maximize $w = 50y_1 + 20y_2 + 30y_3 + 35y_4 + 10y_5 + 90y_6 + 20y_7$

s.t.

$$5y_1 + y_2 + 7y_3 + 5y_4 + 2y_5 + 12y_6 \le 5$$
$$5y_1 + y_2 + 6y_3 + 5y_4 + 4y_5 + 10y_6 + y_7 \le 6$$
$$3y_1 - y_2 - 9y_3 + 5y_4 - 15y_5 - 10y_7 \le 3$$
$$-y_j \le 0 \Rightarrow y_j \ge 0,\ j = 1, 2, \dots, 7$$

From TORA, optimal objective equation is

$$z + 50y_1 + 0y_2 + 90y_3 + 65y_4 + 70y_5 + 110y_6 + 0y_7 + 0s_1 + 20s_2 + 0s_3 = 120$$

(s_1, s_2, s_3) are slack variables.

Thus, $x_1 = 0,\ x_2 = 20,\ x_3 = 0$

Obtaining the solution from the dual is advantgeous computationally because the dual has a smaller number of constraints.

3

Dual: Minimize $w = 30y_1 + 40y_2$

s.t.

$$y_1 + y_2 \ge 5$$
$$5y_1 - 5y_2 \ge 2$$
$$2y_1 - 6y_2 \ge 3$$
$$y_2 \ge 0,\ y_1 \text{ unrestricted}$$

Method 1: $z + 0x_1 + 23x_2 + 7x_3 + 105x_4 + 0x_5 = 150$

Coefficient of $x_4 = 105 \Rightarrow y_1 = 105 + (-100) = 5$

Coefficient of $x_5 = 0 \Rightarrow y_2 = 0$

Method 2:

$$(y_1, y_2) = (5, 0)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$
$$= (5, 0)$$

$w = 30 \times 5 + 40 \times 0 = 150$

4

Dual: Maximize $w = 3y_1 + 6y_2 + 4y_3$

s.t.

$$3y_1 + 4y_2 + y_3 \le 4$$
$$y_1 + 3y_2 + 2y_3 \le 1$$
$$-y_2 \le 0 \Rightarrow y_2 \ge 0$$
$$y_3 \le 0$$
$$y_1 \text{ unrestricted}$$

Method 1: $z - 98.6x_4 - 100x_5 - .2x_6 = 3.4$

Coefficient of $x_4 = -98.6 \Rightarrow y_1 = -98.6 + 100 = 1.4$

Coefficient of $x_5 = -100 \Rightarrow y_2 = -100 + 100 = 0$

Coefficient of $x_6 = -.2 \Rightarrow y_3 = -.2$

Method 2:

$$(y_1, y_2, y_3) = (4, 1, 0)\begin{pmatrix} .4 & 0 & -.2 \\ -.2 & 0 & .6 \\ 1 & -1 & 1 \end{pmatrix}$$
$$= (1.4, 0, -.2)$$

$w = 3 \times 1.4 + 6 \times 0 + 4 \times -.2 = 3.4$

5

Dual: Minimize $w = 4y_1 + 8y_2$

s.t.

$$y_1 + y_2 \ge 2$$
$$y_1 + 4y_2 \ge 4$$
$$y_1 \ge 4$$
$$y_2 \ge -3$$

Method 1: $z + 2x_1 + 0x_2 + 0x_3 + 3x_4 = 16$

Coefficient of $x_3 = 0 \Rightarrow y_1 = 0 + 4 = 4$

Coefficient of $x_4 = 3 \Rightarrow y_2 = 3 + (-3) = 0$

Method 2:

$$(y_1, y_2) = (4, 4)\begin{pmatrix} 1 & -.25 \\ 0 & .25 \end{pmatrix} = (4, 0)$$

$w = 4 \times 4 + 8 \times 0 = 16$

6

Dual: Minimize $w = 3y_1 + 4y_2$

s.t.

$$y_1 + 2y_2 \ge 1$$
$$2y_1 - y_2 \ge 5$$
$$y_1 \ge 3,\ y_2 \text{ unrestricted}$$

Method 1: $z + 2x_2 + 0x_3 + 99x_4 = 5$

Coefficient of $x_3 = 0 \Rightarrow y_1 = 0 + 3 = 3$

Coefficient of $x_4 = 99 \Rightarrow y_2 = 99 + (-100) = -1$

Method 2:

$$(y_1, y_2) = (3, 1)\begin{pmatrix} 1 & -.5 \\ 0 & .5 \end{pmatrix} = (3, -1)$$

$w = 3 \times 3 + 4(-1) = 5$

7

Maximize $z = x_1 + x_2$

s.t.

$$-3x_1 + 3x_2 \le 12$$
$$-3x_1 + 2x_2 \le -4$$
$$3x_1 - 5x_2 \le 2$$
$$x_1 \text{ unrestricted},\ x_2 \ge 0$$

TORA solution:

$x_1 = 3.4737$, $x_2 = 1.6842$, $Z = 5.1579$

Dual: Minimize $w = 12y_1 - 4y_2 + 2y_3$

s.t.

$$y_1 - 3y_2 + 3y_3 = 1$$
$$3y_1 + 2y_2 - 5y_3 \ge 1$$
$$y_1, y_2, y_3 \ge 0$$

From TORA, the optimal objective row is

$$w - 3.0526y_2 - 1.684y_4 - 96.5263y_5 - 98.3158y_6 = 5.1579$$

(y_5 and y_6 are artificial variables)

Coefficient of $y_5 = -96.5263 \Rightarrow x_1 = -96.5263 + 100 = 3.4737$

Coefficient of $y_6 = -98.3158 \Rightarrow x_2 = -98.3158 + 100 = 1.6842$

8

(a)

Primal

min $z = 5x_1 + 2x_2$

s.t.

$$x_1 - x_2 \ge 3$$
$$2x_1 + 3x_2 \ge 5$$
$$x_1, x_2 \ge 0$$

Dual

max $w = 3y_1 + 5y_2$

s.t.

$$y_1 + 2y_2 \le 5$$
$$-y_1 + 3y_2 \le 2$$
$$y_1 \ge 0$$
$$y_2 \ge 0$$

Feasible solutions:

$x_1 = 3$, $x_2 = 0$, $z = 15$ $\quad$ $y_1 = 3$, $y_2 = 1$, $w = 14$

Range: $14 \le$ Optimum value ≤ 15

(b)

max $z = x_1 + 5x_2 + 3x_3$

s.t.

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 - x_2 = 4$$
$$x_1, x_2, x_3 \ge 0$$

min $w = 3y_1 + 4y_2$

s.t.

$$y_1 + 2y_2 \ge 1$$
$$2y_1 - y_2 \ge 5$$
$$y_1 \ge 3$$
$$y_2 \text{ unrestricted}$$

Feasible solutions:

$x_1 = 2$, $x_2 = 0$, $x_3 = 1$, $z = 5$ $\quad$ $y_1 = 3$, $y_2 = 0$, $w = 9$

Range: $5 \le$ optimum value ≤ 9

continued...

(c)

max $z = 2x_1 + x_2$

s.t.

$$x_1 - x_2 \le 10$$
$$2x_1 \le 40$$
$$x_1, x_2 \ge 0$$

min $w = 10y_1 + 40y_2$

s.t.

$$y_1 + 2y_2 \ge 2$$
$$-y_1 \ge 1$$
$$y_1 \ge 0$$
$$y_2 \ge 0$$

Feasible Solution:

$x_1 = 20$, $x_2 = 20$, $z = 60$ $\quad$ No feasible solution.

Primal is unbounded because the primal is feasible and the dual has no feasible solution.

(d)

max $z = 3x_1 + 2x_2$

s.t.

$$2x_1 + x_2 \le 3$$
$$3x_1 + 4x_2 \le 12$$
$$x_1, x_2 \ge 0$$

min $w = 3y_1 + 12y_2$

s.t.

$$2y_1 + 3y_2 \ge 3$$
$$y_1 + 4y_2 \ge 2$$
$$y_1, y_2 \ge 0$$

Feasible solutions:

$x_1 = x_2 = 1$, $z = 5$ $\quad$ $y_1 = 2$, $y_2 = 0$, $w = 6$

Range: $5 \le$ optimum value ≤ 6

9

min $z = 5x_1 + 2x_2$

s.t.

$$x_1 - x_2 \ge 3$$
$$2x_1 + 3x_2 \ge 5$$
$$x_1, x_2 \ge 0$$

max $w = 3y_1 + 5y_2$

s.t.

$$y_1 + 2y_2 \le 5$$
$$-y_1 + 3y_2 \le 2$$
$$y_1, y_2 \ge 0$$

(a) ($x_1 = 3$, $x_2 = 1$; $y_1 = 4$, $y_2 = 1$):

Both primal and dual are infeasible

(b) ($x_1 = 4$, $x_2 = 1$; $y_1 = 1$, $y_2 = 0$):

Primal feasible, $z = 22$

Dual feasible, $w = 3$

Since $z \ne w$, solutions are not optimal.

