ELEMENTS OF POWER SYSTEM ANALYSIS

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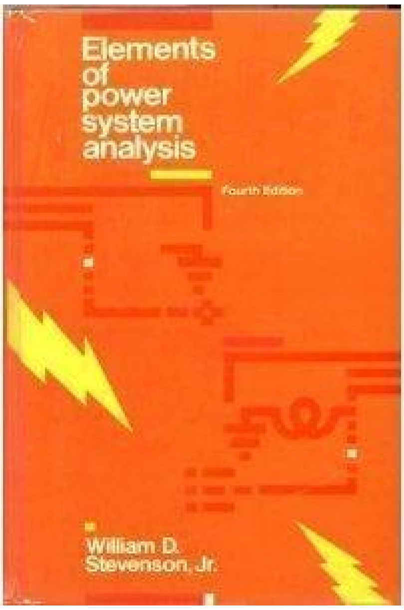

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PREFACE

This book originated from a set of mimeographed notes which the author prepared for a course taught to senior electrical engineering students over a period of years. The primary purpose of the book is to present the more important problems of power system analysis in a teachable form which is theoretically sound and adequately developed for a senior or introductory graduate course. An attempt has been made to awaken the interest of the student and to acquaint him with modern practice as it pertains to the analysis under discussion.

Every author is faced with the problem of selection of subject matter to be included. In this text sufficient material has been presented to conduct a course lasting throughout an academic year. On the other hand, the book is designed so that a judicious selection of material may be made to fit the text to courses of two quarters or one semester. The author has attempted to treat the various subjects so that many sections may be omitted without the loss of continuity and without handicapping the student. For several years the author has experimented with the omission of various parts of the text and hopes that others will communicate to him the results of any selection of material they find advantageous.

An attempt has been made to make the text more teachable by the gradual introduction of some material that might otherwise prove troublesome to the student. For instance, per-unit computations are introduced on a small scale in the chapter on generalized circuit constants and again in the development of a universal circle diagram before they are treated in detail sufficient for their exclusive use in fault calculations and stability problems. The text contains a large number of illustrative examples showing the details of the solution of almost every type of problem. The examples should be considered a part of the body of the text. Many explanations are incorporated in the solution of a problem. Students will find that a careful study of the examples is both necessary and profitable.

The large number of footnotes should encourage the student to supplement his work by additional reading. The footnotes are also intended as an acknowledgment of some of the sources to which the author is indebted for many of the ideas presented. The generosity of many companies in furnishing information can be inferred from the acknowledgments appearing in the text. The author is indebted to Professor C. H. Willis of Princeton University for persuading him to undertake this work, to the late Professor C. G. Brennecke, former Head of the Department of Electrical Engineering at North Carolina State College, for his unfailing words of encouragement, and to Professor Arthur R. Eckels of North Carolina State College for the many valuable suggestions he offered during the year he taught from the mimeographed notes. Many students over a period of years have taken an active interest in the work, and their suggestions have been helpful. The author especially wishes to acknowledge the constant encouragement received at every step of his teaching career from Dr. Webster N. Jones, Vice-President of Carnegie Institute of Technology.

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ELEMENTS OF POWER SYSTEM ANALYSIS

CHAPTER 1

GENERAL BACKGROUND

1.1 The Function of Electric Power Systems. The degree of development of sources of energy to accomplish useful work is one of the measures of industrial progress. The discovery of sources of energy in nature, the transportation of energy in its various forms from one place to another, and the conversion of energy to a more serviceable form are essential parts of an industrial economy. An electric power system is one of the tools of converting and transporting energy.

The only means of transporting energy in the form of electricity is over transmission lines. Gas is transported by pipelines. Railroads, ships, and pipelines carry oil over long distances. Coal is shipped long distances by rail and water. When coal is the primary source of electric energy, the electric transmission line becomes a competitor of railroads and ships in transporting energy. The choice of location of a steam generating station near a coal mine or near a load center, provided there is a good water supply at both places, may depend upon the difference in cost of transmitting electric energy and transporting coal from the mine to the load. Pipelines are increasing rapidly and are becoming a major competitor of the electric transmission line by providing low-cost transportation of energy. Hydroelectric power is inexpensive only if the cost of its transmission is low. The economy of transporting energy in one form instead of another is influenced by whether the demand for the energy is continuous or intermittent, by the distance involved, and by the cost and practicability of storage facilities. The determining factor is the final cost, including transportation charges, of the energy in the desired form.[1] The enormous growth of electric power systems since World War II testifies to the economic soundness of such

An electric power system is especially advantageous in the development of water power. Water power must be converted at the site where it is available, and an electric power system makes the energy derived

[1] For a comparison of the cost of transporting energy in the form of coal, oil, gas, and electricity, see R. E. Pierce and E. E. George, "Economy of Long-Distance Energy Transmission," *Trans. AIEE*, vol. 67, pp. 1089–1094, 1948.

from water power available at remote points. The water power is converted to electric power at the source and transported by transmission lines to the point where it is converted to the desired form, such as light, heat, mechanical energy, or chemical energy. The transmission line cannot store energy, and all the energy furnished at the generating station is converted simultaneously at the load, except for the losses in the system.

An electric power system consists of three principal components: the generating stations, the transmission lines, and the distribution systems. The transmission lines are the connecting links between all the generating stations and the distribution systems. A distribution system connects all the individual loads in a given area to the transmission lines. A well-developed power system integrates a large number of generating stations so that their combined output is readily available throughout the region served. The locations of hydro stations are fixed by the presence of water power, but the choice of sites for steam stations is more flexible. Steam stations are usually spotted throughout the system so that there is at least one generating plant near each large load center. Thus, hydro stations often require the transmission of large amounts of power over long distances, but steam plants usually require transmission over shorter distances. The growth of loads may not be under the control of the power company, but often the availability of cheap power encourages the growth of loads in such favored areas. One job of the power engineer is to predict the future demand for power so that suitably located generating stations and well-coordinated, flexible, and reliable transmission systems will be ready to supply the demand through enlarged distribution systems as required by the load. As the system grows, more energy sources must be exploited to satisfy the increasing demand, and more transmission lines must be built to link the new generating stations to each other, to an increasing number of distribution points, and to other power systems.[2]

1.2 The Growth of Electric Power Systems. The development of a-c systems began in the United States in 1885 when George Westinghouse bought the American patents covering the a-c transmission system developed by L. Gaulard and J. D. Gibbs of Paris. William Stanley, an early associate of Westinghouse, tested transformers in his laborat[illegible] in Great Barrington, Massachusetts. There, in the winter o[illegible], Stanley installed the first experimental a-c distributi[illegible] which supplied 150 lamps in the town. The first a-c tr[illegible]e in the United States was put into operation in 1890 to [illegible]ectric energy

[2] For a description of the development of a large [illegible] power system, see P. Sporn, "The Integrated Power System," McGraw-Hill [illegible] Company, [illegible] New York, 1950.

generated by water power a distance of 13 miles from Willamette Falls to Portland, Oregon.[3]

The first transmission lines were single-phase, and the energy was usually consumed for lighting only. Even the first motors were single-phase, but on May 16, 1888, Nikola Tesla presented a paper describing two-phase induction and synchronous motors.[4] The advantages of polyphase motors were apparent immediately, and a two-phase a-c distribution system was demonstrated to the public at the Columbian Exposition in Chicago in 1893. Thereafter, the transmission of electric energy by alternating current, especially three-phase alternating current, gradually replaced d-c systems. In January, 1894, there were five polyphase generating plants in the United States, of which one was two-phase and the others three-phase.[5]

One reason for the early acceptance of a-c systems was the transformer, which makes possible the transmission of electric energy at a voltage higher than the voltage of generation or utilization. A higher voltage of transmission requires less line current for the transmission of a given amount of power and, therefore, results in lower I^2R losses in the line. An a-c generator is a simpler device than a d-c generator, and this is an additional advantage of a-c systems.

Although most of the electric energy consumed in the United States is transmitted as alternating current, experiments have been carried on for a number of years in this country on a system composed of a-c generators feeding a d-c transmission line through a transformer and an electronic rectifier. In this system an electronic inverter changes the direct current to alternating current at the end of the line so that the voltage can be reduced by a transformer. Direct-current transmission has been more popular in Europe, and most of the recent literature on d-c transmission has been published in Germany, England, and Russia. Direct-current transmission overcomes some of the disadvantages of a-c systems, as will become apparent as the characteristics of a-c systems are studied. The disadvantage of elaborate inverting and rectifying equipment makes d-c

[3] Much interesting material about the early development of electric equipment and apparatus can be found in the volumes of *Transactions of the American Institute of Electrical Engineers* for the period. For instance, a good description of the Willamette-Portland line is given in C. F. Scott, "Long Distance Transmission for Light and Power," *Trans. AIEE*, vol. 9, pp. 425–442, 1892. For a book describing the early discoveries and developments which gave impetus to the electrical industry, see M. MacLaren, "The Rise of the Electrical Industry during the Nineteenth Century," Princeton University Press, Princeton, N.J., 1943.

[4] See Nikola Tesla, "A New System of Alternating-current Motors and Transformers," *Trans. AIEE*, vol. 5, pp. 309–324, 1888.

[5] See Louis Bell, "Practical Properties of Polyphase Apparatus," *Trans. AIEE*, vol. 9, p. 27, 1894.

transmission less economical than a-c systems for distances less than 450 or 500 miles, and there is some doubt whether d-c transmission will ever be as reliable as alternating current.

The Federal Power Commission publishes monthly reports on various aspects of the generation and transmission of electric energy. The first of a continuous series of annual reports giving data collected by the Federal Power Commission appeared in 1920. Table 1.1 gives statistics on the total installed capacity of generators and on the annual production

TABLE 1.1 INSTALLED ELECTRICAL CAPACITY AND ANNUAL PRODUCTION OF ELECTRIC ENERGY IN THE UNITED STATES*

Year	Installed capacity, kw	Annual energy production, kwhr
1920	12,713,608	39,404,639,000
1930	32,384,363	91,111,548,000
1940	39,926,881	141,837,010,000
1950	68,919,040	329,141,343,000

* Source: Federal Power Commission.

of electric energy in the United States at ten-year intervals since 1920. Although these statistics record the growth of power systems in the first half of the twentieth century, statistics alone do not show the impact of the two world wars on the electrical industry. World War I revealed the need for interconnection of power systems operating on a standard frequency in order to furnish larger blocks of power than were available from individual systems. Both wars dramatized the role of electricity in building military power, and both were followed by a greater demand for electric energy. Prior to World War II, the greatest net increase in one year in the installed capacity of generating stations was 3,791,000 kw, in 1925. This figure was not surpassed until 1948, when the net increase in one year was 4,237,831 kw. Since then, statistics on yearly growth indicate a doubling of installed capacity every ten years. Annual energy production is also expected to double every ten years and reach 1 trillion kwhr in 1965.

In the early days of a-c power transmission in the United States, the operating voltage increased rapidly. In 1890 the Willamette-Portland line was operated at 3,300 volts. In 1907 a line was operating [illegible] Voltage rose to 150 kv in 1913, 220 kv in 1923, 244 kv [illegible], and 287 kv on the line from Hoover Dam to Los Angeles, which began service in 1936. In 1952, construction was completed on a small portion of a 330-kv system of the American Gas and Electric Company.[6] The choice

[6] See P. Sporn, E. L. Peterson, I. W. Gross, and H. P. [illegible]lair, "The [illegible] kv Extra-high-voltage Transmission System of the American Gas and Electric Company," *Trans. AIEE*, vol. 70, pp. 64–72, 1951.

of line voltage is principally a matter of balancing the initial investment in line construction and apparatus with the cost of operation. Up to a certain point, increasing the voltage results in lower losses for a given size of conductor or in a smaller conductor for a given power loss. Much of the saving achieved in conductor cost by designing for higher voltage is lost because of the increased loss in the surrounding air, which is ionized by the high voltage gradient at the wire, and because of the increased cost of insulators, transformers, switches, and circuit breakers. The cost of the latter items increases so rapidly at the higher voltages that some maximum voltage exists above which it is not economical to design transmission lines at present. Radio influence is also a factor affecting the selection of voltage. The final determination of a system voltage of 330 kv for the above-mentioned line resulted from studies of tests on a 500-kv experimental line near the Tidd station of the Ohio Power Company. The American Gas and Electric Service Corporation, in cooperation with eight manufacturers of high-voltage equipment, obtained data on insulators, line conductors, switchgear, transformers, lightning arresters, instruments, radio influence, and the effects of atmospheric ionization at voltages up to 500 kv. Through such tests and through experience in operating a 600-mile, 400-kv line in Sweden, the economic and technical limitations of high-voltage transmission are being studied.[7]

Until 1917, electric systems were usually operated as individual units because they started as isolated systems and spread out only gradually to cover the whole country. The demand for large blocks of power and increased reliability suggested the interconnection of neighboring systems. Interconnection is advantageous economically because fewer machines are required as a reserve for operation at peak loads (reserve capacity) and fewer machines running without load are required to take care of sudden, unexpected jumps in load (spinning reserve). The reduction in machines is possible because one company can usually call on neighboring companies for additional power. Interconnection also allows a company to take advantage of the most economical sources of power, and a company may find it cheaper to buy power than to generate it in an obsolete plant. Interconnection has increased to the point where power is exchanged between the systems of different companies as a matter of routine. Figure 1.1 is the map of a small transmission system. The map shows eight points of interconnection with other systems. The continued service of systems depending on water power for a large part of their generation is possible in times of unusual and extreme water

[7] See B. G. Rathsman and G. Jancke, "Experience Gained with the Swedish 400-kv Power Transmission and the Novel Features of the System," *Trans. AIEE*, vol. 72, pt. 3, pp. 1089–1095, 1953.

shortage only because of the power obtained from other systems through interconnections.

Interconnection of systems brought many new problems, most of which have been solved satisfactorily. Interconnection increases the amount of current which flows when a short circuit occurs on a system and requires the installation of breakers able to interrupt a larger current. The disturbance caused by a short circuit on one system may spread to interconnected systems unless proper relays and circuit breakers are provided at the point of interconnection. Not only must the interconnected systems have the same nominal frequency, but also the synchronous machines of one system must remain in step with the synchronous machines of interconnected systems.

Planning the operation, improvement, and expansion of a power system requires load studies, fault calculations, and stability studies. We shall consider the general nature of these types of problems and then proceed to acquire some of the fundamental concepts in the theory of transmission lines before considering these problems in detail.

1.3 Load Studies. A load study is the determination of the voltage, current, power, and power factor or reactive power at various points in an electric network under existing or contemplated conditions of normal operation. Load studies are essential in planning the future development of the system because satisfactory operation of the system depends on knowing the effects of interconnections with other power systems, of new loads, new generating stations, and new transmission lines before they are installed.

Longhand calculations of the effect of changes in a complex system are so tedious and time-consuming that an a-c calculating board is the best means of making a load study to determine the effect of contemplated changes. A calculating board is a small-scale single-phase replica of the actual system. It consists of a number of sources of a-c voltage which may be adjusted in magnitude and phase and of a number of resistances, inductances, and capacitances, all of which are adjustable. The voltage sources and circuit elements can be connected by plugging arrangements to represent the actual network by the equivalent circuits of its component parts scaled down to convenient size. Measurements made on the calculating board are easily converted, by multiplying factors, to values that would exist on the actual network, or meters [illegible] scales may be provided to read system quantities directly. Photographs of an a-c calculating board and of some of its circuit elements are shown in Chap. 8, where the boards are described in more detail.

