INTEGRATING FACTOR

Exact ODE:

$$M\,dx + N\,dy = 0 \quad \text{exact ODE}$$

General solution is

$$\underbrace{\int M\,dx}_{\text{keeping 'y' constant}} + \underbrace{\int N\,dy}_{\text{term not containing } x} = C$$

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

How to make equation exact?

Integrating factor:

If $M\,dx + N\,dy = 0$ is not exact, then it can always be made by multiplying some function of x and y, such a multiplier is called integrating factor.

Let $M\,dx + N\,dy = 0$ —— ① is not exact.

Suppose $I.F = f(x,y)$

then $f(x,y)\,M\,dx + f(x,y)\,N\,dy = 0$ — ②

Now ② becomes exact ODE.

Methods for finding integrating factor.

Method 1: Inspection Method:

e.g. $x\,dy - y\,dx$ —— ①

$M = -y, \; N = x$ $\quad \therefore M\,dx + N\,dy = 0$

$$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ eq-① is not exact.

Multiply eq. ① by $1/x^2$

$$\frac{dy}{x} - \frac{y}{x^2}\,dx = 0 \quad — ②$$

For ② $M = \frac{-y}{x^2}$, $N = \frac{1}{x}$

$$\frac{\partial M}{\partial y} = \frac{-1}{x^2} , \quad \frac{\partial N}{\partial x} = \frac{-1}{x^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Now eq. ② is exact , I.F $= 1/x^2$

If we multiply ① by $1/y^2$

$$\frac{x}{y^2}\,dy - \frac{y}{y^2}\,dx = 0 \quad — ③$$

$M = -1/y$ $\quad N = x/y^2$

$$\frac{\partial M}{\partial y} = \frac{1}{y^2} \quad \frac{\partial N}{\partial x} = \frac{1}{y^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Now eq. ③ becomes exact, I.F $= 1/y^2$

If we multiply $1/xy$ to ①

$$\frac{dy}{y} - \frac{dx}{x} = 0 \quad — ④$$

$M = -1/x$ $\quad N = 1/y$

$$\frac{\partial M}{\partial y} = 0 \quad \frac{\partial N}{\partial x} = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Now eq. ④ becomes exact, I.F $= \frac{1}{xy}$.

Method 2:

If the given ODE is

$$Mdx + Ndy = 0$$

then

$$I.F = \frac{1}{Mx+Ny}, \text{ providing } Mx+Ny \neq 0$$

e.g $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ — ①

$M = x^2y - 2xy^2$ $\qquad$ $N = -x^3 + 3x^2y$

$\frac{\partial M}{\partial y} = x^2 - 4xy$ $\qquad$ $\frac{\partial N}{\partial x} = -3x^2 + 6xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ① is not exact.

$$I.F = \frac{1}{(x^2y - 2xy^2)x + (-x^3 + 3x^2y)y}$$

$$= \frac{1}{\cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2}$$

$$I.F. = \frac{1}{x^2y^2}$$

Solve yourself.

Method 3:

If $Mdx + Ndy = 0$ is in the form

$y f_1(x,y)dx + x f_2(x,y)dy = 0$

then

$$I.F = \frac{1}{Mx-Ny}, \text{ providing } Mx-Ny \neq 0$$

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e.g $[xy\sin(x.y) + \cos(xy)]\,y\,dx + [xy\sin(xy) - \cos(xy)]\,x\,dy = 0$

Check yourself and solve.

$$\left.\begin{aligned} I.F &= \frac{1}{2xy\cos(xy)} \\ \text{Sol} \Rightarrow\quad & yc\cos(xy) = x \end{aligned}\right\} \text{Try yourself.}$$

Method 4 & 5:

let $Mdx + Ndy = 0$ be the ODE if

(i) $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \phi(x)$, then $I.F = e^{\int \phi(y)dy}$

or

(ii) $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \phi(y)$, then $I.f = e^{\int \phi(y)dy}$

e.g ① Solve $(x^2+y^2)dx - 2xy\,dy = 0$ — ①

Sol:

$M = x^2 + y^2 \qquad N = -2xy$

$\dfrac{\partial M}{\partial y} = 2y \qquad \dfrac{\partial N}{\partial x} = -2y$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, eq. ① is not exact