(c) ($x_1 = 3$, $x_2 = 0$; $y_1 = 5$, $y_2 = 0$):

Primal feasible, $z = 15$

Dual feasible, $w = 15$

Since $z = w$, solutions are optimal

Set 4.2d

1

From TORA using $M = 100$:

	x_1	x_2	x_3	x_4	x_5	
z	-205	88	-304	0	0	-800
x_4	1	2	1	1	0	10
x_5	2	-1	3	0	1	8
z	$-7/3$	$-40/3$	0	0	$304/3$	$32/3$
x_4	$1/3$	$7/3$	0	1	$-1/3$	$22/3$
x_3	$2/3$	$-1/3$	1	0	$1/3$	$8/3$

Primal	Dual
Maximize $z = 5x_1 + 12x_2 + 4x_3$	Minimize $w = 10y_1 + 8y_2$
s.t. $x_1 + 2x_2 + x_3 \le 10$	s.t. $y_1 + 2y_2 \ge 5$
$2x_1 - x_2 + 3x_3 = 8$	$2y_1 - y_2 \ge 12$
$x_1, x_2, x_3 \ge 0$	$y_1 + 3y_2 \ge 4$
	$y_1 \ge 0$
	y_2 unrestricted

Iteration 1: x_5 artificial, $M = 100$

$$\text{Inverse} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}, \quad C_B = (0, 4)$$

Constraints:

$$LHS = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 2 & -1 & 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/3 & 7/3 & 0 & 1 & -1/3 \\ 2/3 & -1/3 & 1 & 0 & 1/3 \end{pmatrix}$$

$$RHS = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 10 \\ 8 \end{pmatrix} = \begin{pmatrix} 22/3 \\ 8/3 \end{pmatrix}$$

Objective row:

$$\text{Dual values } (y_1, y_2) = (0, 4)\begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} = (0, 4/3)$$

Variable	Objective coefficient
x_1	$y_1 + 2y_2 - 5 = 0 + 2(4/3) - 5 = -7/3$
x_2	$2y_1 - y_2 - 12 = 2(0) - (4/3) - 12 = -40/3$
x_3	$y_1 + 3y_2 - 4 = 0 + 3(4/3) - 4 = 0$
x_4	$y_1 - 0 = 0 - 0 = 0$
x_5	$y_2 - (-M) = 4/3 - (-100) = 304/3$

2

Dual:

Minimize $w = 21y_1 + 21y_2$

Subject to

$$2y_1 + 7y_2 \ge 4$$
$$7y_1 + 2y_2 \ge 14$$
$$y_1, y_2 \ge 0$$

(a) $\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/7 & 0 \\ -2/7 & 1 \end{pmatrix}\begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \end{pmatrix} \Rightarrow$ feasible

$$(y_1, y_2) = (14, 0)\begin{pmatrix} 1/7 & 0 \\ -2/7 & 1 \end{pmatrix} = (2, 0)$$

$$\left.\begin{aligned} &\text{obj coeff } x_1 = 2y_1 + 7y_2 - 4 \\ &\qquad = 2\times 2 + 7\times 0 - 4 = 0 \\ &\text{obj coeff of } x_3 = y_1 - 0 = 2 - 0 = 2 \end{aligned}\right\} \Rightarrow \text{optimal}$$

(b) Feasibility:

continued...

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 1 & -7/2 \end{pmatrix}\begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 10.5 \\ -\frac{105}{2} \end{pmatrix} \Rightarrow \text{infeasible}$$

Optimality:

$$(y_1, y_2) = (14, 0)\begin{pmatrix} 0 & 1/2 \\ 1 & -7/2 \end{pmatrix} = (0, 7)$$

obj coeff of x_1: $2y_1 + 7y_2 - 4 = 2\times 0 + 7\times 7 - 4 = 45 > 0$

obj coeff of x_4: $y_2 - 0 = 7 - 0 > 0$

Solution is optimal but infeasible

(c) Feasibility:

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 7/45 & -2/45 \\ -2/45 & 7/45 \end{pmatrix}\begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 7/3 \end{pmatrix} \Rightarrow \text{feasible}$$

Optimality:

$$(y_1, y_2) = (14, 4)\begin{pmatrix} 7/45 & -2/45 \\ -2/45 & 7/45 \end{pmatrix} = (2, 0)$$

$$\left.\begin{aligned} &\text{obj coeff of } x_3: y_1 - 0 = 2 - 0 > 0 \\ &\text{obj coeff of } x_4: y_2 - 0 = 0 - 0 = 0 \end{aligned}\right\} \text{optimal}$$

Solution is optimal and feasible

(d) Feasibility:

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -7/2 & 1 \end{pmatrix}\begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} \frac{21}{2} \\ -\frac{105}{2} \end{pmatrix} \Rightarrow \text{infeasible}$$

Optimality:

$$(y_1, y_2) = (4, 0)\begin{pmatrix} 1/2 & 0 \\ -7/2 & 1 \end{pmatrix} = (2, 0)$$

$$\left.\begin{aligned} &\text{Obj coeff of } x_2: 7y_1 + 2y_2 - 14 = 0 \\ &\text{Obj coeff of } x_3: y_1 - 0 = 2 - 0 = 2 \end{aligned}\right\} \text{optimal}$$

Solution optimal but infeasible

3

Dual:

minimize $w = 30y_1 + 60y_2 + 20y_3$

subject to

$$y_1 + 3y_2 + y_3 \ge 3$$
$$2y_1 + 4y_3 \ge 2$$
$$y_1 + 2y_2 \ge 5$$
$$y_1, y_2, y_3 \ge 0$$

(a) Feasibility:

$$\begin{pmatrix} x_4 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 30 \\ 60 \\ 20 \end{pmatrix} = \begin{pmatrix} 0 \\ 50 \\ 20 \end{pmatrix} \text{ feasible}$$

Optimality:

$$(y_1, y_2, y_3) = (0, 5, 0)\begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (0, 5/2, 0)$$

Obj coeff of x_1: $y_1 + 3y_2 + y_3 - 3 = 0 + 3(\frac{5}{2}) + 0 - 3 = 9/2$

Obj coeff of x_2: $2y_1 + 4y_3 - 2 = 2\times 0 + 4\times 0 - 2 = -2 < 0$

Solution feasible but not optimal

continued...

b) Feasibility:

$$\begin{pmatrix}x_2\\x_3\\x_1\end{pmatrix}=\begin{pmatrix}1/4&-1/8&1/8\\3/2&-1/4&-3/4\\-1&1/2&1/2\end{pmatrix}\begin{pmatrix}30\\60\\20\end{pmatrix}=\begin{pmatrix}5/2\\15\\10\end{pmatrix}\Rightarrow \text{feasible}$$

Optimality:

$$(y_1,y_2,y_3)=(2,5,3)\begin{pmatrix}1/4&-1/8&1/8\\3/2&-1/4&-3/4\\-1&1/2&1/2\end{pmatrix}=(5,0,-2)$$

obj coeff of x_4 : $y_1-0=5$

obj. coeff of x_5 : $y_2-0=0$

obj. coeff of x_6 : $y_3-0=-2 \Rightarrow$ not optimal

(c) Feasibility:

$$\begin{pmatrix}x_2\\x_3\\x_6\end{pmatrix}=\begin{pmatrix}1/2&-1/4&0\\0&1/2&0\\-2&1&1\end{pmatrix}\begin{pmatrix}30\\60\\20\end{pmatrix}=\begin{pmatrix}0\\30\\20\end{pmatrix}\Rightarrow \text{feasible}$$

Optimality:

$$(y_1,y_2,y_3)=(2,5,0)\begin{pmatrix}1/2&-1/4&0\\0&1/2&0\\-2&1&1\end{pmatrix}=(1,2,0)$$

$$\left.\begin{array}{l}\text{obj coeff of } x_1: y_1+3y_2+y_3-3=1+6+0-3=4\\ \text{obj coeff of } x_4: y_1-0=1-0=1\\ \text{obj coeff of } x_5: y_2-0=2-0=2\end{array}\right\}\text{optimal}$$