By altering the connections of the a-c board, the effect of any change in the system is determined just as rapidly as the connections can be made and the meters read. For instance, capacitors are often placed in

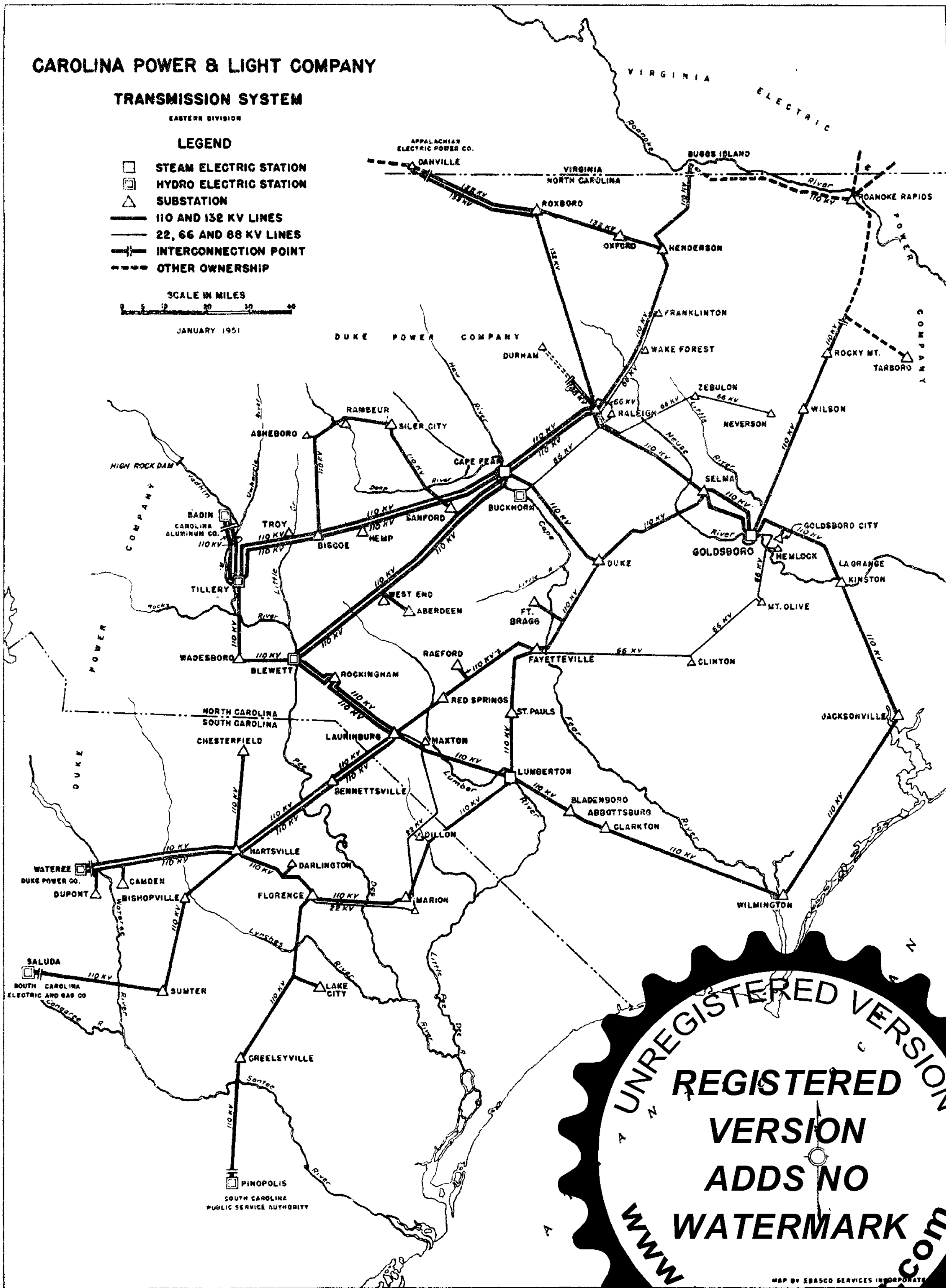


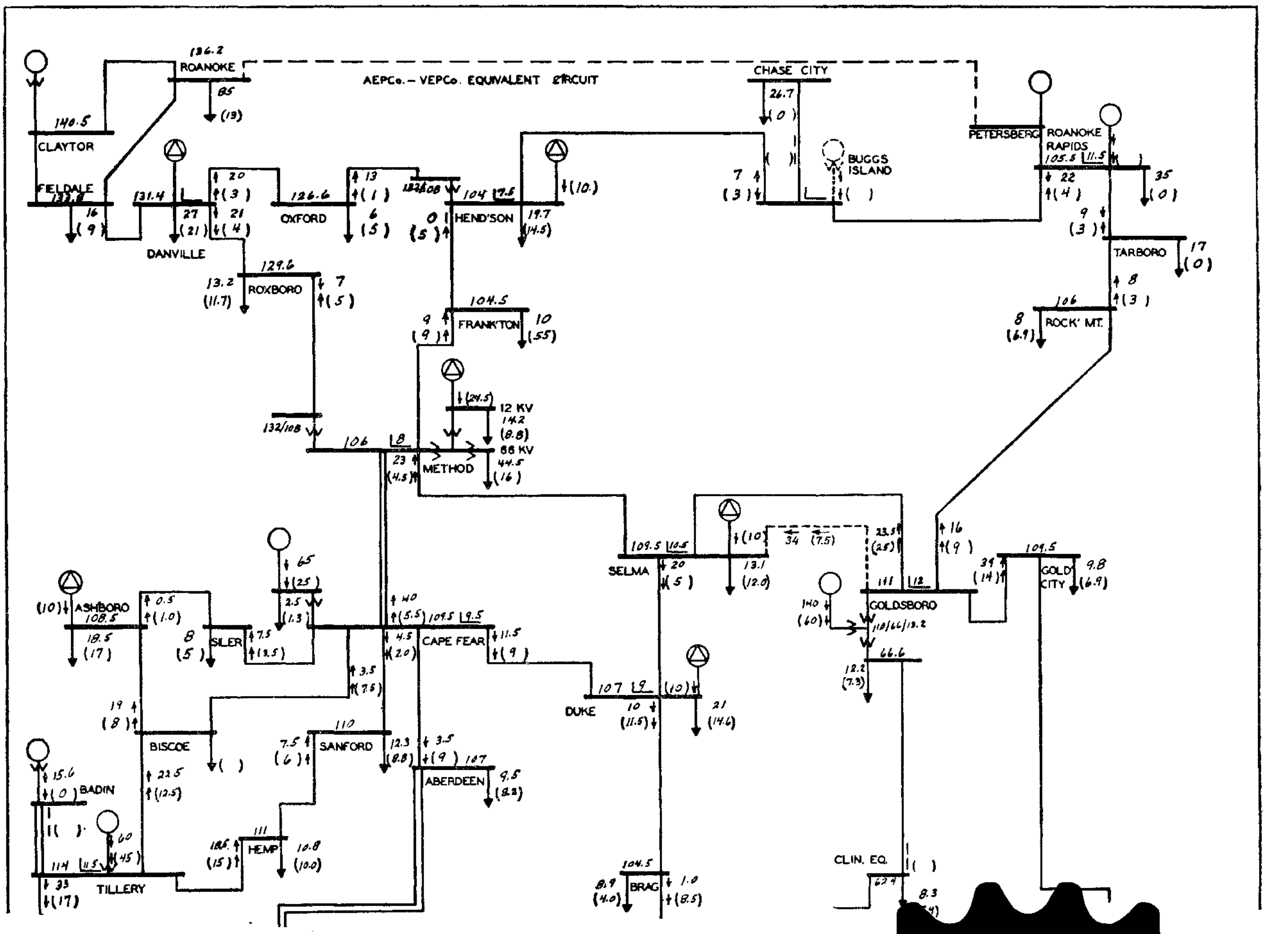
FIG. 1.1 Map of a transmission system.

parallel with a load or at various points along a line in order to improve the power factor and thereby decrease the current drawn by a load having a low lagging power factor. The installation of a capacitor of proper size at the proper location will often raise the voltage of that part of the system. If the voltage is low at some point in a power system, a load study can be made on an a-c calculating board to determine the best size and most favorable location for the capacitor. This is done by reading the voltages on the replica of the system for a number of different capacitor sizes and locations. A load study concerned with the improvement of voltage may indicate that building an additional transmission line somewhere in the system is the best solution to the problem. Load studies serve to determine the best location for a proposed new generating station or substation and to determine the best location for new lines or synchronous condensers.

The results of a load study made on an a-c calculating board for the system of Fig. 1.1 are shown in Fig. 1.2. At the time the study was made, a new generating plant was under construction at Goldsboro, and only one 110-kv line was available to carry power west from Goldsboro. The study was made to compare the operation of a proposed additional 110-kv line from Goldsboro to the Selma substation with a proposed line from Goldsboro to the Duke substation. Figure 1.2 is a diagram of the system with each transmission line represented by a single line. Substations are identified by name, and generating stations and synchronous condensers are indicated by symbols. Data obtained from the calculating board are recorded on the diagram. Voltages in kilovolts are recorded at some substations, and the numbers beside the arrows show the flow of real and reactive power in megawatts and megavars. Positive reactive power is the power drawn by an inductive load. The figures for reactive power are enclosed in parentheses. The data of Fig. 1.2 were taken with the board arranged to represent contemplated future loads with a new 110-kv line from Goldsboro to Selma. The new line is shown by dashes.

The study was repeated with the new line from Goldsboro to Duke instead of to Selma, and a portion of the results is shown in the diagram of Fig. 1.3. The studies show that the new line carries more load when routed from Goldsboro to Selma than when routed to Duke and that the difference between the two routings is slight in the rest of the system. In addition, the distance from Goldsboro to Selma is shorter than to Duke, and the right of way to Selma was already owned by the company. As a result of the load study, the new line was built from Goldsboro to Selma, as shown on the map of Fig. 1.1.

Load studies on a calculating board are valuable not only for planning additions to a system but also for determining the best operating procedure for the existing system. As the load on a power system changes


AEPCo. – VEPCo. EQUIVALENT CIRCUIT
ROANOKE
CLAYTOR
FIELDALE
DANVILLE
OXFORD
HEND'SON
CHASE CITY
BUGGS ISLAND
PETERSBERG
ROANOKE RAPIDS
TARBORO
ROCK' MT.
ROXBORO
FRANKTON
12 KV
66 KV
METHOD
SELMA
GOLDSBORO
GOLD CITY
ASHBORO
SILER
CAPE FEAR
BISCOE
SANFORD
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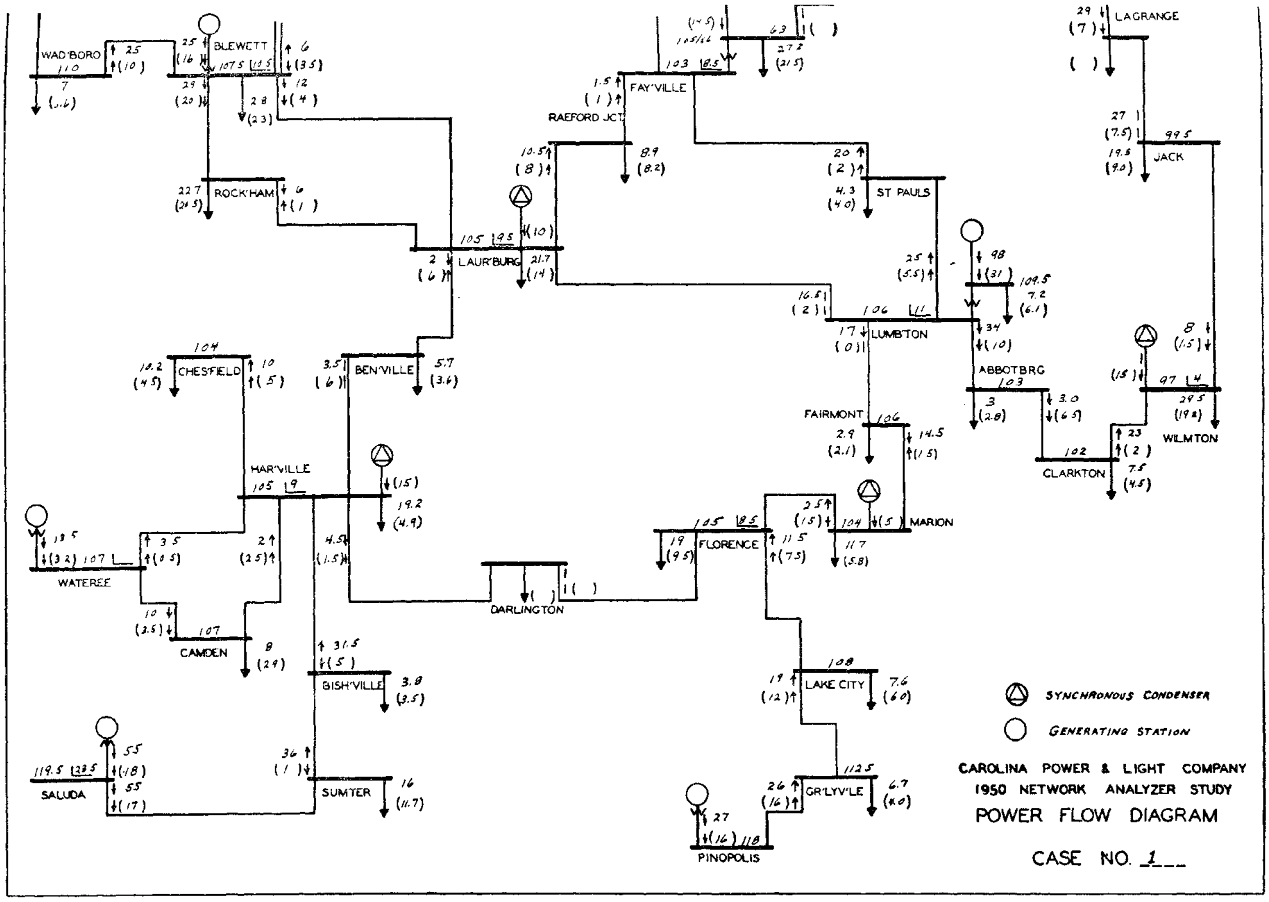


FIG. 1.2 Results of a load study made on an a-c calculating board for the system of Fig. 1.1 to show the effect of a proposed line (shown by dashes) from Goldsboro to Selma. Numbers beside the arrows show the flow of real and reactive power in megawatts and megavars (reactive power in parentheses). Numbers at each bus indicate the voltage in kilovolts.

throughout the day or from day to day, the system dispatcher must know from which generating stations to supply the load so as to obtain the best voltage regulation and the most economical operation. Operating schedules are prepared after making load studies. Load studies can also be made to determine the best operating procedure in the event of the loss of one or more generating stations or transmission lines.

1.4 Fault Calculations. The American Institute of Electrical Engineers defines a fault in a wire or cable as follows: "A wire or cable fault is a partial or total failure in the insulation or continuity of a conductor."[8] Most faults on transmission lines of 115 kv and higher are

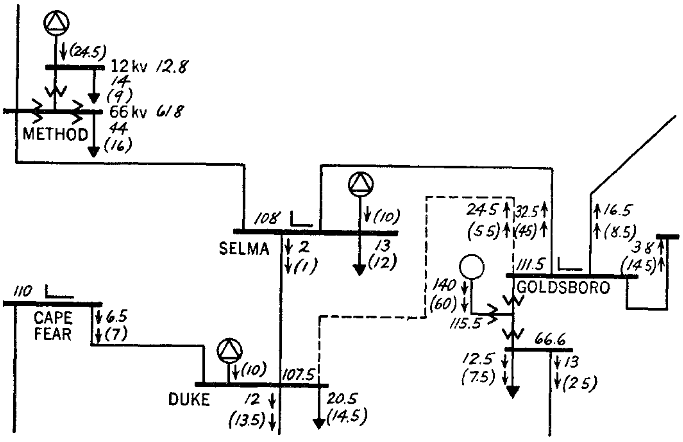


FIG. 1.3 Section of an a-c calculating-board study for the system of Fig. 1.1 to show the effect of a proposed line (shown by dashes) from Goldsboro to Duke. Numbers beside the arrows show the flow of real and reactive power in megawatts and megavars (reactive power in parentheses). Numbers at each bus indicate the voltage in kilovolts.

caused by lightning which results in the flashover of insulators. The high voltage between a conductor and the grounded supporting tower causes ionization which provides a path to ground for the charge induced by the lightning stroke. Once the ionized path to ground is established, the resultant low impedance to ground allows the flow of power current from the conductor to ground and through the ground to the grounded neutral of a transformer or generator, thus completing the circuit. Line-to-line faults not involving ground are less common. The operation of circuit breakers to isolate the faulted portion of the line from the rest of the system interrupts the flow of current in the ionized path and allows deionization to take place. After an interval of about 20 cycles to allow deionization, breakers can usually be reclosed without the reestablish-

[8] "American Standard Definitions of Electrical Terms," 35.40.213, American Institute of Electrical Engineers, New York, 1942.

ment of the arc. A report by Sporn and Muller[9] of nine years' experience with 1,634 circuit miles of line, most of which operated at 132 kv, shows a total of 635 cases of flashover, of which 570 were successfully reclosed by ultrahigh-speed reclosing breakers. Of the 635 cases, eight proved to be permanent faults where successful reclosure would have been impossible. The permanent faults were caused by lines being on the ground, by insulator strings breaking because of ice loads, by permanent damage to towers, and by lightning-arrester failures. Experience has shown that between 70% and 80% of transmission-line faults are single line-to-ground faults, which arise from the flashover of only one line to the tower and ground. The smallest number of faults, roughly 5%, involve all three phases and are called three-phase faults. Other types of transmission-line faults are line-to-line faults, which do not involve ground, and double line-to-ground faults. All the above faults except the three-phase type are unsymmetrical and cause an unbalance between the phases.

