Separable ODE's:

A first order differential equation is said to be separable if, after solving it for the derivative

$$\frac{dy}{dx} = F(x,y)$$

the right-hand side can be factored as "a formula of just x" times "a formula of just y"

$$F(x,y) = f(x)\cdot g(y)$$

If this factoring is not possible, the equation is not separable.

Moreover, a first-order differential equation is ~~only~~ separable if and only if it can be written as

$$\frac{dy}{dx} = f(x)\cdot g(y)$$, where f and g are known functions.

Example: Consider the differential equation

$$\frac{dy}{dx} - x^2y^2 = x^2$$

Sol:

Solving for the derivative

$$\frac{dy}{dx} = x^2 + x^2y^2$$

and factorizing out the x^2 on R.H.S gives

$$\frac{dy}{dx} = x^2(1+y^2)$$

which is in form

$$\frac{dy}{dx} = f(x)\, g(y)$$

with $f(x) = \underbrace{x^2}_{\text{no } y\text{'s}}$, and $g(y) = \underbrace{(1+y^2)}_{\text{no } x\text{'s}}$

Hence the given equation is separable.

Example: $\frac{dy}{dx} - x^2y^2 = 4$

Sol: Solving for the derivative
$$\frac{dy}{dx} = x^2y^2 + 4$$

The R.H.S of this clearly cannot be factored into a function of just 'x' times a function of just 'y'. Thus the given equation is not separable.

Example 1 (Pg.12)

Solve the equation
$$y' = 1 + y^2$$

Sol:

The given ODE is separable, because it can be written as
$$\frac{dy}{dx} = 1 + y^2$$
$$\frac{dy}{1+y^2} = dx$$

integrating both sides
$$\int \frac{1}{1+y^2}\,dy = \int 1 \cdot dx$$

Here $a = 1$, $x^2 = y^2$, we have

$$\frac{\tan^{-1} y}{1} = x + C$$

$$\tan^{-1} y = x + c$$

$$\Rightarrow y = \tan(x + c)$$

As we know

$$\int \frac{1}{a^2+x^2}\,dx = \frac{\tan^{-1} x}{a} + c$$

$$\therefore \int \frac{1}{1+y^2}\,dy = \frac{\tan^{-1} y}{1}$$

$$\int \frac{1}{1+y^2}\,dy = \tan^{-1} y$$

Problem Set 1.3

2- $y^3 y' + x^3 = 0$

Sol:

Given eq. is separable

$$y^3 \frac{dy}{dx} = -x^3$$

$$\Rightarrow y^3 dy = -x^3 dx$$

integrating both sides

$$\int y^3 dy = -\int x^3 dx$$

$$\frac{y^4}{4} = -\frac{x^4}{4} + c$$

$$y^4 = -x^4 + 4c$$

$$y^4 = -x^4 + C$$

$$\Rightarrow \boxed{y = \sqrt[4]{-x^4 + C}}$$

3- $y' = \sec^2 y$

Sol:

The given eq. is separable

$$\frac{dy}{dx} = \sec^2 y$$

$$\frac{dy}{\sec^2 y} = dx$$

$$\cos^2 y \, dy = dx$$

integrating both sides

$$\int \cos^2 y \, dy = \int 1 \cdot dx$$

$$\int \cos^2 y \, dy = x + C \quad \text{—— (1)}$$

Now,

$$\int \cos^2 y \, dy = \int \frac{1+\cos 2y}{2} \, dy$$

$$= \int \frac{1}{2} dy + \int \frac{1}{2} \cos 2y \, dy$$

$$= \frac{y}{2} + \int \frac{1}{2} \cos 2y \, dy$$

by substituting $t = 2y \Rightarrow dt = 2dy \Rightarrow dy = \frac{dt}{2}$

$$= \frac{y}{2} + \frac{1}{2} \int \cos t \, \frac{dt}{2}$$

$$= \frac{y}{2} + \frac{1}{4} \sin t + C$$

$$\int \cos^2 y \, dy = \frac{y}{2} + \frac{1}{4} \sin 2y + C \quad \text{(2)} \qquad \because t = 2y$$

using (2) in (1)

$$\boxed{\frac{y}{2} + \frac{1}{4} \sin 2y = x + C}$$

4- $y' \sin 2\pi x = \pi y \cos 2\pi x$

Sol: The equation is separable

$$\frac{dy}{y} = \frac{\pi \cos 2\pi x}{\sin 2\pi x} dx$$

integrating both sides

$$\int \frac{1}{y} dy = \int \frac{\pi \cos 2\pi x}{\sin 2\pi x} dx$$

$$\ln|y| = \pi \int \frac{\cos 2\pi x}{\sin 2\pi x} dx \quad \text{—— (1)}$$

Now, $\pi \int \frac{\cos 2\pi x}{\sin 2\pi x} dx$

Let $2\pi x = t \Rightarrow dt = 2\pi dx \Rightarrow dx = dt/2\pi$

$$= \pi \int \frac{\cos t}{\sin t} \cdot \frac{dt}{2\pi}$$

$$= \frac{\pi}{2\pi} \int \frac{\cos t}{\sin t} dt$$

$$= \frac{\cancel{\pi}}{2\cancel{\pi}} \ln|\sin t|$$

$$\pi \int \frac{\cos 2\pi x}{\sin 2\pi x} dx = \frac{1}{2} \ln|\sin 2\pi x| \quad \text{—— (2)}$$

