بابل

# CHAPTER # 2

## SECOND ORDER LINEAR ODE'S

### Homogenous linear ODE of Second order:

A second order linear ODE is
$$y'' + P(x)y' + q(x)y = r(x) \quad \text{--- (1)}$$
Then
(i) If $r(x) = 0$, eq. (1) is called homogeneous linear ODE.
(ii) If $r(x) \neq 0$, eq. (1) is called non-homogeneous linear ODE.

e.g (a) $y'' + 2y' + y = 0$ Homogeneous
(b) $y'' + y = \tan x$ non-homogeneous

### Superposition Principle:

A second order homogeneous linear ODE is
$$y'' + P(x)y' + q(x)y = 0 \quad \text{--- (1)}$$
let $y_1$ and $y_2$ be solution of (1), then linear combination of $y_1$ and $y_2$ i-e $c_1y_1 + c_2y_2$ is also solution of (1)

### Example: 1 (Page 47)

Let $\cos x$ and $\sin x$ are solution of homogeneous linear ODE
$$y'' + y = 0$$
Sol: Verification:
let $y_1 = \cos x$ and $y_2 = \sin x$
$y_1'' = -\cos x$ $\quad y_2'' = -\sin x$
Now,
$$y_1'' + y_1 = -\cos x + \cos x = 0$$
$$y_2'' + y_2 = -\sin x + \sin x = 0$$
$\cos x$ and $\sin x$ are solution of $y'' + y = 0$, verified

To verify $y = c_1 y_1 + c_2 y_2$,

By superposition principle,

$y = c_1 \cos x + c_2 \sin x$ is also a solution of

$$y'' + y = 0$$

Verification:

$$y'' + y = (c_1 \cos x + c_2 \sin x)'' + (c_1 \cos x + c_2 \sin x)$$

$$= -c_1 \cancel{\cos} x - c_2 \cancel{\sin} x + c_1 \cancel{\cos} x + c_2 \cancel{\sin} x$$

$$y'' + y = 0$$

Thus $y = c_1 \cos x + c_2 \sin x$ is also solution of

$$y'' + y = 0$$

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Fundamental theorem for the homogeneous linear ODE:

Statement: For a homogeneous linear ODE

$$y'' + P(x) y' + q(x) = 0 \quad \text{--- (1)}$$

then any linear combination of two solutions of eq. (1) is again a solution of eq. (1)

Proof:

Let $y'' + P(x) y' + q(x) y = 0$ — (1) be the second order linear ODE and let $y_1$ and $y_2$ be the solution of (1), then we shall show that $c_1 y_1 + c_2 y_2$ again be the solution of (1).

$$\left.\begin{aligned} y_1'' + P(x) y_1' + q(x) y_1 = 0 \\ y_2'' + P(x) y_2' + q(x) y_2 = 0 \end{aligned}\right\} \text{--- (2)}$$

Let $y = c_1 y_1 + c_2 y_2$

$$y'' + P(x) y' + q(x) y = (c_1 y_1 + c_2 y_2)'' + P(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2)$$

$= c_1 y_1'' + c_2 y_2'' + P(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2)$
$= c_1 (y_1'' + P(x) y_1' + q(x) y_1) + c_2 (y_2'' + P(x) y_2' + q(x) y_2)$

by using (2)

$= c_1 \cdot 0 + c_2 \cdot 0$

$= 0$

$\therefore \; y = c_1 y_1 + c_2 y_2$ is solution of (1)

**Note:** Theorem is applicable for only homogeneous linear ODE.

**Example 2: (A non-homogeneous linear ODE) (Page 48)**

Let $y_1 = 1 + \cos x$ and $y_2 = 1 + \sin x$ are solutions of $y'' + y = 1$ — (1)

**Sol:**

We will check validation of superposition principle

$$y_1'' + y_1 = (1 + \cos x)'' + (1 + \cos x)$$
$$= 0 - \cancel{\cos x} + 1 + \cancel{\cos x}$$
$$y_1'' + y_1 = 1$$

Similarly,

$$y_2'' + y_2 = (1 + \sin x)'' + (1 + \sin x)$$
$$= 0 - \cancel{\sin x} + 1 + \cancel{\sin x}$$
$$y_2'' + y_2 = 1$$

$\Rightarrow y_1$ and $y_2$ are solutions of (1).

Now for

$$y = c_1 y_1 + c_2 y_2$$

$$y'' + y = (c_1 y_1 + c_2 y_2)'' + (c_1 y_1 + c_2 y_2)$$
$$= [c_1(1 + \cos x) + c_2(1 + \sin x)]'' + [c_1(1 + \cos x) + c_2(1 + \sin x)]$$
$$= (c_1 + c_1 \cos x + c_2 + c_2 \sin x)'' + (c_1 + c_1 \cos x + c_2 + c_2 \sin x)$$

$= 0 - c_1 \cancel{\cos x} + 0 - c_2 \cancel{\sin x} + c_1 + c_2 + c_1 \cancel{\cos x} + c_2 \cancel{\sin x}$

$y'' + y = c_1 + c_2$

if $c_1 = 1,\ c_2 = 2$

$\Rightarrow c_1 + c_2 = 3$

$y'' + y = 3 \neq 1$

Thus $y = c_1 y_1 + c_2 y_2$ is not a solution of (1)

Hence fundamental theorem is not applicable for non-homogeneous ODE.