The current which flows in different parts of a power system immediately after the occurrence of a fault differs from that flowing a few cycles later just before circuit breakers are called upon to open the line on both sides of the fault, and both these currents differ widely from the current which would flow under steady-state conditions if the fault were not isolated from the rest of the system by the operation of circuit breakers. Two of the factors upon which the proper selection of circuit breakers depends are the current flowing immediately after the fault occurs and the current which the breaker must interrupt. Fault calculations consist of determining these currents for various types of faults at various locations in the system. The data obtained from fault calculations also serve to determine the settings of relays which control the circuit breakers.

For simple systems, analytic calculations of fault currents are practical, but for the more complex systems the engineer must call upon the calculating board. If great accuracy is not required and the system can be assumed to be composed of purely inductive reactances or of impedances of nearly equal phase angles only, a d-c calculating board with resistances replacing the inductive reactances can be used instead of the more costly a-c board.

Analysis by symmetrical components is a powerful tool which we shall study later and which makes the calculation of unsymmetrical faults almost as easy as the calculation of three-phase faults. A knowledge of symmetrical components is necessary whether the fault calculations are carried out analytically or on a calculating board.

1.5 Stability Studies. The current which flows in an a-c generator or synchronous motor depends on the magnitude of its generated, or

[9] P. Sporn and C. A. Muller, "Nine Years' Experience with Ultrahigh-speed Reclosing of High-voltage Transmission Lines," *Trans. AIEE*, vol. 64, pp. 225–228, 1945.

internal, voltage, on the phase angle of its internal voltage with respect to the phase angle of the internal voltage of every other machine in the system, and on the characteristics of the network and loads. For example, two a-c generators operating in parallel but without any external circuit connections other than the paralleling circuit will carry no current if their internal voltages are equal in magnitude and in phase. If their internal voltages are equal in magnitude but different in phase, the voltage of one subtracted from the voltage of the other will not be zero, and a current will flow, as determined by the difference in voltages and the impedance of the circuit. One machine will supply power to the other, which will run as a motor rather than as a generator.

The phase angles of the internal voltages depend upon the relative positions of the rotors of the machines. If synchronism were not maintained among the generators of a power system, the phase angles of their internal voltages would be changing constantly with respect to each other, and satisfactory operation would be impossible.

The phase angles of the internal voltages of synchronous machines remain constant only as long as the speeds of the various machines remain constant at the speed which corresponds to the frequency of the reference phasor.[10] When the load on any one generator or on the system as a whole changes, the current in the generator or throughout the system changes. If the change in current does not result in a change in magnitude of the internal voltages of the machines, the phase angles of the internal voltages must change. Thus, momentary changes in speed are necessary to obtain adjustment of the phase angles of the voltages with respect to each other, since the phase angles are determined by the relative positions of the rotors. When the machines have adjusted themselves to the new phase angles, or when some disturbance causing a momentary change in speed has been removed, the machines must operate again at synchronous speed. If any machine does not remain in synchronism with the rest of the system, large circulating currents result, and, in a properly designed system, the operation of relays and circuit breakers removes the machine from the system. The problem of stability is the problem of maintaining the synchronous operation of the generators and motors of the system. Power system engineers have devoted much thought and effort to stability studies since about 1925.[11]

Stability studies are classified by whether they involve steady-st[illegible] or

[10] Phasors are often called vectors and are the coplanar direct[illegible] which symbolically represent sine functions. Phasors are the graphical representations of the complex expressions of voltage and current.

[11] See for instance AIEE Subcommittee on Interconnection and Stability Factors, "First Report of Power System Stability," *Elec. Eng.*, v[illegible] pp. 261–282, February, 1937.

transient conditions. There is a definite limit to the amount of power an a-c generator is capable of delivering and to the load which a synchronous motor can carry. Instability results from attempting to increase the mechanical input to a generator or the mechanical load on a motor beyond this definite amount of power called the *stability limit.* A limiting value of power is reached even if the change is made gradually. Disturbances on a system, caused by suddenly applied loads, by the occurrence of faults, by the loss of excitation in the field of a generator, and by switching, may cause loss of synchronism, even if the change in the system caused by the disturbance would not exceed the stability limit if the change were made gradually. The limiting value of power is called the *transient stability limit* or the *steady-state stability limit* according to whether the point of instability is reached by a sudden or a gradual change in conditions of the system.

Fortunately, engineers have found methods of improving stability and of predicting the limits of stable operation under both steady-state and transient conditions. Stability studies of a two-machine system are less complex than studies of multimachine systems, but many of the methods of improving stability can be seen by the analysis of a two-machine system. The a-c calculating board is a great help in predicting the stability limits of a complex system and in comparing various methods of increasing stability, but the same calculations must be made for each machine represented on the board as are made for the machines in a simpler system which is more suited to analytic calculations.

1.6 The Power System Engineer. This chapter has attempted to sketch some of the history of the basic developments of electric power systems and to describe some of the analytic studies which are important in planning the operation, improvement, and expansion of a modern power system. The power system engineer should know the methods of making load studies, fault analyses, and stability studies, for such studies affect the design and operation of the system and the selection of apparatus for its control. Before we can consider these problems in more detail, we must study some fundamental concepts relating to power systems in order to appreciate how these fundamental concepts affect the larger problems.

CHAPTER 2

INDUCTANCE OF TRANSMISSION LINES

2.1 Introduction. An electric transmission line has four parameters which affect its ability to fulfill its function as part of a power system. These parameters are resistance, inductance, capacitance, and conductance. In this chapter we shall study inductance, and we shall consider the other parameters in the two following chapters.

When current flows in an electric circuit we explain some of the properties of the circuit by the magnetic and electric fields which are present. Figure 2.1 shows an open two-wire line and its associated magnetic and electric fields. The lines of magnetic flux form closed loops linking the circuit, and the lines of electric flux originate on the positive charges on one conductor and terminate on the negative charges on the other conductor. Variation of the current in the conductors causes a change in the number of lines of magnetic flux linking the circuit. Any change in the flux linking a circuit induces a voltage in the circuit, and the induced voltage is proportional to the rate of change of flux. Inductance is the property of the circuit that relates the voltage induced by changing flux to the rate of change of current.

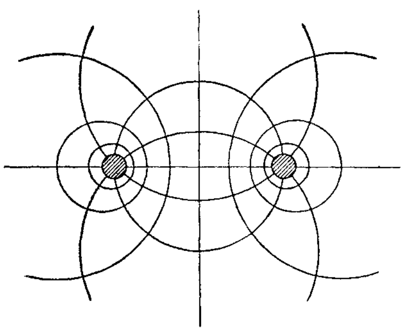


FIG. 2.1 Magnetic and electric fields associated with a two-wire line.

2.2 Definition of Inductance. Two fundamental equations serve to explain and define inductance. The first equation relates induced voltage to the rate of change of flux linking a circuit. The induced voltage is

$$e = \frac{d\psi}{dt} \tag{2.1}$$

where e is the induced voltage in volts and ψ is the number of *flux linkages* of the circuit in weber-turns. The number of weber-turns is the product of each weber of flux and the number of turns of the circuit linked. For the two-wire line of Fig. 2.1 each line of flux links the circuit only once, and 1 volt is induced if the rate of change of flux is 1 weber/sec. If we had been considering a coil instead of the circuit of Fig. 2.1, most of the lines of flux produced would have linked more than one turn of the coil. If the flux linking 100 turns of a coil changed at the rate of 1 weber/sec, the induced voltage in each turn would be 1 volt, but the induced voltage in the coil would be 100 volts since the turns are in series. Therefore, the induced voltage is proportional to the rate of change of flux *linkages*.

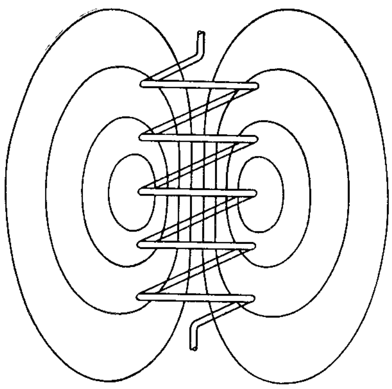


FIG. 2.2 Flux linking a coil.

A coil having five turns is shown in Fig. 2.2. The closed loops represent some of the magnetic flux linking the turns of the coil. Two of these loops are seen to link only one turn of the coil. They contribute a total of two flux linkages. Two other loops link three turns and therefore contribute six flux linkages. Four loops link all five turns to give twenty flux linkages. Thus, for the loops shown, there are $2 + 6 + 20 = 28$ flux linkages. If each loop or line of flux represents 1 weber, the unit of flux linkages is a weber-turn, and the coil has 28 weber-turns. Decreasing this flux to zero at a uniform rate in 1 sec would induce 28 volts in the coil.

When the current in a circuit is changing, its associated magnetic field (which is described by the flux linkages) must be changing. If constant permeability is assumed for the medium in which the magnetic field is set up, the number of flux linkages is directly proportional to the current, and therefore the induced voltage is proportional to the rate of change of current. Thus our second fundamental equation is

$$e = L\frac{di}{dt} \qquad \text{volts} \tag{2.2}$$

where L, the constant of proportionality, is the inductance of the circuit in henrys, e is the induced voltage in volts, and di/dt is the rate of change of current in amperes per second. Equation (2.2) may be used where the permeability is not constant, but in such a case the inductance is not a constant.

When Eqs. (2.1) and (2.2) are solved for L, the result is

$$L = \frac{d\psi}{di} \qquad \text{henrys} \tag{2.3}$$

If the flux linkages of the circuit vary linearly with current, which means the magnetic circuit has a constant permeability,

$$L = \frac{\psi}{i} \qquad \text{henrys} \tag{2.4}$$

from which arises the definition of the self-inductance of an electric circuit as the flux linkages of the circuit per unit of current. The inductance of one conductor of a circuit is equal to the flux linkages of the conductor per unit current in the conductor. In a two-wire line the number of flux linkages of the circuit is the sum of the flux linkages of each wire. In the rationalized mks system of units, L in henrys is equal to weber-turns per ampere. In terms of inductance the flux linkages are

$$\psi = Li \qquad \text{weber-turns} \tag{2.5}$$

In Eq. (2.5), if i is instantaneous current, ψ represents instantaneous flux linkages. When the current is alternating, the flux linkages are alternating, and the rms value of the flux linkages is the product of the inductance and the rms current. Thus

$$\psi_{\text{rms}} = LI \qquad \text{weber-turns}^1 \tag{2.6}$$

The rms voltage drop due to the flux linkages is

$$V = j\omega LI \qquad \text{volts} \tag{2.7}$$

$$V = j\omega\psi_{\text{rms}} \qquad \text{volts} \tag{2.8}$$

Mutual inductance between two circuits is defined as the flux linkages of one circuit due to the current in the second circuit per ampere of current in the second circuit. If the current I_2 produces ψ_{12} flux linkages with circuit 1, the mutual inductance is

$$M_{12} = \frac{\psi_{12}}{I_2} \qquad \text{henrys}$$

The voltage drop in circuit 1 caused by the flux linkages of circuit 2 is

$$V_1 = j\omega M_{12} I_2 = j\omega\psi_{12} \qquad \text{volts}$$

Mutual inductance is important in considering the influence of power lines on telephone lines and in considering the coupling between parallel power lines.

2.3 Partial Flux Linkages. Only flux lines external to the conductors have been shown in Fig. 2.1. Some of the magnetic field [illegible] the conductors although the amount of internal flux [illegible] so small that it can be neglected at high frequencies, as we sh[illegible] Chap. 4.

[1] We shall designate instantaneous currents and voltages [illegible] lower-case letters and rms values by capital letters. Maximum values will be de[illegible]nated by capital letters with the subscript m or max. Thus the instantaneous fl[illegible]kages are [illegible] weber-turns, the maximum flux linkages are LI_m, and the rms value i[illegible] LI.

The changing lines of flux inside the conductors also contribute to the induced voltage of the circuit and, therefore, to the inductance. The correct value of inductance due to internal flux may be computed as the ratio of flux linkages to current by taking into account the fact that each line of internal flux links only a fraction of the total current. The flux linkages of the internal flux in a tubular element are the product of flux in the element and the ratio of the current encircled by the tubular element to the total current in the conductor. Thus a line of flux which encircles only half the current in a conductor contributes only half a flux linkage. Partial flux linkages are those linkages produced by flux which links only part of the current. The total number of flux linkages due to internal flux is the summation of all the partial linkages. The summation of all the partial flux linkages in weber-turns divided by the current in the circuit in amperes is the inductance in henrys due to internal flux.

The principle outlined above for computing inductance is applicable to inductance resulting from external as well as internal flux. By this principle inductance is defined as flux linkages per ampere, and the value of flux linkages is the summation of flux times the fraction of the total current linked. The fraction is less than one for lines of flux inside the conductor and greater than one for flux surrounding several turns of a coil. For Fig. 2.2 the fractions of current linked by the lines of flux shown are 1, 3, and 5. The method of computing flux linkages by multiplying each line of flux by the fraction of current enclosed should become increasingly clear as the topic of inductance is developed further.

We shall show later that the method of partial flux linkages is valid for computing the internal inductance of a cylindrical wire by deriving internal inductance in another manner and comparing the results of the two methods.

2.4 Inductance of a Conductor Due to Internal Flux. In order to obtain an accurate value for the inductance of a transmission line, it is necessary to consider the flux inside each conductor as well as the external flux. Let us consider the long, cylindrical conductor whose cross section is shown in Fig. 2.3. We will assume that the return path for the current in this conductor is so far away that it does not appreciably affect the magnetic field of the conductor shown. Then the lines of flux are concentric with the conductor.

The magnetomotive force (mmf) in ampere-turns around [illegible] path is equal to the current in amperes enclosed by the [illegible] The mmf is also equal to the integral of the tangential compone[illegible] magnetic field intensity around the path.[2] Thus

$$\text{mmf} = \oint H \cdot ds = I \qquad \text{amp[illegible]s} \tag{2.9}$$

[2] See for instance W. H. Timbie and V. Bush, "Princip[illegible] Electric[illegible]," 4th ed., pp. 428-432, John Wiley & Sons, Inc., New Yor[illegible].

where H is the magnetic field intensity in ampere-turns per meter, s is the distance along the path in meters, I is the current in amperes enclosed,[3] and the dot between H and ds indicates that the value of H is the component of the field intensity tangent to ds.

