Equations of motion for a system of Particles:-

Consider a system of n particles whose individual masses are constant. Let $m_1, m_2, m_3 — m_n$ are the masses of individual particle and $\vec{r}_1, \vec{r}_2, \vec{r}_3 — r_n$ are the position vectors from fixed origin outside the system.

Consider an ith particle, so equation of motion of the ith particle is

$$\vec{F}_i = m_i \vec{a}_i$$

$\vec{F}_i$ is the force acting on ith particle which is sum of external forces and Internal force

$$\vec{F}_i = \vec{F}_i^{(e)} + \sum_{j=1}^{n} \vec{F}_{ij}$$

$$\Rightarrow \vec{F}_i^{(e)} + \sum_{\substack{j=1 \\ i \neq j}}^{n} \vec{F}_{ij} = m_i \ddot{\vec{r}}_i \longrightarrow (1)$$

If there are 'n' particles, we have one such equation for each particle so total n number of such equation so eq. of motion for system of n-particles can be found by taking sum over all particles

$$\sum_{i=1}^{n} \vec{F}_i^{(e)} + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{F}_{ij} = \sum_{i=1}^{n} m_i \ddot{\vec{r}}_i \longrightarrow (2)$$

$$\therefore \sum_{i=1}^{n} \vec{F}_i^{(e)} = \vec{F}_1^{(e)} + \vec{F}_2 + \vec{F}_3^{(e)} + \vec{F}_4^{e} + \cdots + \vec{F}_n^{(e)} = \vec{F}^{(e)}$$

$$\sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n} \vec{F}_{ij} = \sum_{i<j}^{n} \vec{F}_{ij} + \sum_{i>j}^{n} \vec{F}_{ij}$$

$$= \sum_{i<j}^{n} \vec{F}_{ij} + \sum_{i<j}^{n} \vec{F}_{ji}$$

$$= \sum_{i<j}^{n} \vec{F}_{ij} + \sum_{i<j}^{n} (-\vec{F}_{ij}) \qquad \because \vec{F}_{ji} = -\vec{F}_{ij}$$

$$= \sum_{i<j}^{n} (\vec{F}_{ij} - \vec{F}_{ij}) = 0$$

and According to definition of Centre of mass $\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{\sum m_i \vec{r}_i}{M}$

$$M\vec{R} = \sum m_i \vec{r}_i \Rightarrow M\ddot{\vec{R}} = \sum m_i \ddot{\vec{r}}_i$$

Put value in eq (2)

$$\vec{F}^{e}_{to} = M\ddot{\vec{R}} \Rightarrow \boxed{\vec{F} = M\ddot{\vec{R}}}$$

Conservation of linear Momentum

Consider system of n-particles, total linear momentum is equal to sum of linear momentum of all the particles.

Linear momentum of i^{th} particle is

$$\vec{P}_i = m_i \vec{V}_i = m_i \dot{\vec{r}}_i$$

Linear momentum of system of particles

$$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots + \vec{P}_n = \sum_{i=1}^{n} \vec{P}_i$$

$$\vec{P} = \sum_{i=1}^{n} m_i \dot{\vec{r}}_i$$

$$\frac{d\vec{P}}{dt} = \sum_{i=1}^{n} \frac{d}{dt}(m_i \dot{\vec{r}}_i) = \sum_{i=1}^{n} m_i \ddot{\vec{r}}_i$$

$$\dot{P} = \frac{dP}{dt} = M\ddot{\vec{R}} \Rightarrow \frac{d\vec{P}}{dt} = \vec{F}$$

(2) If $\vec{F} = 0 \Rightarrow \frac{dP}{dt} = 0$

$$\boxed{P = \text{const}}$$

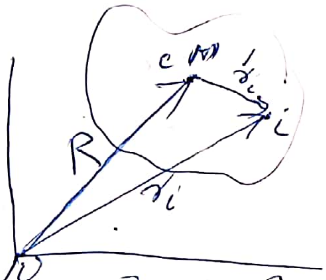
Angular Momentum of the System

The angular momentum of ith particle relative to the origin $\vec{L}_i = \vec{r}_i \times \vec{P}_i$