4

Constraints:

$$LHS=\begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}\begin{pmatrix}3&1&-1&0&0\\4&3&0&-1&0\\1&2&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0&-3/5&1/5&0\\0&1&4/5&-3/5&0\\0&0&-1&1&1\end{pmatrix}$$

$$RHS=\begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}\begin{pmatrix}3\\6\\3\end{pmatrix}=\begin{pmatrix}3/5\\6/5\\0\end{pmatrix}$$

Objective coefficients:

$$(y_1,y_2,y_3)=(2,1,0)\begin{pmatrix}3/5&-1/5&0\\-4/5&3/5&0\\1&-1&1\end{pmatrix}$$

$$=(2/5,1/5,0)$$

obj coeff of $x_3=-y_1-0=-2/5$

obj coeff of $x_4=-y_2-0=-1/5$

$z=2\times 3/5+1\times 6/5=12/5$

	x_1	x_2	x_3	x_4	x_5	
z	0	0	$-2/5$	$-1/5$	0	$12/5$
x_1	1	0	$-3/5$	$1/5$	0	$3/5$
x_2	0	1	$4/5$	$-3/5$	0	$6/5$
x_5	0	0	-1	1	1	0

continued...

5

(a)

(i) $\begin{pmatrix}x_4\\x_3\end{pmatrix}=\begin{pmatrix}1&-1/3\\0&1/3\end{pmatrix}\begin{pmatrix}10\\2\end{pmatrix}=\begin{pmatrix}28/3\\2/3\end{pmatrix}$

$z=4\times 2/3=8/3$

(ii) $\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}2/5&-1/5\\1/5&2/5\end{pmatrix}\begin{pmatrix}10\\2\end{pmatrix}=\begin{pmatrix}18/5\\14/5\end{pmatrix}$

$z=5\times\frac{14}{5}+12\times\frac{18}{5}=57.2$

(iii) $\begin{pmatrix}x_2\\x_3\end{pmatrix}=\begin{pmatrix}3/7&-1/7\\1/7&2/7\end{pmatrix}\begin{pmatrix}10\\2\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}$

$z=12\times 4+4\times 2=56$

Solution in (b) is the best

(b)

$y_1,y_2=(12,5)\begin{pmatrix}2/5&-1/5\\1/5&2/5\end{pmatrix}=\left(\frac{29}{5},-\frac{2}{5}\right)$

obj coeff of x_3: $y_1+3y_2-4=\frac{29}{5}+3\left(-\frac{2}{5}\right)-4=\frac{3}{5}$

obj coeff of x_4: $y_1-0=\frac{29}{5}-0=\frac{29}{5}$

Solution is optimal.

6

Inverse $=\begin{pmatrix}1&0\\-1&1\end{pmatrix}$

(a) $\begin{pmatrix}x_1\\x_5\end{pmatrix}=\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}=\begin{pmatrix}30\\10\end{pmatrix}$

Thus, $b_1=30$, $b_2=40$

(b) Optimal dual solution:

$(y_1,y_2)=(5,0)\begin{pmatrix}1&0\\-1&1\end{pmatrix}=(5,0)$

(c) $(d,e)=(y_1,y_2)=(5,0)$

$a=5y_1-5y_2-2=5\times5-5\times0-2=23$

$\begin{pmatrix}b\\c\end{pmatrix}=\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}5\\-5\end{pmatrix}=\begin{pmatrix}5\\-10\end{pmatrix}$

7

objective value:

in dual $=b_1y_1+b_2y_2+b_3y_3$

in primal $=c_1x_1+c_2x_2$

$$\begin{pmatrix}x_3\\x_2\\x_1\end{pmatrix}=\begin{pmatrix}1&1&-1\\0&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}2\\6\\2\end{pmatrix}$$

Thus, $b_1=4$, $b_2=6$, $b_3=8$

continued...

$$(y_1, y_2, y_3) = (0, c_2, c_1)\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$
$$= (0, c_2 - c_1, c_1)$$

$$\left.\begin{array}{l} \text{Obj coeff of } x_3 = 0 = y_1 - 0 \\ \text{obj coeff of } x_4 = 3 = y_2 - 0 \\ \text{Obj coeff of } x_5 = 2 = y_3 - 0 \end{array}\right\} y_1 = 0,\ y_2 = 3,\ y_3 = 2$$

Thus, $c_2 - c_1 = 3$ and $c_1 = 2 \Rightarrow c_1 = 2, c_2 = 5$

Now we can determine the objective value as follows:

$$\text{Dual} = b_1 y_1 + b_2 y_2 + b_3 y_3$$
$$= 4\times 0 + 6\times 3 + 8\times 2 = 34$$
$$\text{Primal} = c_1 x_1 + c_2 x_2$$
$$= 2\times 2 + 5\times 6 = 34$$

8

Dual:

Minimize $w = 4y_1 + 8y_2$

subject to

$$y_1 + y_2 \geq 2$$
$$y_1 + 4y_2 \geq 4$$
$$y_1 \geq 4$$
$$y_2 \geq -3$$

For basic (x_1, x_2), we have

$$\left.\begin{array}{l} y_1 + y_2 - 2 = 0 \\ y_1 + 4y_2 - 4 = 0 \end{array}\right\} \Rightarrow y_1 = \frac{4}{3},\ y_2 = \frac{2}{3}$$

$$\text{Obj coeff of } x_3 = y_1 - 4 = \frac{4}{3} - 4 = -\frac{8}{3} < 0$$

The result shows that the solution is not optimal.

9

For a slack starting basic variable, the dual constraint is of the form

$$y \geq 0$$

(assuming primal maximization).

Thus,

Optimal obj coeff. of basic variable $= y - 0$

For artificial starting basic variable, the dual constraint is $y \geq -M$ if the primal is maximization, and $y \leq M$ if the primal is minimization.

Thus,

$$\text{Optimal obj coeff} = \begin{cases} y + M, & \text{for maximization} \\ y - M, & \text{for minimization} \end{cases}$$

1

From TORA output:

	y_1	y_2	y_3	y_4
	.75	.5	0	0
Range:	(20,36)	(4,6.7)	(-1.5,∞)	(1.5,∞)

(a) $\$750 \times (22-24) = -\$1500$

(b) $\Delta Z = \$500(4.5-6) = -\$750$

(c) $\Delta Z = \$0\,(10-2) = \$0$

2

x_1, x_2, x_3, x_4 = daily units of cables 320, 325, 340, and 370

(a) Maximize $z = 9.4x_1 + 10.8x_2 + 8.75x_3 + 7.8x_4$

subject to

$$10.5x_1 + 9.3x_2 + 11.6x_3 + 8.2x_4 \le 4800$$
$$20.4x_1 + 24.6x_2 + 17.7x_3 + 26.5x_4 \le 9600$$
$$3.2x_1 + 2.5x_2 + 3.6x_3 + 5.5x_4 \le 4700$$
$$5x_1 + 5x_2 + 5x_3 + 5x_4 \le 4500$$
$$x_1 \ge 100,\ x_2 \ge 100,\ x_3 \ge 100,\ x_4 \ge 100$$

*** OPTIMUM SOLUTION SUMMARY ***

Title:
Final iteration No: 3
Objective value (max) = 4011.1582

Variable	Value	Obj Coeff	Obj Val Contrib
x1	100.0000	9.4000	939.9999
x2	100.0000	10.8000	1080.0000
x3	138.4181	8.7500	1211.1582
x4	100.0000	7.8000	780.0000

Constraint	RHS	Slack(-)/Surplus(+)
1 (<)	4800.0000	394.3503-
2 (<)	9600.0000	0.0000-
3 (<)	4700.0000	3081.6948-
4 (<)	4500.0000	2307.9097-
LB-x1	100.0000	0.0000+
LB-x2	100.0000	0.0000+
LB-x3	100.0000	38.4181+
LB-x4	100.0000	0.0000+

*** SENSITIVITY ANALYSIS ***

Objective coefficients -- Single Changes:

Variable	Current Coeff	Min Coeff	Max Coeff	Reduced Cost
x1	9.4000	-infinity	10.0847	0.6847
x2	10.8000	-infinity	12.1610	1.3610
x3	8.7500	8.1559	infinity	0.0000
x4	7.8000	-infinity	13.1003	5.3003

Right-hand Side -- Single Changes:

Constraint	Current RHS	Min RHS	Max RHS	Dual Price
1 (<)	4800.0000	4405.6497	infinity	0.0000
2 (<)	9600.0000	8919.9999	10201.7242	0.4944
3 (<)	4700.0000	1618.3052	infinity	0.0000
4 (<)	4500.0000	2192.0903	infinity	0.0000
LB-x1	100.0000	0.0000	133.3333	-0.6847
LB-x2	100.0000	42.1946	127.6423	-1.3610
LB-x3	100.0000	-infinity	138.4181	0.0000
LB-x4	100.0000	56.9826	125.6604	-5.3003

continued...