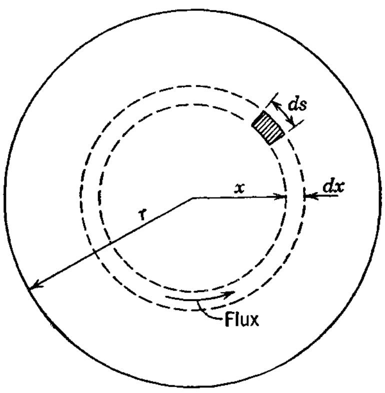


FIG. 2.3 Cross section of a cylindrical conductor.

Let the field intensity at a distance x meters from the center of the conductor be designated H_x. Since the field is symmetrical, H_x is constant at all points equidistant from the center of the conductor. If the integration indicated in Eq. (2.9) is performed around a circular path concentric with the conductor at x meters from the center, H_x is constant over the path and tangent to it. Equation (2.9) becomes

$$\oint H_x\, ds = I_x \tag{2.10}$$

and

$$2\pi x H_x = I_x \tag{2.11}$$

where I_x is the current enclosed. Then, assuming uniform current density,

$$I_x = \frac{\pi x^2}{\pi r^2} I \tag{2.12}$$

where I is the total current in the conductor. Then substituting Eq. (2.12) in Eq. (2.11), we obtain

$$2\pi x H_x = \frac{x^2}{r^2} I \tag{2.13}$$

and

$$H_x = \frac{x}{2\pi r^2} I \qquad \text{amp-turns/meter} \tag{2.14}$$

The flux density x meters from the center of the conductor is

$$B_x = \mu H_x = \frac{\mu x I}{2\pi r^2} \qquad \text{webers/meter}^2 \tag{2.15}$$

where μ is the permeability of the conductor.[4]

[3] If the current is alternating, the maximum value of H is found if the maximum value of the current is used in Eq. (2.9). Similarly, if I is rms current, H is the rms field intensity, and flux computed from rms H is the rms value of flux. The equation is applicable to direct current or instantaneous, maximum, or rms values of alternating current.

[4] In the rationalized mks system of units the permeability of free space is

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry/meter}$$

and the relative permeability μ_r is μ/μ_0.

In the tubular element of thickness dx, the flux $d\phi$ is B_x times the cross-sectional area of the element normal to the flux lines, the area being dx times the axial length. The flux per meter of length is

$$d\phi = \frac{\mu x I}{2\pi r^2}\,dx \qquad \text{webers/meter of length} \tag{2.16}$$

The flux linkages $d\psi$ per meter of length, which are caused by the flux in the tubular element, are the product of the flux per meter of length and the fraction of the current linked. Thus

$$d\psi = \frac{\pi x^2}{\pi r^2}\,d\phi = \frac{\mu I x^3}{2\pi r^4}\,dx \qquad \text{weber-turns/meter} \tag{2.17}$$

Integrating from the center of the conductor to its outside edge to find ψ_{int}, the total flux linkages inside the conductor, we obtain

$$\psi_{int} = \int_0^r \frac{\mu I x^3}{2\pi r^4}\,dx$$

$$\psi_{int} = \frac{\mu I}{8\pi} \qquad \text{weber-turns/meter} \tag{2.18}$$

For a relative permeability of 1, $\mu = 4\pi \times 10^{-7}$ henry/meter, and

$$\psi_{int} = \frac{I}{2} \times 10^{-7} \qquad \text{weber-turns/meter} \tag{2.19}$$

$$L_{int} = \tfrac{1}{2} \times 10^{-7} \qquad \text{henry/meter} \tag{2.20}$$

2.5 Flux Linkages between Two Points External to an Isolated Conductor. As a step in computing inductance due to flux external to a conductor, let us derive an expression for the flux linkages of an isolated conductor due only to that portion of the external flux which lies between two points distant D_1 and D_2 meters from the center of the conductor. In Fig. 2.4, P_1 and P_2 are two points of distances D_1 and D_2 from the conductor which carries a current of I amp. Since the flux paths are concentric circles around the conductor, all the flux between P_1 and P_2 lies within the concentric cylindrical surfaces which pass through P_1 and P_2. At the tubular element which is x meters from the center of the conductor the field intensity is H_x. The mmf around the element is

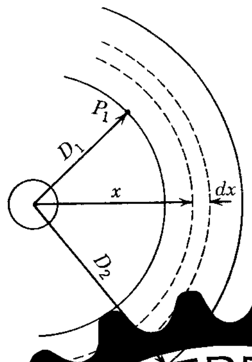


Fig. 2.4 Conductor and external points P_1 and P_2.

$$2\pi x H_x = I \tag{2.21}$$

The field intensity is

$$H_x = \frac{I}{2\pi x} \qquad \text{amp-turns/meter} \tag{2.22}$$

and the flux density in the element is

$$B_x = \frac{\mu I}{2\pi x} \qquad \text{webers/meter}^2 \tag{2.23}$$

The flux $d\phi$ in the tubular element of thickness dx is

$$d\phi = \frac{\mu I}{2\pi x}\, dx \qquad \text{webers/meter of length} \tag{2.24}$$

The flux linkages $d\psi$ per meter are numerically equal to the flux $d\phi$, since flux external to the conductor links all the current in the conductor once and only once. The total flux linkages between P_1 and P_2 are obtained by integrating $d\psi$ from $x = D_1$ to $x = D_2$. We obtain

$$\psi_{12} = \int_{D_1}^{D_2} \frac{\mu I}{2\pi x}\, dx = \frac{\mu I}{2\pi} \ln \frac{D_2}{D_1} \qquad \text{weber-turns/meter} \tag{2.25}$$

or, for a relative permeability of 1,

$$\psi_{12} = 2 \times 10^{-7} I \ln \frac{D_2}{D_1} \qquad \text{weber-turns/meter} \tag{2.26}$$

The inductance due only to the flux included between P_1 and P_2 is

$$L_{12} = 2 \times 10^{-7} \ln \frac{D_2}{D_1} \qquad \text{henrys/meter} \tag{2.27}$$

In Eqs. (2.25) to (2.27), note that "ln" denotes the natural logarithm (base ϵ).[5] Converting henrys per meter to millihenrys per mile and using the logarithm to the base 10, we obtain

$$L_{12} = 0.7411 \log \frac{D_2}{D_1} \qquad \text{millihenrys/mile} \tag{2.28}$$

2.6 Inductance of a Single-phase Two-wire Line. Before proceeding to the more general case of multiconductor lines and three-phase lines, let us consider a simple two-wire line composed of solid, round conductors. Figure 2.5 shows a circuit having two conductors of radii r_1 and r_2. One conductor is the return circuit for the other. First consider only the flux linkages of the circuit caused by the current in conductor 1. A line of flux set up by current in conductor 1 at a distance equal to or greater

[5] Throughout this book "ln" denotes the natural logarithm (base ϵ) and "log" denotes the common logarithm (base 10).

than $D + r_2$ from the center of conductor 1 does not link the circuit and cannot induce a voltage in the circuit. Stated in another manner, such a line of flux links a net current of zero, since the current in conductor 2 is equal in value and opposite in direction to the current in conductor 1. The fraction of the total current linked by a line of flux external to conductor 1 at a distance equal to or less than $D - r_2$ is one. Between $D - r_2$ and $D + r_2$ (that is, over the surface of conductor 2), the fraction of the total current in the circuit linked by a line of flux set up by current in conductor 1 varies from one to zero. Therefore, it is logical to simplify

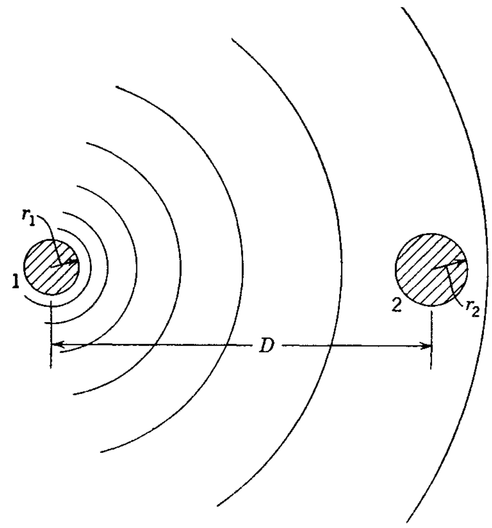


FIG. 2.5 Conductors of different radii and the magnetic field due to current in conductor 1 only.

the problem, when D is much greater than r_1 and r_2 and the flux density through the conductor is nearly uniform, by assuming that all the external flux set up by current in conductor 1 extending to the center of conductor 2 links all the current I and that flux beyond the center of conductor 2 links none of the current. In fact, it can be shown that calculations made on this assumption are correct even when D is small.[6]

The inductance of the circuit due to current in conductor 1 is determined by Eq. (2.27) with the distance D between conductors 1 a[illegible] 2 substituted for D_2 and the radius r_1 of conductor 1 substitu[illegible]. For external flux only

$$L_{1,\,ext} = 2 \times 10^{-7} \ln \frac{D}{r_1} \qquad \text{henrys/[illegible]} \qquad (2.29)$$

[6] See E. W. Kimbark, "Electrical Transmission of Po[illegible]d Signals," [illegible] 65–67, John Wiley & Sons, Inc., New York, 1949.

For internal flux only

$$L_{1,int} = \tfrac{1}{2} \times 10^{-7} \text{ henry/meter} \tag{2.30}$$

The total inductance of the circuit due to the current in conductor 1 only is

$$L_1 = \left(\frac{1}{2} + 2 \ln \frac{D}{r_1}\right) \times 10^{-7} \qquad \text{henrys/meter} \tag{2.31}$$

The expression for inductance may be put in a more concise form by factoring Eq. (2.31) and by noting that $\ln \epsilon^{1/4} = \frac{1}{4}$, whence

$$L_1 = 2 \times 10^{-7}\left(\frac{1}{4} + \ln \frac{D}{r_1}\right) \tag{2.32}$$

$$L_1 = 2 \times 10^{-7}\left(\ln \epsilon^{1/4} + \ln \frac{D}{r_1}\right) \tag{2.33}$$

Upon combining terms, we obtain

$$L_1 = 2 \times 10^{-7} \ln \frac{D}{r_1 \epsilon^{-1/4}} \tag{2.34}$$

If we substitute r_1' for $r_1\epsilon^{-1/4}$,

$$L_1 = 2 \times 10^{-7} \ln \frac{D}{r_1'} \qquad \text{henrys/meter} \tag{2.35}$$

or

$$L_1 = 0.7411 \log \frac{D}{r_1'} \qquad \text{millihenrys/mile} \tag{2.36}$$

The radius r_1' is that of a fictitious conductor assumed to have no internal flux but with the same inductance as the actual conductor of radius r_1. The quantity $\epsilon^{-1/4}$ is equal to 0.7788. Equation (2.35) gives the same value for inductance as Eq. (2.31). The difference is that Eq. (2.35) omits the term to account for internal flux but compensates for it by using an adjusted value for the radius of the conductor. We should note carefully that Eq. (2.31) was derived for a solid, round conductor and that Eq. (2.35) was found by algebraic manipulation of Eq. (2.31). Therefore, the multiplying factor of 0.7788 to adjust the radius in order to account for internal flux applies only to solid, round conductors. We shall consider other conductors later.

Since the current in conductor 2 flows in the direction opposite to that in conductor 1 (or is 180° out of phase with it), the flux lines produced by current in conductor 2 considered alone are in the same direction through the circuit as those produced by current in conductor 1. The resulting flux for the two conductors is determined by the sum of the mmfs of both conductors. For constant permeability, however, the flux linkages (and likewise the inductances) of the two conductors considered separately may be added.

By comparison with Eq. (2.35) the inductance due to current in conductor 2 is

$$L_2 = 2 \times 10^{-7} \ln \frac{D}{r_2'} \qquad \text{henrys/meter} \tag{2.37}$$

and for the complete circuit

$$L = L_1 + L_2 = 4 \times 10^{-7} \ln \frac{D}{\sqrt{r_1' r_2'}} \qquad \text{henrys/meter} \tag{2.38}$$

If $r_1' = r_2' = r'$, the total inductance reduces to

$$L = 4 \times 10^{-7} \ln \frac{D}{r'} \qquad \text{henrys/meter} \tag{2.39}$$

or

$$L = 1.482 \log \frac{D}{r'} \qquad \text{millihenrys/mile} \tag{2.40}$$

Equation (2.40) is the inductance of the two-wire line taking into account the flux linkages caused by current in both conductors, one of which is

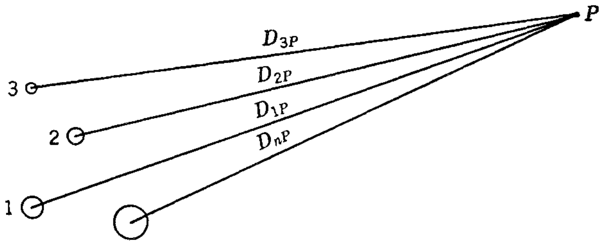


FIG. 2.6 Cross-sectional view of a group of n conductors carrying currents whose sum is zero. Point P is remote from the conductors.

the return path for current in the other. This value of inductance is sometimes called the inductance per loop meter or per loop mile to distinguish it from the inductance of the circuit due to the current in one conductor only. The latter, as given by Eq. (2.36), is one half the total inductance of a single-phase line and is called the inductance per conductor.

2.7 Flux Linkages of One Conductor in a Group. A more general problem than that of the two-wire line is presented by one conductor in a group of conductors where the sum of the currents in all the conductors is zero. Such a group of conductors is shown in Fig. 2.6. Conductors 1, 2, 3, . . . , n carry the currents I_1, I_2, I_3, . . . , I_n. The distances of these conductors from a remote point P are indicated on the figure as D_{1P}, D_{2P}, D_{3P}, . . . , D_{nP}. Let us determine ψ_{1P1}, the flux linkages of conductor 1 due to I_1 including internal flux linkages but excluding all the flux beyond the point P. By Eqs. (2.19) and (2.26),

$$\psi_{1P1} = \left(\frac{I_1}{2} + 2I_1 \ln \frac{D_{1P}}{r_1}\right) 10^{-7} \tag{2.41}$$

$$\psi_{1P1} = 2 \times 10^{-7} I_1 \ln \frac{D_{1P}}{r_1'} \qquad \text{weber-turns/meter} \tag{2.42}$$

The flux linkages ψ_{1P2} with conductor 1 *due to* I_2, but excluding flux beyond point P, is equal to the flux produced by I_2 between the point P and conductor 1 (that is, within the limiting distances D_{2P} and D_{12} from conductor 2). So

$$\psi_{1P2} = 2 \times 10^{-7} I_2 \ln \frac{D_{2P}}{D_{12}} \tag{2.43}$$

The flux linkages ψ_{1P} with conductor 1 *due to all the conductors* in the group, but excluding flux beyond point P, is

$$\psi_{1P} = 2 \times 10^{-7}\left(I_1 \ln \frac{D_{1P}}{r_1'} + I_2 \ln \frac{D_{2P}}{D_{12}} + I_3 \ln \frac{D_{3P}}{D_{13}} + \cdots + I_n \ln \frac{D_{nP}}{D_{1n}}\right) \tag{2.44}$$

which becomes, by expanding the logarithmic terms and regrouping,

$$\psi_{1P} = 2 \times 10^{-7}\left(I_1 \ln \frac{1}{r_1'} + I_2 \ln \frac{1}{D_{12}} + I_3 \ln \frac{1}{D_{13}} + \cdots + I_n \ln \frac{1}{D_{1n}} + I_1 \ln D_{1P} + I_2 \ln D_{2P} + I_3 \ln D_{3P} + \cdots + I_n \ln D_{nP}\right) \tag{2.45}$$

Since the sum of all the currents in the group is zero,

$$I_1 + I_2 + I_3 + \cdots + I_n = 0$$

and, solving for I_n, we obtain

$$I_n = -(I_1 + I_2 + I_3 + \cdots + I_{n-1}) \tag{2.46}$$

Substituting Eq. (2.46) in the second term containing I_n in Eq. (2.45) and recombining some logarithmic terms, we have

$$\psi_{1P} = 2 \times 10^{-7}\left(I_1 \ln \frac{1}{r_1'} + I_2 \ln \frac{1}{D_{12}} + I_3 \ln \frac{1}{D_{13}} + \cdots + I_n \ln \frac{1}{D_{1n}} + I_1 \ln \frac{D_{1P}}{D_{nP}} + I_2 \ln \frac{D_{2P}}{D_{nP}} + I_3 \ln \frac{D_{3P}}{D_{nP}} + \cdots + I_{n-1} \ln \frac{D_{(n-1)P}}{D_{nP}}\right) \tag{2.47}$$

Now letting the point P move infinitely far away so that the set of terms containing logarithms of ratios of distances from P becomes infinitesimal, since the ratios of the distances approach one, we obtain

$$\psi_1 = 2 \times 10^{-7} \left(I_1 \ln \frac{1}{r_1'} + I_2 \ln \frac{1}{D_{12}} + I_3 \ln \frac{1}{D_{13}} + \cdots + I_n \ln \frac{1}{D_{1n}} \right) \quad \text{weber-turns/meter} \tag{2.48}$$

By letting point P move infinitely far away we have included all the flux linkages of conductor 1 in our derivation. Therefore, Eq. (2.48) expresses all the flux linkages of conductor 1 in a group of conductors, provided the sum of all the currents is zero. If the currents are alternating, they must be expressed as instantaneous currents to obtain instantaneous flux linkages or as complex rms values to obtain the rms value of flux linkages as a complex number.