Resultant angular momentum of the system is the vector sum of angular momentum of all individual particles

$$\vec{L} = \sum_{i=1}^{n} \vec{L}_i = \sum_{i=1}^{n} \vec{r}_i \times \vec{P}_i = \sum_{i=1}^{n} \vec{r}_i \times m_i \dot{\vec{r}}_i$$

$$\vec{r}_i = \vec{R} + \vec{r}_i'$$



$$\vec{L} = \sum_{i=1}^{n} (\vec{R} + \vec{r}_i') \times m_i (\dot{\vec{R}} + \dot{\vec{r}}_i')$$

$$= \sum m_i \left[(\vec{R} \times \dot{\vec{R}}) + \underbrace{(\vec{R} \times \dot{\vec{r}}_i') + (\vec{r}_i' \times \dot{\vec{R}})} + (\vec{r}_i' \times \dot{\vec{r}}_i') \right]$$

$$\vec{R} \times \sum_{i=1}^{n} m_i \dot{\vec{r}}_i' + \sum_{i=1}^{n} m_i \vec{r}_i' \times \dot{\vec{R}}$$

$$= \vec{R} \times \frac{d}{dt}\left(\sum m_i r_i'\right) + \sum m_i r_i' \times \dot{R}$$

$$\sum_{i=1}^{n} m_i r_i' = \sum_i m_i (r_i - R)$$

$\because r_i = R + r_i'$

$\therefore r_i' = r_i - R$

$$= \sum_i m_i r_i - R \sum m_i$$

$$= MR - MR = 0$$

$\left[\sum m_i = M, \quad R = \frac{1}{M} \sum m_i r_i \right]$

So 2nd & 3rd terms contribute zero.

$$\Rightarrow \vec{L} = \vec{R} \times \dot{\vec{R}} \sum_i m_i + \sum_i (\vec{r}_i' \times m_i \dot{\vec{r}}_i')$$

$$= \underline{R \times M\dot{R}} + \sum_i (r_i' \times P_i')$$

$$\vec{L} = L_{cm} + L'$$

⇒ Total angular momentum about an origin is the sum of angular momentum of centre of mass about the origin L_{CM} and angular momentum of the system about the centre of mass. (6)

Now $\vec{L}_i = \vec{r}_i \times \vec{P}_i$

$$\frac{dL_i}{dt} = \frac{d}{dt}(\vec{r}_i \times \vec{P}_i)$$

$$\dot{\vec{L}}_i = \dot{\vec{r}}_i \times \vec{P}_i + \vec{r}_i \times \dot{\vec{P}}_i$$

$$= \vec{r}_i \times \dot{\vec{P}}_i \qquad \dot{P}_i = F_i^{(e)} + \sum_{j \neq i}^{n} F_{ij}$$

$$= \vec{r}_i \times \left(F_i^{(e)} + \sum_{j \neq i}^{n} F_{ij}\right)$$

$$= \vec{r}_i \times \vec{F}_i^{(e)} + \vec{r}_i \times \sum_{j \neq i}^{n} F_{ij}$$

$$* \sum_{i,j=1}^{n} \vec{r}_i \times \vec{F}_{ij} = \sum_{i<j}^{n} \vec{r}_i \times \vec{F}_{ij} + \sum_{i>j}^{n} \vec{r}_i \times \vec{F}_{ij}$$
$$= \sum_{i<j} (\vec{r}_i \times \vec{F}_{ij}) + (\vec{r}_j \times \vec{F}_{ji})$$
$$= (\vec{r}_i \times \vec{F}_{ij}) - (\vec{r}_j \times \vec{F}_{ij})$$
$$= \vec{r}_{ij} \times \vec{F}_{ij}$$
$$= 0$$
$$\because \vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$
$$\vec{r}_{ij} \parallel \vec{F}_{ij}$$