(b) Only soldering capacity can be increased because its dual price is positive.

(c) The fact that the dual prices of the lower bounds on x_1, x_2, and x_4 are negative shows that the lower bounds have adverse effect on profitability. Specifically, one unit decrease in the production of cables SC320, SC325, and SC370 will respectively increase the profit by \$.68, \$1.36, and \$5.30 per cable. These values are valid considering the cables one at a time.

(d) Dual price for soldering is \$.49 per minute, valid in the range (8920, 10201.7) minutes. Hence, the \$.49 additional profit per minute is guaranteed only for up to $\frac{10201-9600}{9600} = 6.26\%$ capacity increase.

3

x_1 = number of jackets per week

x_2 = number of handbags per week

Maximize $z = 350x_1 + 120x_2$

Subject to

$$8x_1 + 2x_2 \le 1200$$
$$12x_1 + 5x_2 \le 1850$$
$$x_1, x_2 \ge 0$$

TORA optimum solution:

$x_1 \cong 144$, $x_2 = 25$, $z = \$53312.50$

Resource	Dual price	Range
Leather	\$19.38/m²	(740, 1233.33)
Labor	\$16.25/hr	(1800, 3000)

BagCo should not pay more than \$19.38/m² of leather and \$16.25/hr of labor time.

1

Dual prices: $y_1 = 1$, $y_2 = 2$, $y_3 = 0$ all in \$/min

$$(1-r_1)\,y_1 + 1.25y_2 + y_3 \geq 3$$

Reduced cost of $x_2 = (1-r_1)\times 1 + 1.25\times 2 + 1\times 0 - 3$
$= .5 - r_1$

For x_1 to be just profitable, its reduced cost must be (at least) zero; that is, $.5 - r_1 \leq 0$ or $r_1 \geq .5$.
This means a reduction of at least 50%

2

Dual constraint for fire trucks:

$$y_2 + 3y_3 \geq 4$$

Reduced cost $= y_2 + 3y_3 - 4$
$= 1\times 2 + 3\times 0 - 4 = -2 < 0$

New toy is recommended.

3

x_j = number of units of PP_j, $j = 1,2,3,4$

Maximize $Z = 3x_1 + 6x_2 + 5x_3 + 4x_4$

Subject to

$$2x_1 + 5x_2 + 3x_3 + 4x_4 \leq 5300$$
$$3x_1 + 4x_2 + 6x_3 + 4x_4 \leq 5300$$
$$x_1, x_2, x_3, x_4 \geq 0$$

```
                   *** OPTIMUM SOLUTION SUMMARY ***
---------------------------------------------------------------------
Title: Problem 4.4b-3
Final iteration No: 4
Objective value (max) = 6814.2856
---------------------------------------------------------------
Variable          Value       Obj Coeff    Obj Val Contrib
---------------------------------------------------------------
x1             757.1429       3.0000         2271.4287
x2             757.1428       6.0000         4542.8569
x3               0.0000       5.0000            0.0000
x4               0.0000       4.0000            0.0000
-------------------------------------------------------------
Constraint          RHS            Slack(-)/Surplus(+)
-------------------------------------------------------------
1 (<)          5300.0000             0.0000-
2 (<)          5300.0000             0.0000-

                   *** SENSITIVITY ANALYSIS ***
Objective coefficients -- Single Changes:
---------------------------------------------------------------------
Variable    Current Coeff    Min Coeff     Max Coeff    Reduced Cost
---------------------------------------------------------------------
x1              3.0000         2.9444        4.5000         0.0000
x2              6.0000         4.0000        6.3333         0.0000
x3              5.0000        -infinity      5.1429         0.1429
x4              4.0000        -infinity      5.1429         1.1429

Right-hand Side -- Single Changes:
---------------------------------------------------------------------
Constraint    Current RHS      Min RHS       Max RHS        Dual Price
---------------------------------------------------------------------
1 (<)          5300.0000      3533.3334     6625.0000        0.8571
2 (<)          5300.0000      4240.0000     7949.9998        0.4286
```

continued...

From TORA solution:

Variable	Reduced cost
x_3	.1429
x_4	1.1429

Thus,

$\begin{pmatrix}\text{Rate of deterioration in } z\\ \text{per unit of } x_3\end{pmatrix} = \$.14$

$\begin{pmatrix}\text{Rate of deterioration in}\\ z \text{ per unit of } x_4\end{pmatrix} = \$1.14$

4

Resource	Dual price	Range
Lathe	\$.8571	(5333.33, 6625)
Drill	\$.4286	(4240, 7950)

Reduced cost for x_3
$= .8(3y_1 + 6y_2) - 5$
$= .8(3\times .8571 + 6\times .4286) - 5$
$= -.8857 < 0$

Reduced cost for x_4
$= .8(4y_1 + 4y_2) - 4$
$= .8(4\times .8571 + 4\times .4286) - 4$
$= .1142 > 0$

Only PP_3 will be profitable.
PP_4 needs more than

$$1 - \frac{4}{4\times .8571 + 4\times .4286} = 22.2\%$$

improvement to be profitable

1

(a) No, because A is feasible.
(b) No, because E is feasible. Dual simplex iterations remain infeasible until the last iteration is reached.
(c) $L \rightarrow I \rightarrow F$.

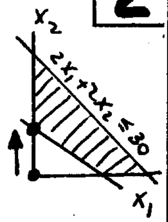
2

(a)

Minimize $z = 2x_1 + 3x_2$

subject to

$$2x_1 + 2x_2 \le 30$$
$$-x_1 - 2x_2 \le -10$$
$$x_1, x_2 \ge 0$$



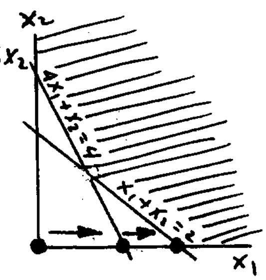
Basic	x_1	x_2	x_3	x_4	Sol^n
z	-2	-3	0	0	0
x_3	2	2	1	0	30
x_4	-1	$\boxed{-2}$	0	1	-10
z	$-1/2$	0	0	$-3/2$	15
x_3	1	0	1	1	20
x_2	$1/2$	1	0	$-1/2$	5

(b)

Minimize $z = 5x_1 + 6x_2$

subject to

$$-x_1 - x_2 \le -2$$
$$-4x_1 - x_2 \le -4$$
$$x_1, x_2 \ge 0$$



Basic	x_1	x_2	x_3	x_4	Sol^n
z	-5	-6	0	0	0
x_3	-1	-1	1	0	-2
x_4	$\boxed{-4}$	-1	0	1	-4
z	0	$-19/4$	0	$-5/4$	5
x_3	0	$-3/4$	1	$-1/4$	-1
x_1	1	$1/4$	0	$-1/4$	1
z	0	-1	-5	0	10
x_4	0	3	-4	1	4
x_1	1	1	-1	0	2

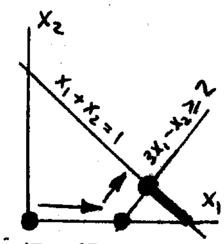
continued...

(c)

Minimize $z = 4x_1 + 2x_2$

subject to

$$x_1 + x_2 \le 1$$
$$x_1 + x_2 \ge 1$$
$$3x_1 - x_2 \ge 2$$
$$x_1, x_2 \ge 0$$



(Convert the equation into two inequalities to fit the dual simplex format.)