2.8 Inductance of Composite-conductor Lines. The commonest conductors for overhead power transmission lines are composed of strands of wire with alternate layers spiraled in opposite directions. Spiraling alternate layers in opposite directions prevents unwinding and makes the outer radius of one layer coincide with the inner radius of the next. Stranding provides flexibility with large cross-sectional area. The number of strands depends on the number of layers and on whether all the strands are the same diameter. The total number of strands in concentrically stranded cables, where the total annular space is filled with strands of uniform diameter, is 7, 19, 37, 61, 91, or more. A general formula for the total number of strands in such cables is

$$\text{Number of strands} = 3x^2 - 3x + 1$$

where x is the number of layers, including the single center strand. A 500,000-circular-mil conductor may be composed of 37 strands having individual diameters of 0.1162 in. or of 19 strands having individual diameters of 0.1622 in. Table A.1 in the Appendix lists the characteristics of concentrically stranded conductors of hard-drawn copper and of conductors having 12, 3, and single strands. The strands of copper conductors are usually uniform in diameter and composed of copper only.

Figure 2.7 shows a typical steel-reinforced aluminum cable (ACSR). The conductor shown has 19 steel strands forming a central core, around which are two layers of aluminum strands. There are 30 aluminum strands in the two outer layers. The conductor stranding is specified as 30 Al/19 St, or simply 30/19. Various tensile strengths, current capacities, and conductor sizes are obtained by using different combinations of steel and aluminum. Table A.2 in the Appendix, which gives the characteristics of ACSR, indicates the sizes of both the aluminum and steel strands, the number of strands of each, and the number of layers for the usual types of strnading. A type of conductor known as "expanded" ACSR has a filler such as paper separating the inner steel strands from the outer aluminum strands.

Steel wires coated with a thick layer of copper are used to obtain high tensile strength combined with good current-carrying capacity. Sometimes cables are composed of copper strands in the outer layers and copper-coated steel wires in the inner layers. Hollow copper conductors are sometimes used on high-voltage lines. One type of hollow copper conductor consists of interlocked sections of copper forming a spiral

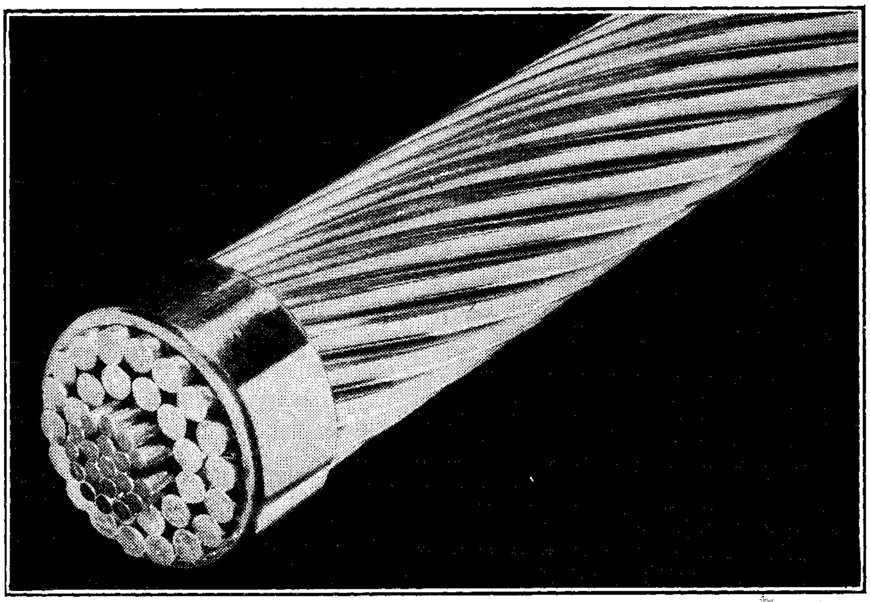


FIG. 2.7 Steel-reinforced aluminum conductor, 19 steel strands, 30 aluminum strands. (*Aluminum Company of America.*)

along the axial length of the conductor. Such a conductor is self-supporting and has some degree of flexibility. Another type of hollow conductor consists of copper strands with the inner layer twisted in the direction opposite to that of a twisted copper I beam around which the strands are spiraled.

Stranded conductors come under the general classification [illegible] conductors, which means conductors composed of two [illegible] elements or strands electrically in parallel. We are now rea[illegible] to study the inductance of a transmission line composed of c[illegible]site conductors, but we shall limit ourselves to the case where all th[illegible] strands are identical and share the current equally. The method ca[illegible] expanded to apply to all types of conductors containing strands of [illegible]erent sizes and conductivities,[7] but this will not be done here since values of internal inductance of specific conductors are generally available from the various manufacturers and can be found in handbooks. The method to be developed indicates the approach to the more complicated problems of nonhomogeneous conductors and unequal division of current between strands. The method is applicable to the determination of inductance of lines consisting of circuits electrically in parallel since two conductors in parallel can be treated as strands of a single composite conductor.

Figure 2.8 shows a single-phase line composed of two conductors. In order to be more general, each conductor forming one side of the line is shown as an arbitrary arrangement of an indefinite number of conductors. The only restrictions are that the parallel filaments are cylindrical and share the current equally. Conductor X is composed of n identical, parallel filaments, each of which carries the current I/n. Conductor Y, which is the return circuit for the current in conductor X, is composed of m identical, parallel filaments, each of which carries the current $-I/m$. Distances between the elements will be designated by the letter D with appropriate subscripts. Applying Eq. (2.48) to filament a of conductor X, we obtain for flux linkages of filament a

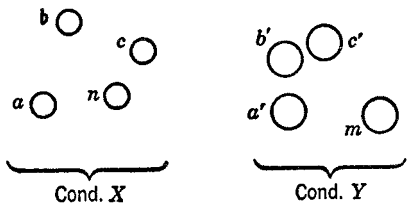


FIG. 2.8 Single-phase line consisting of two composite conductors.

$$\begin{aligned}\psi_a = {} & 2 \times 10^{-7} \frac{I}{n} \left(\ln \frac{1}{r'_a} + \ln \frac{1}{D_{ab}} + \ln \frac{1}{D_{ac}} + \cdots + \ln \frac{1}{D_{an}} \right) \\ & - 2 \times 10^{-7} \frac{I}{m} \left(\ln \frac{1}{D_{aa'}} + \ln \frac{1}{D_{ab'}} + \ln \frac{1}{D_{ac'}} + \cdots + \ln \frac{1}{D_{am}} \right) \qquad (2.49)\end{aligned}$$

from which

$$\psi_a = 2 \times 10^{-7} I \ln \frac{\sqrt[m]{D_{aa'} D_{ab'} D_{ac'} \cdots D_{am}}}{\sqrt[n]{r'_a D_{ab} D_{ac} \cdots D_{an}}} \qquad \text{weber-turns/meter} \quad (2.50)$$

Dividing Eq. (2.50) by the current I/n, we find that the inductance of filament a is

$$\begin{aligned} L_a &= \frac{\psi_a}{I/n} \\ &= 2n \times 10^{-7} \ln \frac{\sqrt[m]{D_{aa'} D_{ab'} D_{ac'} \cdots D_{am}}}{\sqrt[n]{r'_a D_{ab} D_{ac} \cdots D_{an}}} \qquad \text{henrys/meter} \quad (2.51)\end{aligned}$$

[7] See for instance L. F. Woodruff, "Electric Power Transmission," Chap. II, John Wiley & Sons, Inc., New York, 1938.

Similarly, the inductance of filament b is

$$L_b = \frac{\psi_b}{I/n}$$
$$= 2n \times 10^{-7} \ln \frac{\sqrt[m]{D_{ba'}D_{bb'}D_{bc'} \cdots D_{bm}}}{\sqrt[n]{D_{ba}r'_bD_{bc} \cdots D_{bn}}} \quad \text{henrys/meter} \tag{2.52}$$

The average inductance of the filaments of conductor X is

$$L_{\text{av}} = \frac{L_a + L_b + L_c + \cdots + L_n}{n} \tag{2.53}$$

Conductor X is composed of n filaments electrically in parallel. If all the filaments had the same inductance, the inductance of the conductor would be $1/n$ times the inductance of one filament. Here all the filaments have different inductances, but the inductance of all of them in parallel is $1/n$ times the average inductance. Thus the inductance of conductor X is

$$L_X = \frac{L_{\text{av}}}{n} = \frac{L_a + L_b + L_c + \cdots + L_n}{n^2} \tag{2.54}$$

Substituting the logarithmic expression for inductance of each filament in Eq. (2.54) and combining terms, we obtain

$$L_X = 2 \times 10^{-7}$$
$$\left[\ln \frac{\sqrt[mn]{(D_{aa'}D_{ab'}D_{ac'} \cdots D_{am})(D_{ba'}D_{bb'}D_{bc'} \cdots D_{bm}) \cdots (D_{na'}D_{nb'}D_{nc'} \cdots D_{nm})}}{\sqrt[n^2]{(D_{aa}D_{ab}D_{ac} \cdots D_{an})(D_{ba}D_{bb}D_{bc} \cdots D_{bn}) \cdots (D_{na}D_{nb}D_{nc} \cdots D_{nn})}}\right]$$
$$\text{henrys/meter} \tag{2.55}$$

where r'_a, r'_b, and r'_n have been replaced by D_{aa}, D_{bb}, and D_{nn}, respectively, to make the expression appear more symmetrical.

Note that the numerator of the argument of the logarithm in Eq. (2.55) is the mnth root of mn terms, which are the products of the distances from all the n filaments of conductor X to all the m filaments of conductor Y. For each filament in conductor X there are m distances to filaments in conductor Y, and there are n filaments in conductor X. The product of m distances for each of n filaments results in mn terms. The mnth root of the product of the mn distances is called the *geometric mean distance* between conductor X and conductor Y. It is abbreviated D_m or GMD and is also called the *mutual* GMD between the two conductors. Geometric mean distance is a mathematical concept which we will discuss later in more general terms.

The denominator of the argument of the logarithm in Eq. (2.55) is the n^2 root of n^2 terms. There are n filaments, and for each filament there are n terms consisting of r' for that filament times the distances from that filament to every other filament in conductor X. Thus we account for n^2 terms. Sometimes r'_a is called the distance from filament a to itself, especially when it is designated as D_{aa}. With this in mind the terms under the radical in the denominator may be described as the product of the distances from every filament in the conductor to itself and to every other filament. The n^2 root of these terms is called the *self* GMD of conductor X, and the r' of a separate filament is called the self GMD of the filament, for reasons which we shall see later when we discuss GMD as a mathematical concept. Sometimes self GMD is called *geometric mean radius* or GMR. Self GMD may be abbreviated D_s.

In terms of D_m and D_s, Eq. (2.55) becomes

$$L_X = 2 \times 10^{-7} \ln \frac{D_m}{D_s} \qquad \text{henrys/meter} \tag{2.56}$$

$$L_X = 0.7411 \log \frac{D_m}{D_s} \qquad \text{millihenrys/mile} \tag{2.57}$$

If we compare Eq. (2.57) with Eq. (2.36), the similarity between them is apparent. The equation for the inductance of one conductor of a composite-conductor line is obtained by substituting in Eq. (2.36) the GMD between conductors of the composite-conductor line for the distance between the solid conductors of the single-conductor line and by substituting the self GMD of the composite conductor for the self GMD (r') of the single conductor. Equation (2.57) gives the inductance of one conductor of a single-phase line. The conductor is composed of all the strands which are electrically in parallel. The inductance is the total number of flux linkages of the composite conductor per unit of line current. Equation (2.36) gives the inductance of one conductor of a single-phase line for the special case where the conductor is a solid, round wire.

The inductance of conductor Y is determined in a similar manner, and the inductance of the line is

$$L = L_X + L_Y$$

2.9 Geometric Mean Distance. In the preceding section we derived an expression for the inductance of a composite-conductor line. We found in the expression for inductance due to the current in one condu[illegible] a term which is the geometric mean of the distances between [illegible] of the one conductor and the wires of the return cond[illegible]. Another term in the expression is the geometric mean of distances between wires of the same conductor only. Geometric mean distan[illegible] a mathematical concept which is helpful in calculating inductance[illegible]

By definition the GMD from one point to a g[illegible] other points is the geometric mean of the distances from the on[illegible] point to each of the other points. For instance, the GMD from an external point to four points on a circle is the geometric mean of the four distances shown in Fig. 2.9. Here the geometric mean of the distances is

$$\text{GMD} = \sqrt[4]{D_1 D_2 D_3 D_4}$$

If the number of points on the circle is increased without limit, the geometric mean of the distances from the external point to the points on the circle approaches the GMD from the point to the circle. It is equal to the distance from the point to the center of the circle.[8] The GMD from any point on a circle to all other points on a circle is equal to the radius of the circle.

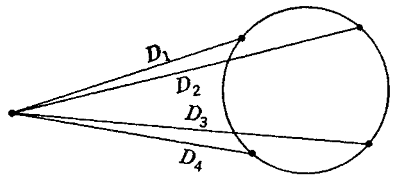


FIG. 2.9 Distances from an external point to four points on a circle.

The concept of the GMD from a point to an area is important and can be visualized by dividing the area into a large number of equal elements and taking the geometric mean of the distances from the point to the elements of area. If there are n elements, the geometric mean of the distances is the nth root of the product of the n distances. The GMD from the point to the area is the limit approached by the GMD from the point to the elements of the area as the number of elements increases without limit.

To find the GMD between two areas, each area is divided into a number of equal elements, say m equal elements for one area and n equal elements for the other. The GMD between the areas is the limit of the mnth root of the mn products of the distances between the m elements of one area and the n elements of the other area as m and n increase without limit. Figure 2.10 shows the six distances between two of the m equal elements into which one area is divided and three of the n equal elements into which the other area is divided. To find the GMD between the areas all distances between elements must be considered, and the number of elements in each area must be infinite. The GMD between two circular areas can be shown to be equal to the distance between their centers.

The self GMD of an area is the limit of the geometric mean of the distances between all the pairs of elements in that area as the number

[8] See E. B. Rosa and F. W. Grover, "Formulas and Tables for the Calculation of Mutual and Self Inductance," Scientific Paper 169, *Bull. Bur. Standards*, vol. 8, no. 1, pp. 1–237, 1912. Other formulas for GMD have been taken from the same source. See also J. C. Maxwell, "A Treatise on Electricity and Magnetism," vol. II, pp. 298–301, Clarendon Press, Oxford, 1881.

of elements increases without limit. The self GMD of a circular area can be shown to be equal to the radius of the circle times $\epsilon^{-1/4}$. Since r' in our formulas for the inductance of a round wire is the radius of the wire times $\epsilon^{-1/4}$, we can see the reason for calling r' the self GMD of the wire.