Summing above expression over i for whole system

$$\dot{\vec{L}} = \sum_i \dot{\vec{L}}_i = \sum_i (\vec{r}_i \times \vec{F}_i^{(e)}) + \sum_{i,j \neq i} \vec{r}_i \times \vec{F}_{ij} \; * \qquad \because F_{ji} = -F_{ij}$$

$$\Rightarrow \dot{\vec{L}} = \sum_i \left[\vec{r}_i \times \vec{F}_i^{(e)}\right] = \sum_{i=1}^{n} N_i^{(e)}$$

$$\dot{L} = N^{(e)}$$

If $N^{(e)} = 0 \Rightarrow \frac{dL}{dt} = 0 \Rightarrow L = \text{const}$

If net external torques about a given axis is zero, the total angular momentum of the system about that axis remains constant.

Conservation of Energy for system of Particles

Work done in moving the ith particle from position 'A' to position 'B' is — (1)

$$W_{A\to B} = \int_A^B \vec{F}_i \cdot d\vec{r}_i$$

Work done to move 'n' number of system of particles

$$W_{A\to B} = \sum_{i=1}^{n} \int_A^B \vec{F}_i \cdot d\vec{r}_i$$

Assume all work done is stored in form of K.E.

$$W_{A\to B} = \sum_{i=1}^{n} \int_A^B m_i \ddot{\vec{r}}_i \cdot d\vec{r}_i = \sum_{i=1}^{n} \int m_i \frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i$$

$$= \sum_{i=1}^{n} \int_A^B m_i \frac{d\vec{v}_i}{d\vec{r}_i} \cdot \frac{d\vec{r}_i}{dt} \, dt$$

$$= \sum_{i=1}^{n} \int_A^B m_i \left(\frac{d\vec{v}_i}{d\vec{r}_i} \cdot d\vec{v}_i \right) d\vec{r}_i$$

$$= \sum_{i=1}^{n} \int_A^B \frac{d}{dr_i} \left(\frac{1}{2} m_i v_i^2 \right) dr_i$$

$$= \sum_{i=1}^{n} \int_A^B d\left(\frac{1}{2} m_i v_i^2 \right) = \int_A^B d\left(\sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 \right)$$

$$= \int_A^B dT$$

$$\boxed{W_{A\to B} = T_B - T_A} \longrightarrow (1)$$

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Again assumed the same number of or numbers of particles. work done to move system of particles from position $\vec{A}$ to position $\vec{B}$, (2)

$$W_{A\to B} = \sum_{i=1}^{n}\int_{A}^{B} \vec{F_i}\cdot d\vec{r_i}$$

$$\vec{F_i} = \vec{F_i}^{(e)} + \sum_{\substack{j=1\\ j\neq i}}^{n} \vec{F_{ij}}$$

$$W_{A\to B} = \sum_{i=1}^{n}\int_{A}^{B}\Big(\vec{F_i}^{(e)} + \sum_{\substack{j=1\\ j\neq i}}^{n}\vec{F_{ij}}\Big)\cdot d\vec{r_i}$$

$$= \sum_{i=1}^{n}\int_{A}^{B}\vec{F_i}^{(e)}\cdot d\vec{r_i} + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\int_{A}^{B}\vec{F_{ij}}\cdot d\vec{r_i} \rightarrow (2)$$

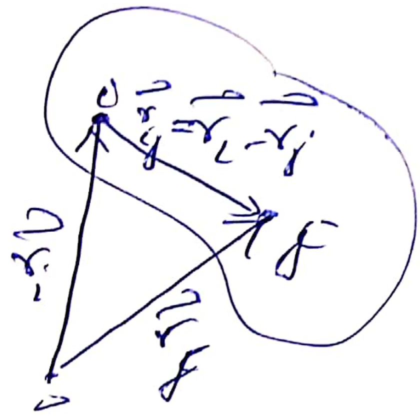
1st term of eq (2) If $\vec{F_i}^{(e)}$ are conservative then $\vec{F_i} = -\nabla_i V_i(r_i)$

$$\sum_{i=1}^{n}\int_{A}^{B}\vec{F_i}^{(e)}\cdot d\vec{r_i} = -\sum_{i=1}^{n}\int_{A}^{B}\bar{\nabla}_i V_i\cdot d\vec{r_i}$$