**Example 3: (A Nonlinear ODE) (Page 48)**

let $y_1 = x^2$ and $y_2 = 1$ are solutions of non-linear ODE

$$y'' y - x y' = 0 \quad \text{— (1)}$$

**Solution:**

Verification:

$y_1'' y_1 - x y_1' = (x^2)'' \cdot x^2 - x(x^2)'$

$= 2x^2 - x \cdot 2x = 2x^2 - 2x^2$

$y_1'' y_1 - x y_1' = 0$

Similarly,

$y_2'' y_2 - x y_2' = (1)'' \cdot 1 - x(1)'$

$= 0 \cdot 1 - x \cdot 0$

$y_2'' y_2 - x y_2' = 0$

Now,

$y = c_1 y_1 + c_2 y_2$

$y'' y - x y' = (c_1 y_1 + c_2 y_2)''(c_1 y_1 + c_2 y_2) - x(c_1 y_1 + c_2 y_2)'$

$= (c_1 x^2 + c_2)''(c_1 x^2 + c_2) - x(c_1 x^2 + c_2)'$

$= 2c_1(c_1 x^2 + c_2) - x(2c_1 x) \neq 0$

$\Rightarrow y = c_1 x^2 + c_2$ is not solution of eq. (1)

Thus fundamental theorem is not applicable for non-linear ODE.

Solve the initial value problem.

Example: $y''+2y'+2y=0 \quad y(0)=2,\ y'(0)=1$

Sol:

Auxiliary eq. is

$m^2+2m+2=0$

$a=1,\ b=2,\ c=2$

$m=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\dfrac{-2\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)}$

$=\dfrac{-2\pm\sqrt{4-8}}{2}$

$=\dfrac{-2\pm\sqrt{-4}}{2}$

$=\dfrac{-2\pm 2i}{2}$

$=-1\pm 1i$

$\alpha \quad \beta$

or

$b^2-4ac=(2)^2-4(1)(2)$

$=4-8$

$=-4$

$\Rightarrow b^2-4ac<0$

we know

$y=e^{\alpha x}(c_1\cos\beta(x)+c_2\sin\beta(x))$

$\alpha=-1,\ \beta=1$

$y(x)=e^{-x}(c_1\cos(x)+c_2\sin(x))$ — ①

using condition $y(0)=2$

$y(0)=e^0(c_1\cos(0)+c_2\sin(0))$

$\boxed{2=c_1}$ put in ①

$\Rightarrow y(x)=e^{-x}(2\cos(x)+c_2\sin(x))$ — ②

taking derivative

$y'(x)=e^{-x}(-2\sin(x)+c_2\cos(x))-(2\cos(x)+c_2\sin x)$

Now using condition $y'(0)=1$

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$y'(0) = e^{0}(-2\sin(0) + C_2\cos(0)) - (2\cos(0) + C_2\sin(0))$

$1 = C_2 - 2$

$\Rightarrow C_2 = 1 + 2$

$\Rightarrow \boxed{C_2 = 3}$ Put in 1 2

$y(x) = e^{-x}(2\cos x + 3\sin x)$

Example 4 : (Page 49)

Solve the initial value problem

$y'' + y = 0 \quad y(0) = 3, \; y'(0) = -0.5$

Sol:

The auxiliary eq. is

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1} = 0 \pm 1i$

$m = 0 \pm 1i$,

we know

$y(x) = e^{\alpha x}(C_1\cos B(x) + C_2\sin B(x))$

$\alpha = 0, \; B = 1$

$y(x) = e^{0(x)}(C_1\cos x + C_2\sin x)$

$y(x) = C_1\cos x + C_2\sin x$ — 1

using initial condition $y(0) = 3$

$y(0) = C_1\cos(0) + C_2\sin(0)$

$\boxed{3 = C_1}$ Put in 1

$y(x) = 3\cos x + C_2\sin x$ — 2

taking derivative

$y'(x) = -3\sin x + C_2\cos x$

Now using initial condition $y'(0) = -0.5$

$y'(0) = -3\sin(0) + C_2\cos(0)$

$\boxed{-0.5 = C_2}$ put in 2

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$y(x) = 3\cos x - 0.5 \sin x$.

Reduce to first order and solve

(i) $y'' + y' = 0$ — ①

Sol:

First we need to reduce the eq. to first order.

Let $z = y' \Rightarrow z' = y''$

① $\Rightarrow$ $z' - z = 0$

$$\frac{dz}{dx} = z$$

$$\frac{dz}{z} = dx$$

Now, variables are separated, integrating both side.

$$\int \frac{dz}{z} = \int 1 \, dx$$

$$\ln|z| = x + c$$

$$e^{\ln|z|} = e^{x+c}$$

$$z = e^{x} \cdot e^{c}$$

$$z = c_1 e^{x} \qquad \because e^{c} = c_1$$

$$\Rightarrow y' = c_1 e^{x} \qquad \because z = y'$$

$$\frac{dy}{dx} = c_1 e^{x}$$

$$dy = c_1 e^{x} dx$$

Variables are separated again, integrating both sides

$$\int 1 \, dy = c_1 \int e^{x} dx$$

$$\boxed{y = c_1 e^{x} + c_2}$$