Since the cross-sectional areas of the filaments of the composite conductors considered in deriving Eq. (2.55) could be elements of areas such as those in Fig. 2.10, the inductance of a line composed of conductors of irregular area can be found by calculating GMD values. The self GMD of each area and the mutual GMD between the two areas must be found. The inductance due to current in each conductor is found by

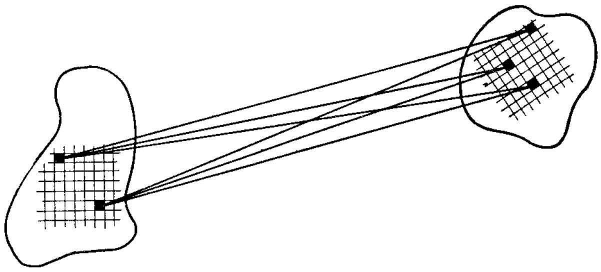


FIG. 2.10 The six distances from two equal elements of one area to three equal elements of another area.

Eq. (2.56) or (2.57), and the two inductances are added to find the inductance of the line. Uniform current density throughout is assumed.

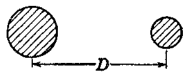
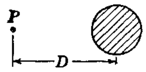
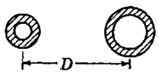
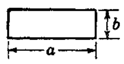
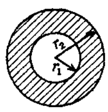
Table 2.1 gives some formulas for self and mutual GMD.

The GMD method does not apply strictly to nonhomogeneous conductors such as ACSR or to cases where the current density is not uniform throughout the conductor. An approximate value for inductance of ACSR is obtained by neglecting entirely the current carried by the steel conductors. The current in the steel conductors is relatively small, and the inductance depends on the amount of current in the conductor, since the permeability is not constant and the flux linkages are not a linear function of current. If inductance is determined experimentally for ACSR or other conductors not having uniform current density, an equivalent self GMD may be found.[9] Let D_s be the equivalent self GMD which when substituted in the inductance formula w[illegible] value of the experimentally determined inductance. S[illegible]

$$L = 2 \times 10^{-7} \ln \frac{D_m}{D_s} \qquad \text{henrys/m[illegible]} \qquad (2.58)$$

[9] See L. F. Woodruff, "Electric Power Transmission," [illegible] ed., p. 30, John Wiley & Sons, Inc., New York, 1938.

TABLE 2.1 GEOMETRIC MEAN DISTANCES

Description	Illustration	Value
Self GMD of a circular area		$D_s = r\epsilon^{-1/4} = 0.7788r$
GMD from one circular area to another		$D_m = D$
GMD from circular line to enclosed area		$D_m = r$
GMD from external point to circular area		$D_m = D$
GMD between n equally spaced points on a circle		$D_m = r\sqrt[n-1]{n}$
GMD from one annular area to another		$D_m = D$
Self GMD of a rectangular area		$D_s = 0.2235(a+b)$ Range of constant is 0.2231 to 0.2237, depending on ratio a/b
Self GMD of an annular area		$\ln \text{GMD} = \ln r_2 - \frac{r_1^4}{(r_2^2 - r_1^2)^2}\ln\frac{r_2}{r_1} + \frac{3r_1^2 - r_2^2}{4(r_2^2 - r_1^2)}$

the equivalent self GMD is

$$D_s = D_m\epsilon^{-10^7L/2} \tag{2.59}$$

where L is in henrys per meter. The value of D_m is not affected by nonuniform distribution of the current inside the conductor as long as the external magnetic field is not changed. Table A.2 in the Appendix lists the self GMD (GMR) of various sizes of ACSR.

2.10 Examples of Inductance Computations. Let us study two examples of the method of calculating inductance by GMD.

Example 2.1

One circuit of a single-phase transmission line is composed of three solid wires, each 0.1 in. in radius. The return circuit is composed of two wires, each 0.2 in. in radius. The arrangement of conductors is shown in Fig. 2.11. Find the inductance due to the current in each side of the line and the inductance of the complete line in millihenrys per mile.

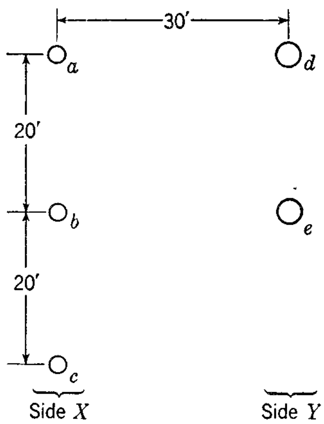


FIG. 2.11 Arrangement of conductors for Example 2.1.

Solution

Find the GMD between sides X and Y.

$$D_m = \sqrt[6]{D_{ad}D_{ae}D_{bd}D_{be}D_{cd}D_{ce}}$$
$$D_{ad} = D_{be} = 30 \text{ ft}$$
$$D_{ae} = D_{bd} = D_{ce} = \sqrt{20^2 + 30^2} = \sqrt{1{,}300}$$
$$D_{cd} = \sqrt{30^2 + 40^2} = 50 \text{ ft}$$
$$D_m = \sqrt[6]{30^2 \times 50 \times 1{,}300^{3/2}}$$
$$= 30^{1/3} \times 50^{1/6} \times 1{,}300^{1/4} = 35.8 \text{ ft}$$

Then find the self GMD for side X.

$$D_s = \sqrt[9]{D_{aa}D_{ab}D_{ac}D_{ba}D_{bb}D_{bc}D_{ca}D_{cb}D_{cc}}$$
$$= \sqrt[9]{\left(\frac{0.1 \times 0.7788}{12}\right)^3 \times 20^4 \times 40^2}$$
$$= \sqrt[3]{\frac{0.1 \times 0.7788}{12}} \times 20^{4/9} \times (2 \times 20)^{2/9}$$
$$= \sqrt[3]{\frac{0.1 \times 0.7788}{12}} \times 20^{2/3} \times 4^{1/9} = 1.605 \text{ ft}$$

and for side Y

$$D_s = \sqrt[4]{\left(\frac{0.2 \times 0.7788}{12}\right)^2 \times 20^2} = 0.509 \text{ ft}$$

The inductance is, by Eq. (2.57),

$$L_X = 0.7411 \log \frac{35.8}{1.605} = 1.00 \text{ millihenry/mile}$$
$$L_Y = 0.7411 \log \frac{35.8}{0.509} = 1.38 \text{ millihenrys/mile}$$
$$L = L_X + L_Y = 2.38 \text{ millihenrys/mile.}$$

Example 2.2

A conductor is composed of seven identical copper strands, each having a radius r as shown in Fig. 2.12. Find the factor by which r should be multiplied to find the self GMD of the conductor. Also find the factor by which the square root of the area of the conductor in circular mils should be multiplied to obtain the self GMD of the conductor.

Solution

First we find the distances D_{12}, D_{13}, and D_{14}, as follows:

$$D_{12} = 2r \qquad D_{14} = 4r$$
$$D_{13} = \sqrt{D_{14}^2 - D_{34}^2} = \sqrt{(4r)^2 - (2r)^2} = 2r\sqrt{3}$$

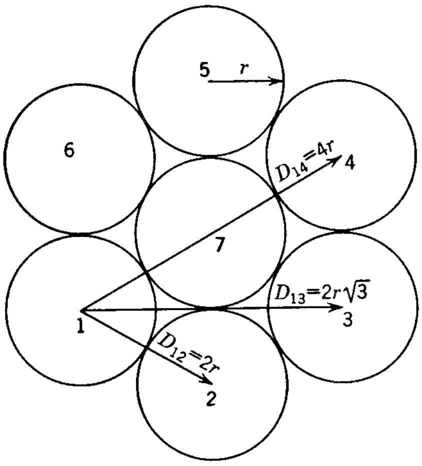


FIG. 2.12 Cross section of a seven-strand conductor for Example 2.2.

The self GMD of the seven-strand conductor is the 49th root of 49 distances. So

$$D_s = \sqrt[49]{(r')^7(D_{12}^2 D_{13}^2 D_{14} D_{17})^6(2r)^6}$$

where $(r')^7$ is the product of the self GMD of one strand and the self GMD values of every other strand. The term $D_{12}^2D_{13}^2D_{14}D_{17}$ is the product of the distances from one outside strand to every other strand. It is raised to the sixth power to account for the six outside strands. The term $(2r)^6$ accounts for the product of the distances from the inner strand to every outside strand. Thus there are seven distances for each of the seven strands. Simplifying the expression for D_s, we obtain

$$D_s = \sqrt[7]{r'} \times \sqrt[49]{(2^2r^2 \times 3 \times 2^2r^2 \times 2^2r \times 2r \times 2r)^6} = \frac{2r\sqrt[7]{3(0.7788)}}{\sqrt[49]{6}}$$
$$= 2.177r$$

To find D_s in terms of total conductor area in circular mils, let

A = total conductor area in circular mils
d = diameter of each strand in mils
r = radius of each strand in mils

Then

$$A = 7d^2 = 28r^2$$

and

$$D_s = \frac{2.177}{\sqrt{28}}\sqrt{A} = 0.4114\sqrt{A} \qquad \text{mils}$$

If a single-phase line consists of two stranded cables similar to the one computed in Example 2.2, it is seldom necessary to calculate the GMD between strands of the two sides, for the GMD would be almost equal to the distance between centers of the cables. The calculation of mutual GMD is important only where the various strands (or conductors) electrically in parallel are separated from each other by distances more nearly approaching the distance between the two sides of the circuit. For instance, in Example 2.1 the conductors in parallel on one side of the line are separated by 20 ft, and the distance between the two sides of the line is 30 ft. Here the calculation of mutual GMD is important. For stranded conductors such as that of Example 2.2, the distance between sides of the line is usually so great that the mutual GMD can be taken as equal to the center-to-center distance with negligible error.

2.11 The Use of Tables. The self GMD of conductors of any number of strands can be computed as in Example 2.2. The engineer seldom has to make such computations, however, since tables listing values of self GMD are generally available for standard conductors. All manufacturers furnish data, including values of self GMD, for their conductors, and tables provide the most practical method of obtaining the desired values, especially for nonhomogeneous conductors such as ACSR. In order to use the tables intelligently the engineer must understand thoroughly the meaning of the tabulated data.

Inductive reactance rather than inductance is usually desired. The inductive reactance of one conductor of a single-phase two-conductor line is

$$X_L = 2\pi f L = 2\pi f \times 0.7411 \times 10^{-3} \log \frac{D_m}{D_s}$$

$$= 4.657 \times 10^{-3} f \log \frac{D_m}{D_s} \qquad \text{ohms/mile} \qquad (2.60)$$

where D_m is the distance between the two conductors. The self GMD D_s may be found in the tables and substituted in the equation. D_m and D_s must be in the same units. Most tables list values of D_s for 60 cps, 25 cps, and direct current. The value of equivalent D_s varies with frequency because current density does not remain uniform throughout the conductor as frequency increases, as we shall see in Chap. 4. As current distribution becomes nonuniform the inductance due to internal flux decreases, and different degrees of nonuniformity are accounted for by different values of D_s at different frequencies. The nonuniform distribution of current due to the frequency of the current is called *skin effect*. In the equations and formulas already discussed in this chapter we neglected skin effect by assuming uniform current density. Our discussion of skin effect in Chap. 4 will show the amount of correction to be applied to the value of internal inductance computed on the assumption of uniform distribution of current.

Some tables give values of inductive reactance in addition to self GMD. One method is to expand the logarithmic term of Eq. (2.60) as follows:

$$X_L = 4.657 \times 10^{-3} f \log \frac{1}{D_s} + 4.657 \times 10^{-3} f \log D_m \qquad \text{ohms/mile} \qquad (2.61)$$

If both D_s and D_m are in feet, the first term in Eq. (2.61) is the inductive reactance of one conductor of a two-conductor line having a distance of 1 ft between conductors, as may be seen by comparing Eq. (2.61) with Eq. (2.57). Therefore, the first term of Eq. (2.61) is called the *inductive reactance at 1-ft spacing*. It depends upon the self GMD of the conductor and the frequency. The second term of Eq. (2.61) is called the *inductive reactance spacing factor*. This second term is independent of the type of conductor and depends on frequency and spacing only. The spacing factor is equal to zero when D_m is 1 ft. If D_m is less than 1 ft, the spacing factor is negative. The procedure for computing inductive reactance is to look up the inductive reactance at 1-ft spacing for the conductor under consideration and to add to this value the inductive reactance spacing factor, both at the desired line frequency.[10] In the Appendix, Tables A.1 and A.2 include values of inductive reactance at 1-ft spacing, and Table A.3 lists values of the inductive reactance spacing factor.

Example 2.3

Find the inductive reactance per mile of a two-conductor single-phase line operating at 60 cps. The conductors are each No. 1/0 seven-strand hard-drawn copper wire spaced 18 ft between centers.

Solution

The area of the stranded conductor is $A = 105{,}500$ circular mils (from Table A.1). From Example 2.2

$$D_s = 0.4114\sqrt{A} \quad \text{in.}$$
$$= \frac{0.4114\sqrt{105{,}500}}{12} \times 10^{-3}\ \text{ft} = 0.01113\ \text{ft}$$

which is the value listed in Table A.1 for D_s at 60 cps. Agreement of calculated and tabulated values indicates that skin effect is negligible for this case.

For one conductor

$$X_L = 4.657 \times 10^{-3} \times 60 \log \frac{18}{0.01113} = 0.897\ \text{ohm/mile}$$

If only D_s is given in the tables, the above method is used. An alternative method follows:

[10] This method of computing inductive reactance was suggested by W. A. Lewis and appeared in C. F. Wagner and R. D. Evans, "Symmetrical Components," McGraw-Hill Book Company, Inc., New York, 1933. See also A. E. Knowlton, "Standard Handbook for Electrical Engineers," pp. [illegible], McGraw-Hill Book Company, Inc., New York, 1941.

$$\text{Inductive reactance at 1-ft spacing} = 4.657 \times 10^{-3} \times 60 \log \frac{1}{0.01113}$$
$$= 0.546 \text{ ohm/mile}$$
$$\text{Inductive reactance spacing factor} = 4.657 \times 10^{-3} \times 60 \log 18$$
$$= 0.351 \text{ ohm/mile}$$
$$\text{Inductive reactance of one conductor} = 0.546 + 0.351$$
$$= 0.897 \text{ ohm/mile}$$

The latter method is preferred if tables are available giving inductive reactance at 1-ft spacing and the inductive reactance spacing factor, for then it is necessary only to add these two values found in the tables.

Since the conductors composing the two sides of the line are identical, the inductive reactance of the line is

$$X_L = 2 \times 0.897 = 1.794 \text{ ohms/mile}$$

2.12 Inductance of Three-phase Lines with Equilateral Spacing. So far in our discussion we have considered only single-phase lines. The equations we have developed are quite easily adapted, however, to the calculation of the inductance of three-phase lines. Figure 2.13 shows the conductors of a three-phase line spaced at the corners of an equilateral triangle. If we assume that there is no neutral wire, or if we assume balanced three-phase currents, $I_a + I_b + I_c = 0$. Equation (2.48) determines the flux linkages of conductor a. So

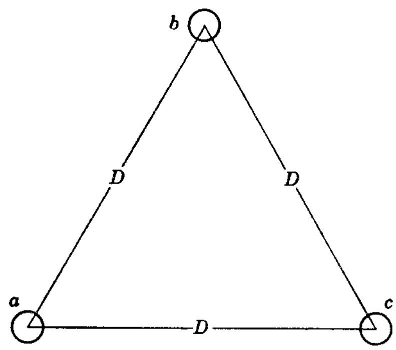


FIG. 2.13 Cross-sectional view of the equilaterally spaced conductors of a three-phase line.

$$\psi_a = 2 \times 10^{-7}\left(I_a \ln \frac{1}{r'} + I_b \ln \frac{1}{D} + I_c \ln \frac{1}{D}\right) \quad \text{weber-turns/meter} \quad (2.62)$$

Since $I_a = -(I_b + I_c)$, Eq. (2.62) becomes

$$\psi_a = 2 \times 10^{-7}\left(I_a \ln \frac{1}{r'} - I_a \ln \frac{1}{D}\right) = 2 \times 10^{-7} I_a \ln \frac{D}{r'} \quad \text{weber-turns/meter} \quad (2.63)$$

and

$$L_a = 2 \times 10^{-7} \ln \frac{D}{r'} \qquad \text{henrys/meter} \quad (2.64)$$

or

$$L_a = 0.7411 \log \frac{D}{r'} \qquad \text{millihenrys/mile} \quad (2.65)$$

Equation (2.65) is the same in form as Eq. (2.36) for a single-phase line. For stranded conductors, D_s replaces r' in the equation. Because of symmetry, the inductances of conductors b and c are the same as the inductance of conductor a. Since each phase consists of only one conductor, Eqs. (2.64) and (2.65) give the inductance per phase of the three-phase line.