$$= -\sum_{i=1}^{n}\int_{A}^{B}\frac{dV_i}{dr_i}dr_i = -\sum_{i=1}^{n}\int_{A}^{B}dV_i$$

$$= -\sum_{i=1}^{n} V_i\Big|_{A}^{B} \rightarrow (3)$$

$$V_{ij} = V_{ij}(r_i, r_j)$$



If internal forces are also conservative then

$$\vec{F}_{ij} = -\bar{\nabla}_i V_{ij}$$

Similarly $\vec{F}_{ji} = -\bar{\nabla}_j V_{ij}$

$$-\vec{F}_{ij} = -\bar{\nabla}_j V_{ij} \Rightarrow \vec{F}_{ij} = \bar{\nabla}_j V_{ij}$$

$$dV_{ij}(r_i, r_j) = \frac{\partial V_{ij}}{\partial r_i} \cdot dr_i + \frac{\partial V_{ij}}{\partial r_j} \cdot dr_j$$

$$= \bar{\nabla}_i V_{ij} \cdot d\bar{r}_i + \bar{\nabla}_j V_{ij} \cdot dr_j$$

$$= -\vec{F}_{ij} \cdot d\bar{r}_i + \vec{F}_{ij} \cdot d\bar{r}_j$$

$$= -\left[\vec{F}_{ij} \cdot (d\bar{r}_i - d\bar{r}_j)\right]$$

$$= -\vec{F}_{ij} \cdot d\vec{r}_{ij}$$

$$-dV_{ij} = \vec{F}_{ij} \cdot d\vec{r}_{ij} \longrightarrow (4)$$

Now consider 2nd term of eq (2)

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \int \vec{F}_{ij} \cdot d\bar{r}_i = \sum_{i<j}^{n} \int \vec{F}_{ij} \cdot d\bar{r}_i + \sum_{i>j}^{n} \int \vec{F}_{ij} \cdot d\vec{r}_i$$

$$\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n}\int \vec{F}_{ij}\cdot d\vec{r}_i = \sum_{i<j}\int \vec{F}_{ij}\, d\vec{r}_i + \sum_{i<j}^{n}\int \vec{F}_{ji}\cdot d\vec{r}_j \quad (4)$$

$$= \sum_{i<j}^{n}\int \vec{F}_{ij}\cdot d\vec{r}_i + \sum_{i<j}^{n}\int (-\vec{F}_{ij})\cdot d\vec{r}_j$$

$$= \sum_{i<j}^{n}\int \vec{F}_{ij}\cdot (d\vec{r}_i - d\vec{r}_j)$$

$$= \sum_{i<j}^{n}\int \vec{F}_{ij}\cdot d\vec{r}_{ij} \rightarrow (5)$$

Put ~~this~~ value of eq (4) in eq (5)

$$= -\sum_{i<j}^{n}\int_{A}^{B} dV_{ij}$$

$$= -\sum_{i<j}^{n} V_{ij}\Big|_{A}^{B} \rightarrow (6)$$

Put value of eq (3) and (6) in eq (2)

$$W_{A\to B} = -\sum_{i=1}^{n} V_i\Big|_{A}^{B} - \sum_{i<j}^{n} V_{ij}\Big|_{A}^{B}$$

$$= -\left(\sum_{i=1}^{n} V_i + \sum_{i<j} V_{ij}\right)\Big|_{A}^{B}$$

$$\therefore \sum_{i=1}^{n} V_i + \sum_{i<j}^{n} V_{ij} = V = \text{total P.E of the system}$$

$$W_{A\to B} = -V\Big|_B^A \quad (8)$$

$$= -(V_B - V_A) \rightarrow (7)$$

Compare eq (8) and (7)

$$T_B - T_A = -(V_B - V_A)$$

$$T_B - T_A = -V_B + V_A$$

$$T_B + V_B = T_A + V_A$$

$$(T+V)_B = (T+V)_A$$

For conservative system, total energy of a system of particles remains conserved