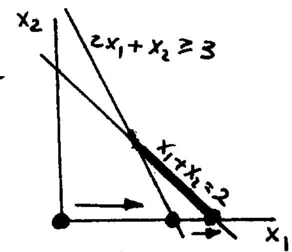
Basic	x_1	x_2	x_3	x_4	x_5	Sol^n
z	-4	-2	0	0	0	0
x_3	1	1	1	0	0	1
x_4	-1	-1	0	1	0	-1
x_5	$\boxed{-3}$	1	0	0	1	-2
z	0	$-4/3$	0	0	$-4/3$	$8/3$
x_3	0	$4/3$	1	0	$1/3$	$1/3$
x_4	0	$\boxed{-4/3}$	0	1	$-1/3$	$-1/3$
x_1	1	$-1/3$	0	0	$-1/3$	$2/3$
z	0	0	0	$-5/2$	$-1/2$	$7/2$
x_3	0	0	1	1	0	0
x_2	0	1	0	$-3/4$	$1/4$	$1/4$
x_1	1	0	0	$-1/4$	$-1/4$	$3/4$

(d)

Minimize $z = 2x_1 + 3x_2$

subject to

$$2x_1 + x_2 \ge 3$$
$$x_1 + x_2 \le 2$$
$$x_1 + x_2 \ge 2$$
$$x_1, x_2 \ge 0$$



Basic	x_1	x_2	x_3	x_4	x_5	Sol^n
z	-2	-3	0	0	0	0
x_3	$\boxed{-2}$	-1	1	0	0	-3
x_4	1	1	0	1	0	2
x_5	-1	-1	0	0	1	-2
z	0	-2	-1	0	0	3
x_1	1	$1/2$	$-1/2$	0	0	$3/2$
x_4	0	$1/2$	$1/2$	1	0	$1/2$
x_5	0	$-1/2$	$\boxed{-1/2}$	0	1	$-1/2$
z	0	-1	0	0	-2	4
x_1	1	1	0	0	-1	2
x_4	0	0	0	1	1	0
x_3	0	1	1	0	-2	1

3

Add the constraints $x_1 + x_3 \le M$

Basic	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
z	-2	1	-1	0	0	0	0	0
s_1	-2	-3	5	1	0	0	0	-4
s_2	1	-9	1	0	1	0	0	-3
s_3	4	6	3	0	0	1	0	8
s_4	$\boxed{1}$	0	1	0	0	0	1	M
z	0	0	1	0	0	0	2	2M
s_1	0	-3	7	1	0	0	2	-4+2M
s_2	0	-9	0	0	1	0	-1	-3-M
s_3	0	6	-1	0	0	1	$\boxed{-4}$	8-4M
x_1	1	0	1	0	0	0	1	M

The second tableau is now optimal but infeasible. We can thus apply the dual simplex to the second tableau. Optimal solution is:

$x_1 = 1.286,\ x_2 = .476,\ x_3 = 0$

$z = 2.095$

4

(a) Add the constraint $x_3 \le M$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
z	0	0	-2	0	0	0	0	0
x_4	1	-2	2	1	0	0	0	-8
x_5	-1	1	1	0	1	0	0	4
x_6	2	-1	4	0	0	1	0	10
x_7	0	0	$\boxed{1}$	0	0	0	1	M
z	0	0	0	0	0	0	2	2M
x_4	1	-2	0	1	0	0	-2	-8-2M
x_5	-1	1	0	0	1	0	-1	4-M
x_6	2	$\boxed{-1}$	0	0	0	1	-4	10-4M
x_7	0	0	1	0	0	0	1	M

Last tableau is optimal but infeasible. Application of the dual simplex method yields the solution:

$x_1 = 56/9,\ x_2 = 26/3,\ x_3 = 14/9$

$z = 28/9$

continued...

(b) Add the constraint $x_1 \le M$

	x_1	x_2	s_1	s_2	s_3	s_4	
z	-1	3	0	0	0	0	0
s_1	1	-1	1	0	0	0	2
s_2	-1	-1	0	1	0	0	-4
s_3	-2	2	0	0	1	0	-3
s_4	$\boxed{1}$	0	0	0	0	1	M
z	0	3	0	0	0	1	M
s_1	0	-1	1	0	0	$\boxed{-1}$	2-M
s_2	0	-1	0	1	0	1	-4+M
s_3	0	2	0	0	1	2	-3+2M
x_1	1	0	0	0	0	1	M

Optimum: $x_1 = 3,\ x_2 = 1\ \ z = 0$

(c) Add the constraint $x_1 \le M$

	x_1	x_2	s_1	s_2	s_3	s_4	
z	1	-1	0	0	0	0	0
s_1	-1	4	1	0	0	0	-5
s_2	1	-3	0	1	0	0	1
s_3	-2	5	0	0	1	0	-1
s_4	$\boxed{1}$	0	0	0	0	1	M
z	0	-1	0	0	0	-1	-M
s_1	0	4	1	0	0	1	-5+M
s_2	0	$\boxed{-3}$	0	1	0	-1	1-M
s_3	0	5	0	0	1	2	-1+2M
x_1	1	0	0	0	0	1	M

Problem has no feasible solution

(d) Add the constraint $x_3 \le M$

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
z	0	0	-2	0	0	0	0	0
s_1	1	-3	7	1	0	0	0	-5
s_2	-1	1	-1	0	1	0	0	1
s_3	3	1	-10	0	0	1	0	8
s_4	0	0	1	0	0	0	1	M
z	0	0	0	0	0	0	2	2M
s_1	1	$\boxed{-3}$	0	1	0	0	-7	-5-7M
s_2	-1	1	0	0	1	0	1	1+M
s_3	3	1	0	0	0	1	10	8+10M
s_4	0	0	1	0	0	0	1	M

Solution is unbounded

5

Method 1: M-technique (or two-phase method)

Starting tableau:

Basic	x_1	x_2	x_3	x_4	S_1	S_2	S_3	R_1	R_2	R_3	Sol'n
z	-6	-7	-3	-5	0	0	0	-M	-M	-M	—
R_1	5	6	-3	4	-1	0	0	1	0	0	12
R_2	0	1	-5	-6	0	-1	0	0	1	0	10
R_3	2	5	1	1	0	0	-1	0	0	1	8

Method 2: Solve the dual problem

Starting tableau:

Basic	y_1	y_2	y_3	S_1	S_2	S_3	S_4	
w	-12	-10	-8	0	0	0	0	0
S_1	5	0	2	1	0	0	0	6
S_2	6	1	5	0	1	0	0	7
S_3	-3	-5	1	0	0	1	0	3
S_4	4	-6	1	0	0	0	1	5

Method 3: Dual simplex

starting tableau:

Basic	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Sol'n
z	-6	-7	-3	-5	0	0	0	0
S_1	-5	-6	3	-4	1	0	0	-12
S_2	0	-1	5	6	0	1	0	-10
S_3	-2	-5	-1	-1	0	0	1	-8

Optimal solution: $x_1 = 0, x_2 = 10, x_3 = x_4 = 0$

$z = 70$

Method	Number of iterations
1	5
2	3
3	

The dual simplex is the best. It follows because it requires the smallest number of iterations and has the smallest number of constraints.

Set 4.4b

1

Basic	x_1	x_2	x_3	x_4	x_5	
z	1	-1	0	0	0	0
x_3	$\boxed{-1}$	4	1	0	0	-5
x_4	1	-3	0	1	0	1
x_5	-2	5	0	0	1	-1
z						
x_1	1	-4	-1	0	0	5
x_4	0	1	1	1	0	-4
x_5	0	-3	-2	0	1	9

In the second iteration, row 2 has all nonnegative coefficients on the left-hand side. This means that the infeasibility of x_4 cannot be removed, and the problem has no feasible solution.

2

	x_1	x_2	x_3	x_4	x_5	x_6	
z	0	0	-2	0	0	0	0
x_4	1	$\boxed{-3}$	7	1	0	0	-5
x_5	-1	1	-1	0	1	0	1
x_6	3	1	-10	0	0	1	8
z	0	0	-2	0	0	0	0
x_2	-1/3	1	-7/3	-1/3	0	0	5/3
x_5	$\boxed{-2/3}$	0	4/3	1/3	1	0	-2/3
x_6	10/3	0	-23/3	1/3	0	1	19/3
z			-2				0
x_1			-4/3				2
x_1			-2				1
x_6			-1				3

Iteration 3 is feasible but nonoptimal. However, x_3 shows that the solution is unbounded.