2.13 Inductance of Three-phase Lines with Unsymmetrical Spacing. When the conductors of a three-phase line are not spaced equilaterally, the problem of finding the inductance becomes more difficult. Then the flux linkages and inductance of each phase are not the same. A different inductance in each phase results in an unbalanced circuit and in induced voltages in adjacent communication lines even when the phase currents are balanced. These undesirable characteristics can be overcome by exchanging the positions of the conductors at regular intervals along the

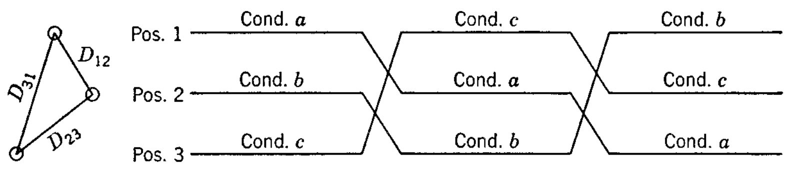


FIG. 2.14 Transposition cycle.

line so that each conductor occupies the original position of every other conductor over an equal distance. Such an exchange of conductor positions is called *transposition*. A complete transposition cycle is shown in Fig. 2.14. The phase conductors are designated a, b, and c, and the positions occupied are numbered 1, 2, and 3. Transposition results in each conductor having the same average inductance over the whole cycle.

If an untransposed telephone line parallels an untransposed power line, the flux produced by the power line induces a voltage of power-line frequency in the telephone line. Transposition of the power line without transposition of the telephone line eliminates interference of the power line with the telephone line except for unbalanced cases where power currents flow in the earth or in overhead ground wires. For balanced three-phase currents in a transposed power line, the magnetic field linking an adjacent telephone line is shifted 120° in time phase with each rotation of the conductor positions in the transposition cycle. Over the length of one transposition cycle of the power line, the net voltage induced in the telephone line is zero, because it is the sum of three induced voltages equal in magnitude and displaced 120° from each other. It is not necessary to transpose a power line to prevent interference with a telephone line, for the same result is accomplished by transposition of the telephone line.

Modern power lines are usually not transposed at regular intervals, although an interchange in the positions of the conductors may be made at switching stations in order to balance more closely the inductance of the phases.[11] Fortunately, the dissymmetry between the phases of an untransposed line is small and may be neglected in the solution of many problems. If the dissymmetry is neglected, the inductance of the untransposed line is calculated as though the line were correctly transposed. The inductive reactance of each phase of the untransposed line is taken as equal to the average value of the inductive reactance of one phase of the same line correctly transposed. The derivations to follow are for transposed lines. The error is small, and the calculations are less laborious if the inductance of an untransposed line is calculated by the same equations.

To find the average inductance of one conductor of a transposed line, the flux linkages of a conductor are found for each position it occupies in the transposition cycle, and the average flux linkages are determined. Let us apply Eq. (2.48) to conductor a of Fig. 2.14 to find the flux linkages of a in position 1, when b is in position 2 and c is in position 3, as follows:

$$\psi_{a1} = 2 \times 10^{-7}\left(I_a \ln \frac{1}{r'} + I_b \ln \frac{1}{D_{12}} + I_c \ln \frac{1}{D_{31}}\right) \quad \text{weber-turns/meter} \tag{2.66}$$

With a in position 2, b in position 3, and c in position 1,

$$\psi_{a2} = 2 \times 10^{-7}\left(I_a \ln \frac{1}{r'} + I_b \ln \frac{1}{D_{23}} + I_c \ln \frac{1}{D_{12}}\right) \quad \text{weber-turns/meter} \tag{2.67}$$

and, with a in position 3, b in position 1, and c in position 2,

$$\psi_{a3} = 2 \times 10^{-7}\left(I_a \ln \frac{1}{r'} + I_b \ln \frac{1}{D_{31}} + I_c \ln \frac{1}{D_{23}}\right) \quad \text{weber-turns/meter} \tag{2.68}$$

The average value of the flux linkages of a is

$$\begin{aligned} \psi_a &= \frac{\psi_{a1} + \psi_{a2} + \psi_{a3}}{3} \\ &= \frac{2 \times 10^{-7}}{3}\left(3I_a \ln \frac{1}{r'} + I_b \ln \frac{1}{D_{12}D_{23}D_{31}} + I_c \ln \frac{1}{D_{12}D_{23}D_{31}}\right) \end{aligned} \tag{2.69}$$

[11] For instance, see E. T. B. Gross and A. H. Weston, "Transposition of High-voltage Overhead Lines and Elimination of Electrostatic Unbalance to Ground," *Trans. AIEE*, vol. 70, pp. 1837–1841, 1951.

With the restriction that $I_a = -(I_b + I_c)$,

$$\psi_a = \frac{2 \times 10^{-7}}{3}\left(3I_a \ln \frac{1}{r'} - I_a \ln \frac{1}{D_{12}D_{23}D_{31}}\right)$$

$$= 2 \times 10^{-7} I_a \ln \frac{\sqrt[3]{D_{12}D_{23}D_{31}}}{r'} \qquad \text{weber-turns/meter} \qquad (2.70)$$

and the *average* inductance per phase is

$$L_a = 2 \times 10^{-7} \ln \frac{D_{eq}}{r'} \qquad \text{henrys/meter}$$

$$L_a = 0.7411 \log \frac{D_{eq}}{r'} \qquad \text{millihenrys/mile} \qquad (2.71)$$

where $$D_{eq} = \sqrt[3]{D_{12}D_{23}D_{31}} \qquad (2.72)$$

Equation (2.71) may be written

$$L_a = 0.7411 \log \frac{D_{eq}}{D_s} \qquad \text{millihenrys/mile} \qquad (2.73)$$

where D_s is the self GMD of the conductor. D_{eq}, the geometric mean of the three distances of the unsymmetrical line, is the equivalent equilateral spacing, as may be seen by the comparison of Eq. (2.71) with Eq. (2.65). We should note the similarity between all the equations for inductance of a conductor. If the inductance is in millihenrys per mile, the factor 0.7411 appears in all the equations, and the denominator of the logarithmic term is always the self GMD of the conductor. The numerator is the distance between wires of a two-wire line, the mutual GMD between sides of a composite-conductor single-phase line, the distance between conductors of an equilaterally spaced line, or the equivalent equilateral spacing of an unsymmetrical line.

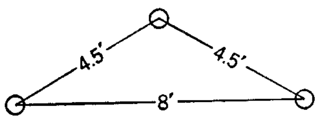


FIG. 2.15 Arrangement of conductors for Example 2.4.

Example 2.4

A single-circuit three-phase line operated at 60 cps is arranged as shown in Fig. 2.15. Each conductor is No. 2 single-strand hard-drawn copper wire. Find the inductance and inductive reactance per phase per mile.

Solution

The diameter of No. 2 wire is 0.258 in.

$$D_s = \frac{0.258 \times 0.7788}{2 \times 12} = 0.00836 \text{ ft}$$

$$D_{eq} = \sqrt[3]{4.5 \times 4.5 \times 8} = 5.45 \text{ ft}$$

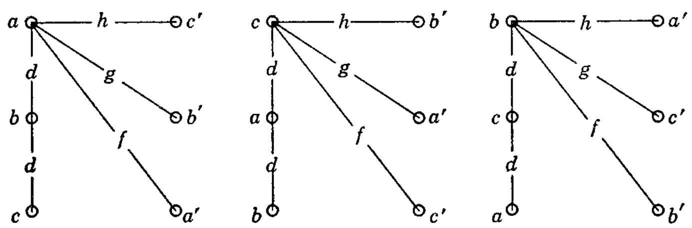
$$L = 0.7411 \log \frac{5.45}{0.00836} = 2.083 \text{ millihenrys/phase/mile}$$

$$X_L = 2\pi 60 \times 2.083 \times 10^{-3} = 0.787 \text{ ohm/phase/mile}$$

or, from Tables A.1 and A.2,

Inductive reactance at 1-ft spacing	=	0.581
Inductive reactance spacing factor for 5.45 ft	=	0.2058
Inductive reactance per phase	=	0.7868 ohm/phase/mile

2.14 Parallel-circuit Three-phase Lines. Two three-phase circuits that are identical in construction and electrically in parallel have the same inductive reactance. The inductive reactance of the single equivalent circuit, however, is half that of each of the individual circuits considered alone only if they are so widely separated that there is negligible mutual inductance between them. If the two circuits are on the same towers, the method of GMD may be used to find the inductance per



(*a*) Phase *a* in position 1 (*b*) Phase *a* in position 2 (*c*) Phase *a* in position 3

Fig. 2.16 Arrangement of the conductors of a double-circuit three-phase line in the three parts of the transposition cycle.

phase by considering all the conductors of any particular phase to be strands of one composite conductor.

Let us consider the two three-phase circuits with flat, vertical spacing shown in Fig. 2.16*a*. One circuit is composed of conductors *a*, *b*, and *c*. The other is composed of conductors *a'*, *b'*, and *c'*. Conductors *a* and *a'* are in parallel and compose phase *a*. Similarly, conductors *b* and *b'* are in parallel composing phase *b*, and *c* and *c'* in parallel compose phase *c*. In the other parts of the transposition cycle, conductors *a* and *a'* take first the positions originally occupied by *b* and *b'* and then the positions occupied by *c* and *c'*, as shown in Figs. 2.16*b* and 2.16*c*. The inductance is lowered if the individual conductors of a phase are separated as widely as possible and if the distances between phases are kept small. This results in a low D_m and a high D_s. It is accomplished in a double-circuit line by having the conductors of two of the phases spaced diagonally, as in Fig. 2.16, rather than horizontally adjacent.

By the method of GMD, the equivalent equilateral spacing is

$$D_{eq} = \sqrt[3]{D_{ab}D_{bc}D_{ca}} \tag{2.74}$$

where D_{ab} = mutual GMD between phases a and b in position 1

$$= \sqrt[4]{dgdg} = \sqrt{dg}$$

D_{bc} = mutual GMD between phases b and c in position 1

$$= D_{ab} = \sqrt{dg}$$

D_{ca} = mutual GMD between phases c and a in position 1 $= \sqrt{2dh}$

Thus

$$D_{eq} = 2^{1/6} d^{1/2} g^{1/3} h^{1/6} \tag{2.75}$$

If the self GMD of each individual conductor of phase a is r', the self GMD in position 1 for the entire phase consisting of conductors a and a' is

$$D_{s1} = \sqrt[4]{r'fr'f} = \sqrt{r'f}$$

In position 2 the self GMD of phase a is

$$D_{s2} = \sqrt[4]{r'hr'h} = \sqrt{r'h}$$

and in position 3 the self GMD of phase a is

$$D_{s3} = \sqrt[4]{r'fr'f} = \sqrt{r'f}$$

The average value of the flux linkages of the phase for the whole transposition cycle determines the average inductance. We saw [in Eqs. (2.50) to (2.55) and again in Eqs. (2.69) and (2.70)] that the average of logarithmic terms is equal to the logarithm of the geometric mean of the arguments of the logarithms. Therefore, the equivalent self GMD of one phase for the transposition cycle is the geometric mean of the three values of self GMD of the phase in the three parts of the transposition cycle. Thus

$$D_s = \sqrt[3]{D_{s1}D_{s2}D_{s3}} \tag{2.76}$$

$$D_s = (r')^{1/2} f^{1/3} h^{1/6} \tag{2.77}$$

Equations (2.76) and (2.77) are the same for all three phases if r' is the same for all three phases, since phases b and c occupy the same positions as phase a for equal distances. The inductance per phase is

$$L = 0.7411 \log \frac{D_{eq}}{D_s} = 0.7411 \log \left[2^{1/6} \left(\frac{d}{r'}\right)^{1/2} \left(\frac{g}{f}\right)^{1/3} \right] \quad \text{millihenrys/phase/mile} \tag{2.78}$$

Equation (2.78), derived by the GMD method, is the inductance *per phase,* since the two parallel conductors of each phase are considered as strands of one conductor. The inductance of an *individual conductor* is twice the combined inductance of the two conductors in parallel, or

$$L = 0.7411 \log \left[2^{1/3} \frac{d}{r'} \left(\frac{g}{f}\right)^{2/3} \right] \quad \text{millihenrys/mile/conductor} \tag{2.79}$$

As the two circuits of Fig. 2.16 are moved farther apart, the ratio g/f approaches one. If the two circuits are very far apart, the mutual inductance between them is negligible, and we would expect the inductance per conductor computed from Eq. (2.79) to approach that of a single circuit. Considering one of the circuits of Fig. 2.16 alone, we obtain

$$D_{eq} = \sqrt[3]{2ddd} = 2^{1/3}d$$

and

$$D_s = r'$$

Substituting the above values in Eq. (2.73), we find

$$L = 0.7411 \log \left(2^{1/3}\frac{d}{r'}\right) \quad \text{millihenrys/mile/conductor} \qquad (2.80)$$

which is the same as Eq. (2.79) if the ratio g/f is one. Therefore, we may consider the ratio g/f as a factor which accounts for the mutual effect of one circuit with flat spacing on a similar parallel circuit.

The preceding discussion shows the application of the GMD method to the computation of the inductance of a flat-spaced parallel-circuit line. Equations (2.75) and (2.77) to (2.80) apply only to flat-spaced parallel lines. It is not practicable to develop the special equations for other arrangements because the equations are complicated. The GMD method is applicable, however, to any circuits electrically in parallel, regardless of arrangement. Equations (2.74) and (2.76) apply to any multicircuit three-phase line if we remember that D_{ab}, D_{bc}, and D_{ca} are mutual GMD values.

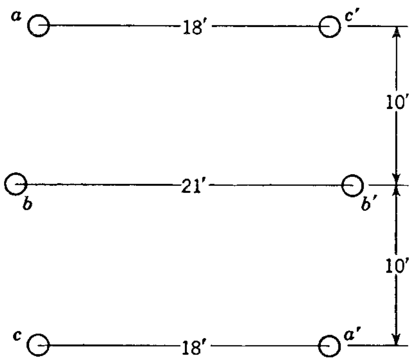


FIG. 2.17 Arrangement of conductors for Example 2.5.

Example 2.5

A three-phase double-circuit line is composed of 19-strand concentric copper conductors of 300,000-circular-mil cross-sectional area. The line is arranged as shown in Fig. 2.17 and is completely transposed. Find the 60-cycle inductive reactance per phase per mile.