using (2) in (1)

$$\ln|y| = \frac{1}{2} \ln|\sin 2\pi x| + c$$

$$e^{\ln|y|} = e^{\frac{1}{2}\ln|\sin 2\pi x| + c}$$

$$|y| = e^{\ln|\sin 2\pi x|^{1/2} + c}$$

$$|y| = e^{\ln|\sin 2\pi x|^{1/2}} \cdot e^{c}$$

$$\boxed{y = c\sqrt{\sin 2\pi x}}$$

5- $yy' + 36x = 0$.

Sol:

The eq. is separable

$$y \frac{dy}{dx} = -36x$$

$$y\, dy = -36\, x\, dx$$

integrating both sides

$$\int y\,dy = -36\int x\,dx$$

$$\frac{y^2}{2} = -36\frac{x^2}{2} + c$$

$$y^2 = -36x^2 + 2c$$

$$y^2 = -36x^2 + C$$

$$\Rightarrow \boxed{y = \sqrt{C - 36x^2}}$$

6- $y' = e^{2x-1}\,y^2$

Solution:

The given eq: is separable

$$\frac{dy}{dx} = e^{2x-1}\,y^2$$

$$\frac{dy}{y^2} = e^{2x-1}\,dx$$

integrating both sides

$$\int \frac{1}{y^2}\,dy = \int e^{2x-1}\,dx$$

using substitution
$t = 2x-1 \Rightarrow dt = 2\,dx$
$\Rightarrow dx = dt/2$

$$\frac{y^{-2+1}}{-1} = \int e^t\,\frac{dt}{2}$$

$$-\frac{1}{y} = \frac{1}{2}e^t + C$$

$$-\frac{1}{y} = \frac{1}{2}e^{2x-1} + C \qquad \because t = 2x-1$$

$$\Rightarrow \boxed{y = -\frac{2}{e^{2x-1} + C}}$$

Q.No. 7, 8, 9, 10 (Do Yourself)

Solve IVP

11- $xy' + y = 0$, $y(4) = 6$

Sol: The eq. is separable

$$\frac{dy}{y} = -\frac{dx}{x}$$

integrating both side

$$\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$$

$$\ln|y| = -\ln|x| + c$$

$$e^{\ln|y|} = e^{-\ln|x| + c}$$

$$y = e^{\ln|x|^{-1}} \cdot e^{c}$$

$$y = c \cdot \frac{1}{x}$$

$$\Rightarrow \quad y = \frac{c}{x} \quad \Rightarrow y(x) = \frac{c}{x} \quad \text{--- (1)}$$

Now using IVP $y(4) = 6$

$$y(4) = \frac{c}{4}$$

$$6 = \frac{c}{4} \quad \Rightarrow c = 24$$

$$(1) \Rightarrow \boxed{y(x) = \frac{24}{x}}$$

12- $y' = 1+4y^2$, $y(1)=0$

Sol:

The eq. is separable

$$\frac{dy}{1+4y^2} = dx$$

integrating both sides

$$\int \frac{1}{1+4y^2}\,dy = \int 1 \cdot dx$$

$$\int \frac{1}{1+(2y)^2}\,dy = x + c$$

use substitution

$t = 2y \Rightarrow dt = 2dy \Rightarrow dy = dt/2$

$$\int \frac{1}{1+t^2}\,\frac{dt}{2} = x+c$$

$$\frac{1}{2}\int \frac{1}{1+t^2}\,dt = x+c$$

$$\frac{1}{2}\tan^{-1} t = x + c$$

$$\frac{1}{2}\tan^{-1}(2y) = x+c \qquad \because\ t = 2y$$

$$\tan^{-1}(2y) = 2(x+c)$$

$$y = \frac{1}{2}\tan(2(x+c)) \quad \text{— (1)}$$

Now using IVP $y(1)=0$

$$y(1) = \frac{1}{2}\tan(2(1+c))$$

$$0 = \frac{1}{2}\tan(2(c+1))$$

$\Rightarrow \tan^{-1}(0) = 2(c+1)$

$0 = 2(c+1)$

$\Rightarrow c = -1$

① $\Rightarrow$ $\boxed{y = \dfrac{\tan(2(x-1))}{2}}$

Q. 13, 14, 15, 16, 17 (Do yourself)