1

$$\text{new RHS} = \begin{pmatrix}430\\480\\400\end{pmatrix}$$

Thus,

$$\begin{pmatrix}x_2\\x_3\\x_6\end{pmatrix} = \begin{pmatrix}1/2 & -1/4 & 0\\0 & 1/2 & 0\\-2 & 1 & 1\end{pmatrix}\begin{pmatrix}430\\480\\400\end{pmatrix} = \begin{pmatrix}95\\240\\20\end{pmatrix}$$

The new solution is feasible with $x_1 = 0$, $x_2 = 95$, $x_3 = 240$. $Z = 3\times 0 + 2\times 95 + 5\times 240 = \$1390$, which is better than the current value of Z

2

(a) $$\begin{pmatrix}x_2\\x_3\\x_6\end{pmatrix} = \begin{pmatrix}1/2 & -1/4 & 0\\0 & 1/2 & 0\\-2 & 1 & 1\end{pmatrix}\begin{pmatrix}460\\500\\400\end{pmatrix} = \begin{pmatrix}105\\250\\-20\end{pmatrix}$$

Solution is infeasible

	x_1	x_2	x_3	x_4	x_5	x_6	
z	4	0	0	1	2	0	1460
x_2	-1/4	1	0	1/2	-1/4	0	105
x_3	3/2	0	1	0	1/2	0	250
x_6	2	0	0	$\boxed{-2}$	1	1	-20
z	5	0	0	0	5/2	1/2	1450
x_2	1/4	1	0	0	0	1/4	100
x_3	3/2	0	1	0	1/2	0	250
x_4	-1	0	0	1	-1/2	-1/2	10

(b) $$\begin{pmatrix}x_2\\x_3\\x_6\end{pmatrix} = \begin{pmatrix}1/2 & -1/4 & 0\\0 & 1/2 & 0\\-2 & 1 & 1\end{pmatrix}\begin{pmatrix}500\\400\\600\end{pmatrix} = \begin{pmatrix}150\\200\\0\end{pmatrix}$$

New solution is feasible. Z=\$1300

(c) $$\begin{pmatrix}x_2\\x_3\\x_6\end{pmatrix} = \begin{pmatrix}1/2 & -1/4 & 0\\0 & 1/2 & 0\\-2 & 1 & 1\end{pmatrix}\begin{pmatrix}300\\800\\200\end{pmatrix} = \begin{pmatrix}-50\\400\\400\end{pmatrix}$$

	x_1	x_2	x_3	x_4	x_5	x_6	
z	4	0	0	1	2	0	1900
x_2	-1/4	1	0	1/2	-1/4	0	-50
x_3	3/2	0	1	0	1/2	0	400
x_6	2	0	0	-2	1	1	400
z	2	8	0	5	0	0	1500
x_5	1	-4	0	-2	1	0	200
x_3	1	2	1	1	0	0	300
x_6	1	4	0	0	0	1	200

continued...

(d) $$\begin{pmatrix}x_2\\x_3\\x_6\end{pmatrix} = \begin{pmatrix}1/2 & -1/4 & 0\\0 & 1/2 & 0\\-2 & 1 & 1\end{pmatrix}\begin{pmatrix}450\\700\\350\end{pmatrix} = \begin{pmatrix}50\\350\\150\end{pmatrix}$$

Solution is feasible. Z = \$1850

3

$$\begin{pmatrix}x_2\\x_3\\x_6\end{pmatrix} = \begin{pmatrix}1/4 & -1/2 & 0 & 0\\-1/8 & 3/4 & 0 & 0\\3/8 & -5/4 & 1 & 0\\1/8 & -3/4 & 0 & 1\end{pmatrix}\begin{pmatrix}28\\8\\1\\2\end{pmatrix} = \begin{pmatrix}3\\5/2\\3/2\\-1\end{pmatrix}$$

	x_1	x_2	s_1	s_2	s_3	s_4	
z	0	0	3/4	1/2	0	0	25
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	5/2
s_3	0	0	3/8	-5/4	1	0	3/2
s_4	0	0	1/8	$\boxed{-3/4}$	0	1	-1
z	0	0	5/6	0	0	2/3	24 2/3
x_1	1	0	1/6	0	0	-2/3	10/3
x_2	0	1	0	0	0	1	2
s_3	0	0	1/6	0	1	-5/3	7/3
s_2	0	0	-1/6	1	0	-4/3	2/3

4

x_1 = lb limestone in weekly mix
x_2 = lb corn in weekly mix
x_3 = lb soybean meal in weekly mix

Minimize $z = .12x_1 + .45x_2 + 1.6x_3$

s.t.

$$x_1 + x_2 + x_3 \geq Q$$
$$.38x_1 + .001x_2 + .002x_3 \geq .008(x_1+x_2+x_3)$$
$$.38x_1 + .001x_2 + .002x_3 \leq .012(x_1+x_2+x_3)$$
$$.09x_2 + .5x_3 \geq .22(x_1+x_2+x_3)$$
$$.02x_2 + .08x_3 \leq .05(x_1+x_2+x_3)$$
$$x_1, x_2, x_3 \geq 0$$

Q = weekly mix

The constraints simplify to

$$x_1 + x_2 + x_3 \geq Q$$
$$.372x_1 - .007x_2 - .006x_3 \geq 0$$
$$.368x_1 - .011x_2 - .01x_3 \leq 0$$
$$-.22x_1 - .13x_2 + .28x_3 \geq 0$$
$$-.05x_1 - .03x_2 + .03x_3 \leq 0$$

Week	1	2	3	4	5	6	7	8
Q (lb)	5200	9600	15000	20000	26000	32000	38000	42000

continued...

First, we solve the problem using $Q = 5200$ lb, feed requirements for week 1. Then we use sensitivity analysis for the remaining weeks.

Week 1 Solution (using TORA)

$$\begin{pmatrix}\text{Basic}\\ \text{vector}\end{pmatrix} = \begin{pmatrix}x_2\\ x_1\\ sx_5\\ x_3\\ sx_{11}\end{pmatrix}, \quad Z = \$4224.74$$

$$\text{inverse} = \begin{pmatrix} .649 & 0 & -3.216 & -2.431 & 0\\ .028 & 0 & 2.637 & -.006 & 0\\ .004 & -1 & 1.000 & .000 & 0\\ .323 & 0 & .579 & 2.438 & 0\\ .011 & 0 & .018 & .146 & 1\end{pmatrix}$$

Solution given Q:

$$\begin{pmatrix}x_2\\ x_1\\ sx_5\\ x_3\\ sx_{11}\end{pmatrix} = (\text{inverse})\begin{pmatrix}Q\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}.649Q\\ .028Q\\ .004Q\\ .323Q\\ .011Q\end{pmatrix}$$

General solution:

$$x_1 = .028Q$$
$$x_2 = .649Q$$
$$x_3 = .323Q$$

$$Z = (.12 \times .028 + .45 \times .649 + 1.6 \times .323)Q = .81221Q$$

5

B^{-1} = inverse

D_i = change in RHS of constraint i, $i = 1, 2, \dots, m$

Simultaneous feasibility conditions:

$$B^{-1}\begin{pmatrix}b_1 + D_1\\ \vdots\\ b_m + D_m\end{pmatrix} \geq \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} \qquad (1)$$

Let $p_i \leq D_i \leq q_i$ be the feasibility range computed from the single-change conditions:

$$B^{-1}\begin{pmatrix}b_1\\ \vdots\\ b_i + D_i\\ \vdots\\ b_m\end{pmatrix} \geq \begin{pmatrix}0\\ 0\\ \vdots\\ 0\end{pmatrix} \qquad (2)$$

Define
$$\Delta_i = \begin{cases} p_i, & \text{if } D_i < 0\\ q_i, & \text{if } D_i > 0\end{cases}$$

Condition (2) holds true for $D_i = \Delta_i$ also.

Now, define $r_i \geq 0$, $i = 0, 1, 2, \dots, m$ such that $r_0 + r_1 + \dots + r_m = 1$. Then

$$B^{-1}\left[r_0\begin{pmatrix}b_1\\ \vdots\\ b_i\\ \vdots\\ b_m\end{pmatrix} + r_1\begin{pmatrix}b_1+\Delta_1\\ \vdots\\ b_i\\ \vdots\\ b_m\end{pmatrix} + \dots + r_m\begin{pmatrix}b_1\\ \vdots\\ b_i\\ \vdots\\ b_m+\Delta_m\end{pmatrix}\right]$$

must also be feasible. The last expression reduces to

$$B^{-1}\left[\begin{pmatrix}b_1\\ \vdots\\ b_m\end{pmatrix} + \begin{pmatrix}r_1\Delta_1\\ \vdots\\ r_m\Delta_m\end{pmatrix}\right] \geq \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} \qquad (3)$$

Next, select $r_i = \frac{D_i}{\Delta_i}$, $i = 1, 2, \dots, m$. Then (3) is the same as condition (1). However, because $r_0 + r_1 + \dots + r_m = 1$, it must be true that $r_1 + r_2 + \dots + r_m \leq 1$. The condition

$$r_1 + r_2 + \dots + r_m \leq 1$$

thus implies that (3), and hence (1), is feasible. The condition is not sufficient because (3) can be satisfied for arbitrary values of $r_0, r_1, \dots,$ and r_m.