Now,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial y}}{N} = \frac{2y+2y}{-2xy} = \frac{\cancel{4}^{\,2}\cancel{y}}{-\cancel{2}x\cancel{y}}$$

$$= \frac{-2}{x}$$

$$I.F = e^{\int -\frac{2}{x} dx} = e^{-2\ln(x)} = e^{\ln(x)^{-2}}$$

$$I.F = 1/x^2$$

Now, Multiplying ① by $1/x^2$

$$\left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}\,dy = 0 \quad \text{— ②}$$

For ②, $M = 1 + \frac{y^2}{x^2}$, $N = -\frac{2y}{x}$

$$\frac{\partial M}{\partial y} = \frac{2y}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{2y}{x^2}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \text{ ② is exact.}$$

Now,

$$\underbrace{\int M\,dx}_{\text{keeping } y \text{ constant}} + \underbrace{\int N\,dy}_{\text{term free from } x} = C$$

$$\int\left(1 + \frac{y^2}{x^2}\right)dx + \int 0\,dy = c$$

$$x - \frac{y^2}{x} = c$$

Example 5 (Page 25)

Solve $(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$ — ①

$y(0) = -1$

Sol:

$M = e^{x+y} + ye^y$, $N = xe^y - 1$

$\frac{\partial M}{\partial y} = e^{x+y} + e^y + ye^y$, $\frac{\partial N}{\partial x} = e^y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \text{ eq. ① is not exact.}$$

Now,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{e^{x+y} + \cancel{e^y} + ye^y - \cancel{e^y}}{xe^y - 1}$$

$$= \frac{e^y(e^x+y)}{xe^y-1} \neq \phi(x)$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\cancel{e^y} - e^{x+y} - \cancel{e^y} - ye^y}{e^{x+y}+ye^y}$$

$$= \frac{-\cancel{e^y}\cancel{(e^x+y)}}{\cancel{e^y}\cancel{(e^x+y)}}$$

$$= -1$$

$$I.F = e^{-\int 1 \cdot dy} = e^{-y}$$

Now, multiply ① by e^{-y}

$$e^{-y}(e^{x+y}+ye^{-y})dx + e^{-y}(xe^y-1)dy = 0$$

$$(e^x+y)dx + (x-e^{-y})dy = 0 \quad -②$$

For ②, $M = e^x+y$, $N = x-e^{-y}$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow ② \text{ is exact}$$

Now, keeping y constant $\quad\rightarrow$ term free from x

$$\int M dx + \int N dy = c$$

$$\int (e^x+y)dx + \int (-e^{-y})dy = c$$

$$f(x,y) = e^x + yx + e^{-y} = c$$

Using $y(0) = -1$

$$f(0,-1) = e^0 + (-1)(0) + e^{-(-1)} = c$$

$$\Rightarrow c = 1+e$$

$$\Rightarrow e^x + yx + e^{-y} = 1+e$$

Method 6:

If $Mdx + Ndy = 0$ is in the form

$$x^a y^b (my\,dx + nx\,dy) + x^{a'} y^{b'} (m'y\,dx + n'x\,dy) = 0$$

then

$$I.F = x^h y^k$$

where h & k are given by

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \quad \text{— (1)}$$

$$\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'} \quad \text{— (2)}$$

where $a, a', b, b', m, m', n, n'$ are constants.

Example $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$

Sol:

$$xy^2dx + 2x^2y^3dx + x^2ydy - x^3y^2dy = 0$$

$$\underbrace{xy^2dx + x^2y\,dy} + \underbrace{2x^2y^3dx - x^3y^2dy} = 0$$

$$xy(ydx + xdy) + x^2y^2(2ydx - xdy) = 0$$

$a = 1,\ b = 1 \qquad a' = 2,\ b' = 2$

$m = 1,\ n = 1 \qquad m' = 2,\ n' = -1$

For finding h & k we have

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \quad (1)$$

$$\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'} \quad \text{— (2)}$$

$$(1) \Rightarrow \frac{1+h+1}{1} = \frac{1+k+1}{1}$$

$$\cancel{2}+h = \cancel{2}+k$$

$$\Rightarrow h = k$$

$$\frac{[illegible]}{2} = \frac{[illegible]}{-1}$$

$$\frac{3+h}{2} = \frac{3+k}{-1}$$

$$-3-h = 6+2k$$

$$-h = 9+2h \qquad \therefore k=h.$$

$$3h = -9$$

$$h = -3$$

$$\Rightarrow k = -3$$

$$\therefore \quad I.F = x^h y^k = x^{-3} y^{-3}$$

$$I.F = \frac{1}{(xy)^3}$$

Further, Try yourself.