6

(a)

$$B^{-1} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

$$Y = (1, 4, 0, 0)\begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

$$= (-1/4, 5/2, 0, 0)$$

$$X_B = B^{-1} b$$

$$= \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 28 \\ 8 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5/2 \\ 3/2 \\ -1/2 \end{pmatrix}$$

The simplex tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	Soluti
z	0	0	-1/4	5/2	0	0	13
x_1	1	0	1/4	-1/2	0	6	3
x_2	0	1	-1/8	3/4	0	0	5/2
x_5	0	0	$\boxed{3/8}$	-5/4	1	0	3/2
x_6	0	0	1/8	-3/4	0	1	-1/2

The tableau is both nonoptimal and infeasible.

(b) Apply the primal simplex to the tableau above, disregarding the x_6-row in the ratio test. Thus, x_3 enters the basic solution and x_5 leaves. The resulting tableau is

continued...

	x_1	x_2	x_3	x_4	x_5	x_6	
z	0	0	0	5/3	2/3	0	14
x_1	1	0	0	1/3	-2/3	0	2
x_2	0	1	0	1/3	1/3	0	3
x_3	0	0	1	-10/3	8/3	0	4
x_6	0	0	0	-1/3	-1/3	1	-1

The tableau is now optimal but infeasible. Application of the dual simplex method should then lead to feasibility while maintaining the tableau optimal.

continued...

1

Current optimum is

$x_1 = 0, \quad x_2 = 100, \quad x_3 = 230$

(a) $4x_1 + x_2 + 2x_3 \le 570$:

Since $4 \times 0 + 1 \times 100 + 2 \times 230 = 560 < 570$, the additional constraint is redundant and the solution remains unchanged.

(b) $4x_1 + x_2 + 2x_3 \le 548$:

The current solution violates the new constraints. We use the dual simplex method to determine the new solution.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
z	4	0	0	1	2	0	0	1350
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	0	100
x_3	$3/2$	0	1	0	$1/2$	0	0	230
x_6	2	0	0	-2	1	1	0	20
x_7	4	$\boxed{1}$	$\boxed{2}$	0	0	0	1	548
z	4	0	0	1	2	0	0	1350
x_2	$-1/4$	1	0	$1/2$	$-1/4$	0	0	100
x_3	$3/2$	0	1	0	$1/2$	0	0	230
x_6	2	0	0	-2	1	1	0	20
x_7	$5/4$	0	0	$\boxed{-1/2}$	$-3/4$	0	1	-12
z	$13/2$	0	0	0	$1/2$	0	2	1326
x_2	$-1/4$	1	0	0	-1	0	1	88
x_3	$3/2$	0	1	0	$1/2$	0	0	230
x_6	-3	0	0	0	4	1	-4	68
x_4	$-5/2$	0	0	1	$3/2$	0	-2	24

Optimum solution:

$x_1 = 0, \quad x_2 = 88, \quad x_3 = 230$

$z = \$1326$

2

Maximize $z = 5x_1 + 6x_2 + 3x_3$

Subject to

$$
\begin{array}{rl}
5x_1 + 5x_2 + 3x_3 \le 50 & (1) \\
x_1 + x_2 - x_3 \le 20 & (2) \\
7x_1 + 6x_2 - 9x_3 \le 30 & (3) \\
5x_1 + 5x_2 + 5x_3 \le 35 & (4) \\
12x_1 + 6x_2 \le 90 & (5) \\
x_2 - 9x_3 \le 20 & (6) \\
x_1, x_2, x_3 \ge 0 &
\end{array}
$$

Start with constraints (1), (3), and (4). The associated solution is

$x_1 = 0, \quad x_2 = 6.2, \quad x_3 = .8$

This solution automatically satisfies the remaining constraints (2), (5), and (6). Hence these constraints are discarded as redundant and the optimum solution for the problem is as given above.

1

$\begin{pmatrix}\text{Basic}\\ \text{vector}\end{pmatrix} = \begin{pmatrix}x_2\\x_3\\x_6\end{pmatrix}$ Inverse $= \begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix}$

Nonbasic variables: x_1, x_4, x_5

(a) $z = 2x_1 + x_2 + 4x_3$

$$(y_1, y_2, y_3) = (1, 4, 0)\begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix} = (1/2, 7/4, 0)$$

Reduced costs:

$x_1: (1/2, 7/4, 0)\begin{pmatrix}1\\3\\1\end{pmatrix} - 2 = 15/4$

$x_4: (1/2, 7/4, 0)\begin{pmatrix}1\\0\\0\end{pmatrix} - 0 = 1/2$

$x_5: (1/2, 7/4, 0)\begin{pmatrix}0\\1\\0\end{pmatrix} - 0 = 7/4$

current solution remains optimal

(b) $z = 3x_1 + 6x_2 + x_3$

$$(y_1, y_2, y_3) = (6, 1, 0)\begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix} = (3, -1, 0)$$

Reduced costs:

$x_1: 1\times3 + 3\times-1 + 1\times0 - 3 = -3 < 0$

$x_4: 1\times3 + 0\times-1 + 0\times0 - 0 = 3$

$x_5: 0\times3 + 1\times-1 + 0\times0 - 0 = -1 < 0$

Solution is not optimal.

	x_1	x_2	x_3	x_4	x_5	x_6	
z	-3	0	0	3	-1	0	830
x_2	-1/4	1	0	1/2	-1/4	0	100
x_3	3/2	0	1	0	1/2	0	230
x_6	$\boxed{2}$	0	0	-2	1	1	20
z	0	0	0	0	1/2	3/2	860
x_2	0	1	1/4	1/4	-1/4	1/8	$102\frac{1}{2}$
x_3	0	0	0	0	1/2	0	215
x_1	1	0	-1	-1	1/2	1/2	10

Optimum Solution: $x_1 = 10$, $x_2 = 102\frac{1}{2}$, $x_3 = 215$

Problem has alternative optima. $z = \$860$

(c) $z = 8x_1 + 3x_2 + 9x_3$

$$(y_1, y_2, y_3) = (3, 9, 0)\begin{pmatrix}1/2 & -1/4 & 0\\ 0 & 1/2 & 0\\ -2 & 1 & 1\end{pmatrix} = (\tfrac{3}{2}, \tfrac{15}{4}, 0)$$

Reduced costs:

$x_1: 1\times\frac{3}{2} + 3\times\frac{15}{4} + 1\times0 - 8 = 19/4$

$x_4: 1\times\frac{3}{2} + 3\times0 + 1\times0 - 0 = 3/2$

continued...

$x_5: 0\times3/2 + 1\times15/4 + 0\times0 - 0 = 15/4$

Solution remains optimal

2

$\begin{matrix}\text{Basic}\\ \text{vector}\end{matrix} = \begin{pmatrix}x_1\\x_2\\x_5\\x_6\end{pmatrix}$, inverse $= \begin{pmatrix}1/4 & -1/2 & 0 & 0\\ -1/8 & 3/4 & 0 & 0\\ 3/8 & -5/4 & 1 & 0\\ 1/8 & -3/4 & 0 & 1\end{pmatrix}$

Dual problem:

Minimize $w = 24y_1 + 6y_2 + y_3 + 2y_4$

Subject to

$$6y_1 + y_2 - y_3 \geq 5$$
$$4y_1 + 2y_2 + y_3 + y_4 \geq 4$$
$$y_1, y_2, y_3, y_4 \geq 0$$

(a) $z = 3x_1 + 2x_2$

$$(y_1, y_2, y_3, y_4) = (3, 2, 0, 0)\begin{pmatrix}1/4 & -1/2 & 0 & 0\\ -1/8 & 3/4 & 0 & 0\\ 3/8 & -5/4 & 1 & 0\\ 1/8 & -3/4 & 0 & 1\end{pmatrix} = (1/2, 0, 0, 0)$$

Reduced costs:

$x_3: y_1 - 0 = 1/2 - 0 = 1/2$

$x_4: y_2 - 0 = 0 - 0 = 0$

Solution remains optimal.

(b) $z = 8x_1 + 10x_2$

$$(y_1, y_2, y_3, y_4) = (8, 10, 0, 0)\begin{pmatrix}1/4 & -1/2 & 0 & 0\\ -1/8 & 3/4 & 0 & 0\\ 3/8 & -5/4 & 1 & 0\\ 1/8 & -3/4 & 0 & 1\end{pmatrix} = (3/4, 7/2, 0, 0)$$

Reduced costs:

$x_3: y_1 - 0 = 3/4 - 0 = 3/4$

$x_4: y_2 - 0 = 7/2 - 0 = 7/2$

Solution remains optimal

(c) $z = 2x_1 + 5x_2$

$$(y_1, y_2, y_3, y_4) = (2, 5, 0, 0)\begin{pmatrix}1/4 & -1/2 & 0 & 0\\ -1/8 & 3/4 & 0 & 0\\ 3/8 & -5/4 & 1 & 0\\ 1/8 & -3/4 & 0 & 1\end{pmatrix} = (-1/8, 11/4, 0, 0)$$

Reduced costs:

$x_3: y_1 - 0 = -1/8 - 0 = -1/8 < 0$

$x_4: y_2 - 0 = 11/4 - 0 = 11/4$

current solution is not optimal.

continued...

Set 4.5c

	x_1	x_2	x_3	x_4	x_5	x_6	
z	0	0	$-1/8$	$11/4$	0	0	$27/2$
x_1	1	0	$1/4$	$-1/2$	0	0	3
x_2	0	1	$-1/8$	$3/4$	0	0	$3/2$
x_5	0	0	$3/8$	$-5/4$	1	0	$5/2$
x_6	0	0	$1/8$	$\boxed{-3/4}$	0	1	$1/2$
z	0	0	0	2	0	1	14
x_1	1	0	0	1	0	-2	2
x_2	0	1	0	0	0	1	2
x_5	0	0	0	1	1	-3	1
x_3	0	0	1	-6	0	8	4

Optimum solution:

$x_1 = 2,\ x_2 = 2,\ x_3 = 4,\ z = 14$

3

Let d_j = change in the objective coefficient c_j, $j = 1, 2, \ldots, n$

The simultaneous changes yield the same optimum if (for maximization)

$$(z_j - c_j - d_j) \geq 0,\ j = 1, 2, \ldots, n \qquad (1)$$

where

z_j = left-hand of constraint dual$_j$ $= \sum_{i=1}^{m} a_{ij} y_i$

Let $u_j \leq d_j \leq v_j$ be the optimality range computed from the single-change condition

$$z_j - c_j - d_j \geq 0 \qquad (2)$$

and define

$$\delta_j = \begin{cases} u_j, & \text{if } d_j < 0 \\ v_j, & \text{if } d_j > 0 \end{cases}$$

Condition (2) holds true also for $d_j = \delta_j$.

Define $r_j \geq 0$, $j = 0, 1, 2, \ldots, n$, such that $r_0 + r_1 + \cdots + r_n = 1$. Then

$$r_0 (z_1 - c_1, \ldots, z_n - c_n) + r_1 (z_1 - c_1 - \delta_1, \ldots, z_n - c_n) + \cdots + r_n (z_1 - c_1, \ldots, z_n - c_n - \delta_n)$$

continued...

must be nonnegative. However, the last expression reduces to

$$(z_1 - c_1, \ldots, z_n - c_n) - (r_1\delta_1, \ldots, r_n\delta_n) \geq 0$$

or

$$z_j - c_j - r_j\delta_j \geq 0,\ j = 1, 2, \ldots, n \qquad (3)$$

Now, set $r_j = \frac{d_j}{\delta_j}$, then (3) is identical to (1), the desired condition. However, since $r_0 + r_1 + \cdots + r_n = 1$ and $r_0 \geq 0$, then for optimality we must have

$$r_1 + r_2 + \cdots + r_n \leq 1$$

1

Dual constraint for toy trains is

$$y_1 + 3y_2 + y_3 \ge 3$$

where $y_1 = 1$, $y_2 = 2$, and $y_3 = 0$

new reduced cost for x_1 is

$$\frac{P}{100}(y_1 + 3y_2 + y_3) - 3.$$

For toy trains to be just profitable, we must have

$$\frac{P}{100}(1 + 3\times 2 + 1\times 0) - 3 \ge 0$$

or $P \ge 42.86\%$

2

$$x_1\text{-reduced cost} = .5y_1 + y_2 + .5y_3 - 3$$
$$= .5\times 1 + 1\times 2 + .5\times 0 - 3 = -.5$$

$$x_1\text{-column} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} .5 \\ 1 \\ .5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

	x_1	x_2	x_3	x_4	x_5	x_6	
z	$-1/2$	0	0	1	2	0	1350
x_2	0	1	0	1/2	$-1/4$	0	100
x_3	1/2	0	1	0	1/2	0	230
x_6	$\boxed{1/2}$	0	0	-2	1	1	20
z	0	0	0	-1	3	1	1370
x_2	0	1	0	1/2	$-1/4$	0	100
x_3	0	0	1	$\boxed{2}$	$-1/2$	-1	210
x_1	1	0	0	-4	2	2	40
z	0	0	1/2	0	11/4	1/2	1475
x_2	0	1	$-1/4$	0	$-1/8$	1/4	$47\frac{1}{2}$
x_4	0	0	1/2	1	$-1/4$	$-1/2$	105
x_1	1	0	2	0	1	0	460

3

(a) New dual constraint for fire engines is

$$3y_1 + 2y_2 + 4y_3 \ge 5,\ y_1 = 1,\ y_2 = 2,\ y_3 = 0$$

Reduced cost $= 3\times 1 + 2\times 2 + 4\times 0 - 5$
$= 2 > 0$

Fire engines are not profitable

continued...

(b) Reduced cost $= 3\times 1 + 2\times 2 + 4\times 0 - 10 = -3$

$$\begin{pmatrix}\text{Tableau}\\ \text{column}\end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
z	4	0	0	-3	1	2	0	1350
x_2	$-1/4$	1	0	1	1/2	$-1/4$	0	100
x_3	3/2	0	1	1	0	1/2	0	230
x_7	2	0	0	0	-2	1	1	20
z	13/4	3	0	0	5/2	5/4	0	1650
x_4	$-1/4$	1	0	1	1/2	$-1/4$	0	100
x_3	7/4	-1	1	0	$-1/2$	3/4	0	130
x_7	2	0	0	0	-2	1	1	20

4

x_3 = daily tons of new exterior paint

Maximize $z = 5x_1 + 4x_2 + 3.5x_3$

subject to

$$6x_1 + 4x_2 + 3/4x_3 \le 24$$
$$x_1 + 2x_2 + 3/4x_3 \le 6$$
$$-x_1 + x_2 + x_3 \le 1$$
$$x_2 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

New dual constraint: $\frac{3}{4}y_1 + \frac{3}{4}y_2 + y_3 \ge 3.5$

Dual solution: $y_1 = 3/4$, $y_2 = 1/2$, $y_3 = 0$

Reduced cost $= \frac{3}{4}(3/4 + 1/2) + 0 - 3.5 = -41/16$

$$\begin{pmatrix}\text{Constraint}\\ \text{column}\end{pmatrix} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3/4 \\ 3/4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/16 \\ 15/32 \\ 13/16 \\ -15/32 \end{pmatrix}$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
z	0	0	$-41/16$	3/4	1/2	0	0	21
x_1	1	0	$-3/16$	1/4	$-1/2$	0	0	3
x_2	0	1	15/32	$-1/8$	3/4	0	0	3/2
x_6	0	0	13/16	3/8	$-5/4$	1	0	5/2
x_7	0	0	$-15/32$	1/8	$-3/4$	0	1	1/2
z	0	5.47	0	.07	4.6	0	0	29.2
x_1	1	.4	0	.2	$-.2$	0	0	3.6
x_3	0	2.13	1	$-.27$	1.6	0	0	3.2
x_6	0	$-.73$	0	.47	-1.8	1	0	1.4
x_7	0	1	0	0	0	0	1	2.0

Optimum solution:

$x_1 = 3.6$ tons, $x_2 = 0$, $x_3 = 3.2$ tons

$z = \$29,200$