Solution

From Table A.1, for the specified conductor, D_s = 0.01987 ft
Distance from a to b in original position = $\sqrt{10^2 + 1.5^2}$ = 10.1 ft
Distance from a to b' in original position = $\sqrt{10^2 + 19.5^2}$ = 21.9 ft
Distance from a to a' in original position = $\sqrt{20^2 + 18^2}$ = 26.9 ft.

$$D_s = (\sqrt{26.9r'}\,\sqrt{21r'}\,\sqrt{26.9r'})^{1/3} = \sqrt{0.01987}\,(26.9 \times 21 \times 26.9)^{1/6}$$
$$= 0.141 \times 4.98 = 0.702 \text{ ft}$$
$$D_{ab} = D_{bc} = \sqrt[4]{(10.1 \times 21.9)^2} = 14.88 \text{ ft}$$
$$D_{ca} = \sqrt[4]{(18 \times 20)^2} = 18.97 \text{ ft}$$
$$D_{eq} = \sqrt[3]{14.88 \times 14.88 \times 18.97} = 16.1 \text{ ft}$$
$$L = 0.7411 \log \frac{16.1}{0.702} = 1.01 \text{ millihenrys/mile/phase}$$
$$X_L = 2\pi 60 \times 1.01 \times 10^{-3} = 0.38 \text{ ohm/mile/phase}$$

2.15 Summary. Tables are helpful in computing the inductance and inductive reactance of a transmission line. If the self GMD of the conductor is obtained from a table, we can find the inductance of a single-circuit line by Eq. (2.57) if the line is single-phase, or by Eq. (2.73) if the line is three-phase. These two equations are the same except that the numerator of the argument of the logarithm of Eq. (2.73) for the three-phase line is the distance of equivalent equilateral spacing, rather than a single GMD as in Eq. (2.57) for the single-phase line. When more elaborate tables giving reactance at 1-ft spacing for various conductors and inductive reactance spacing factors are available, the inductance can be found by adding two values obtained from the tables.

For multicircuit lines tables may be used as described above except that additional calculations are necessary to apply the principle of GMD.

PROBLEMS

2.1 A hollow, cylindrical conductor has an outside diameter of 1.100 in. and a wall thickness of 0.130 in. Find the flux density at a distance of 0.485 in. from the center of the conductor when the current is 500 amp. Neglect the effect of the return circuit.

2.2 Derive the formula for the internal inductance in henrys per meter of a hollow conductor having an inside radius r_1 and an outside radius r_2. In what units should r_1 and r_2 be expressed?

2.3 Determine the formula for the inductance in henrys per meter of a single-phase line consisting of the hollow conductors described in Prob. 2.2 if the spacing between conductors is D ft. In what units should r_1 and r_2 be expressed? Compare the formula with the self GMD of an annular area given in Table 2.1.

2.4 Compute the 60-cycle inductive reactance at 1-ft spacing in ohms per mile for the hollow conductor whose dimensions are given in Prob. 2.1.

2.5 Find the self GMD of a seven-strand conductor if the center strand is removed and replaced by a strand of zero conductivity. Express the result in terms of the radius r of an individual strand.

2.6 Find the self GMD of a three-strand conductor in terms of the radius r of an individual strand.

2.7 Find the self GMD of each of the unconventional conductors shown in Fig. 2.18 in terms of the radius r of an individual strand and in terms of the square root of A, where A is the total area of the composite conductor in circular measure. Assume that all strands have the same radius and the same current density.

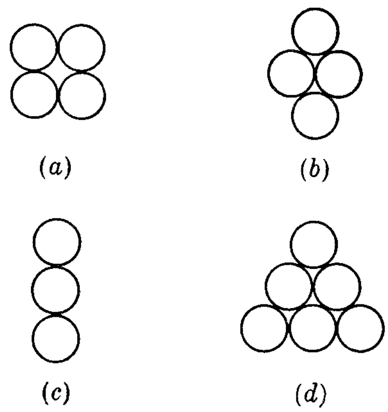


Fig. 2.18 Cross-sectional views of unconventional conductors for Prob. 2.7.

2.8 Compute the 60-cycle inductive reactance at 1-ft spacing in ohms per mile of a cable consisting of 12 equal strands around a nonconducting core. The diameter of each strand is 0.0936 in. The outside diameter of the cable is 0.470 in.

2.9 The outside diameter of the single layer of aluminum strands of No. 6 AWG ACSR conductor is 0.198 in. The diameter of each strand is 0.0661 in. Determine the 60-cycle inductive reactance at 1-ft spacing. Neglect the effect of the center strand of steel, but compare the result with the values given in Table A.2.

2.10 The 60-cycle inductive reactance at 1-ft spacing of a solid conductor is 0.595 ohm/mile. Find the reactance for a spacing of 6 ft, and determine the cross-sectional area of the wire in circular mils.

2.11 The 60-cycle inductive reactance per conductor of a single-phase line having solid conductors spaced 4 ft apart is 0.791 ohm/mile. Specify the 25-cycle inductive reactance at 1-ft spacing for the conductors. What is the cross-sectional area of the conductors in circular mils?

2.12 The distance between conductors of a single-phase line is 10 ft. Each conductor is composed of seven equal strands. The diameter of each strand is 0.1 in. Find the inductance of the line in henrys per mile.

2.13 A single-phase 60-cycle power line is supported on a horizontal crossarm. The spacing between conductors is 8 ft. A telephone line is supported on a horizontal crossarm 6 ft below the power line. The conductors of the telephone line are No. 14 AWG solid copper spaced 2 ft between centers. The conductors of the power line are No. 2 AWG solid copper. Find the mutual inductance between the circuits and the voltage per mile induced in the telephone line if the current in the power line is 150 amp.

2.14 If the power and telephone lines described in Prob. 2.13 are in the same horizontal plane and the distance between the nearest conductors of th[illegible] two [illegible] is 60 ft, find the mutual inductance between the circuits and the voltage pe[illegible] in the telephone line for 150 amp in the power line.

2.15 The conductors of a three-phase line are equilaterally [illegible]ed. Each conductor is a solid wire having a diameter of 0.162 in. The [illegible] is 8 ft. Find the inductance per phase in millihenrys per mile.

2.16 A three-phase line is designed with equilateral [illegible] of 8 ft. It is decided to build the line with horizontal spacing ($D_{13} = 2D_{12} =$ [illegible]). The [illegible] are transposed. What should be the spacing between adja[illegible]t conductors in order to obtain the same inductance as in the original design?

2.17 A single-phase circuit consists of three conductors on one side of the line and one on the other side. The arrangement is shown in Fig. 2.19. The three conductors composing one side of the line are transposed. Find the inductance per mile of this line.

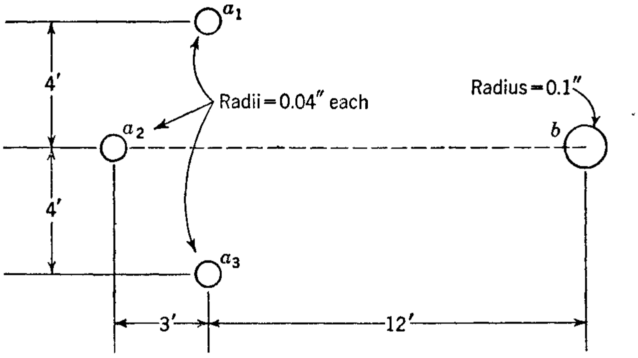


Fig. 2.19 Arrangement of conductors for Prob. 2.17.

2.18 Six conductors of 19-strand hard-drawn copper with an area of 300,000 circular mils are arranged as shown in Fig. 2.17. The vertical spacing, however, is 13 ft, the longer horizontal distance is 28 ft, and the shorter horizontal distances are 22 ft. If the line is operated single-phase with conductors a, b, and c in parallel forming one side of the line and conductors a', b', and c' forming the other side, find the inductance per mile of the line. Assume equal current in all conductors.

2.19 A 132-kv three-phase double-circuit power line is arranged with the conductors of each circuit in a vertical plane. The distance between adjacent conductors of the same circuit is 12 ft. The horizontal spacing between circuits is 24 ft. The conductors are 556,500 circular-mil ACSR, 30/7. Compute the inductance per phase of the double-circuit line. Compare the inductance of one conductor of the double-circuit line with the inductance of one conductor of a single circuit alone with the same vertical spacing.

2.20 If the line of Prob. 2.18 is operated three-phase, find the inductance per phase and per conductor.

2.21 Each phase of a three-phase line consists of three solid conductors. The diameter of each conductor is 0.26 in., and the spacing is shown in Fig. 2.20. Phases are designated a, b, and c. There is complete transposition both as to phases and as to individual conductors in each phase. Find the inductance per phase in millihenrys per mile.

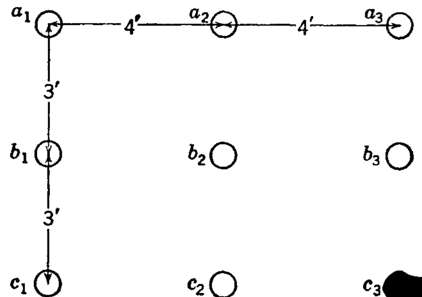


Fig. 2.20 Arrangement of conductors for P[illegible]

CHAPTER 3

CAPACITANCE OF TRANSMISSION LINES

3.1 Introduction. The potential difference between the conductors of a transmission line causes the conductors to be charged in the same manner as the plates of a capacitor are charged when there is a potential difference between the plates of the capacitor. The capacitance between conductors is the charge per unit of potential difference. Capacitance is a constant depending on the size and spacing of the conductors. For power lines less than about 50 miles long, the effect of capacitance is slight and is usually neglected. For longer lines of higher voltage, capacitance becomes increasingly important.

An alternating voltage impressed between the conductors of a transmission line causes the charge on the conductors to increase and decrease with the increase and decrease of the instantaneous value of the voltage. The movement of charge is a current, and the current caused by the alternate charging and discharging of a line due to an alternating voltage is called the charging current of the line. Charging current flows in a transmission line even when it is open-circuited. It affects the voltage drop along the line as well as the efficiency and power factor of the line and the stability of the system of which the line is a part.

3.2 Electric Field of a Long, Straight Conductor. Just as the magnetic field is important in considering inductance, so the electric field is important in studying capacitance. In the preceding chapter we saw (Fig. 2.1) both the magnetic and electric fields of a two-wire line. Lines of electric flux originate on the positive charges of one conductor and terminate on the negative charges of the other conductor. The total electric flux emanating from a conductor is numerically equal to the coulombs of charge on the conductor. Electric flux density is the electric flux per square meter and is measured in coulombs per square meter.

If a long, straight, cylindrical conductor has a uniform charge throughout its length and is isolated from other charges so that the charge is uniformly distributed around its periphery, the flux is radial. All points equidistant from such a conductor are points of equipotential and have the same electric flux density. Figure 3.1 shows such an isolated conductor carrying a uniformly distributed charge. The electric flux density at x meters from the conductor may be computed by imagining a cylindrical surface concentric with the conductor and x meters in radius. Since all parts of the surface are equidistant from the conductor, which has a uniformly distributed charge, the cylindrical surface is a surface of equipotential, and the electric flux density on the surface is equal to the flux leaving the conductor per meter of length divided by the area of the surface in an axial length of 1 meter. The electric flux density is

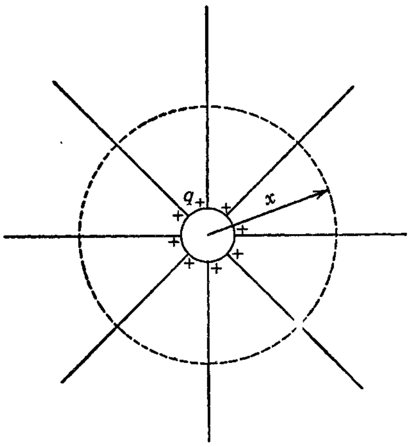


FIG. 3.1 Lines of electric flux originating on the positive charges uniformly distributed over the surface of an isolated cylindrical conductor.

$$D = \frac{q}{2\pi x} \qquad \text{coulombs/meter}^2 \qquad (3.1)$$

where q is the charge on the conductor in coulombs per meter of length and x is the distance in meters from the conductor to the point where the electric flux density is computed. The electric field intensity, or voltage gradient, is equal to the electric flux density divided by the permittivity[1] of the medium. Therefore, the electric field intensity is

$$\mathcal{E} = \frac{q}{2\pi x k} \qquad \text{volts/meter} \qquad (3.2)$$

3.3 The Potential Difference between Two Points Due to a Charge. The potential difference between two points is the work in newton-meters (joules) necessary to move a coulomb of charge between the two points. Electric field intensity is a measure of the force on a charge in the field. Electric field intensity in volts per meter is equal to the force in newtons on a coulomb of charge at the point considered. Between two points the line integral of the force in newtons acting on a coulomb of positive charge is the work done in moving the charge from the point of lower potential to the point of higher potential and is the potential difference between the two points.

Consider a long, straight wire carrying a positive charge of q coulombs/meter, as shown in Fig. 3.2. Points P_1 and P_2 are located at distances

[1] In the rationalized mks system of units the permittivity of free space k_0 is 8.85×10^{-12} farad/meter. Relative permittivity k_r is the ratio of the actual permittivity k of a material to the permittivity of free space. Thus, $k_r = k/k_0$.

D_1 and D_2 meters from the center of the wire. The positive coulomb on the wire will exert a repelling force on a positive charge placed in the field. Since the force repels a positive charge in the field and since D_2 in this case is greater than D_1, work must be done on a positive coulomb to move it from P_2 to P_1, and P_1 is at a higher potential than P_2. The difference in potential is the amount of work done. On the other hand, if the coulomb moves from P_1 to P_2, it expends energy, and the amount of work, or energy, in newton-meters is the voltage *drop* from P_1 to P_2. The potential difference is independent of the path followed. The simplest way to compute the voltage drop between the two points is to compute the voltage between the equipotential surfaces passing through P_1 and P_2 by integrating the field intensity over a radial path between the equipotential surfaces. Thus the voltage drop between P_1 and P_2 is

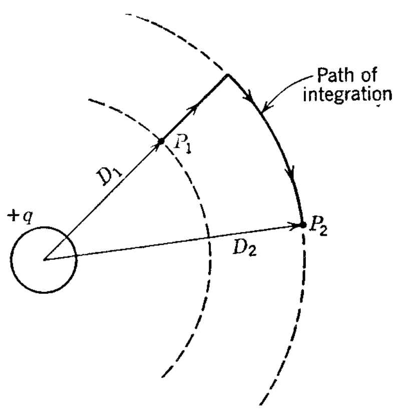


FIG. 3.2 Path of integration between two points external to a cylindrical conductor having a uniformly distributed positive charge.

$$V_{12} = \int_{D_1}^{D_2} \mathcal{E}\, dx = \int_{D_1}^{D_2} \frac{q}{2\pi k x}\, dx = \frac{q}{2\pi k} \ln \frac{D_2}{D_1} \qquad \text{volts} \tag{3.3}$$

where q is the charge on the wire in coulombs per meter of length. Note that the voltage drop between two points, as given by Eq. (3.3), may be positive or negative depending on whether the charge causing the potential difference is positive or negative and on whether the voltage drop is computed from a point near the conductor to a point farther away, or vice versa. The sign of q may be either positive or negative, and the logarithmic term is either positive or negative depending on whether D_2 is greater or less than D_1.

3.4 Capacitance of a Two-wire Line. In Sec. 3.1 the capacitance between the two conductors of a two-wire line was defined as the charge on the conductors per unit of potential difference between them. In the form of an equation, capacitance is

$$C = \frac{q}{V} \qquad \text{farads/meter} \tag{3.4}$$

where q is the charge on the line in coulombs per meter and V is the potential difference between the conductors in volts. The capacitance between two conductors may be found by substituting in Eq. (3.4) the expression for V in terms of q found by Eq. (3.3). The voltage V_{ab} between the two conductors of the two-wire line shown in Fig. 3.3 may be found by determining the potential difference between the two conductors of the line, first by computing the voltage drop due to the charge q_a on conductor a and then by computing the voltage drop due to the charge q_b on conductor b. By the principle of superposition the voltage drop from conductor a to conductor b due to the charges on both conductors is the sum of the voltage drops caused by each charge alone.

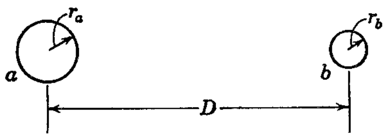


Fig. 3.3 Cross section of a parallel-wire line.

Consider the charge q_a on conductor a, and assume that conductor b is uncharged and merely an equipotential surface in the electric field created by the charge on a. The equipotential surface of conductor b and the equipotential surfaces due to the charge on a are shown in Fig. 3.4. The distortion of the equipotential surfaces near conductor b is caused by the fact that conductor b is also an equipotential surface. Equation (3.3) was derived by assuming all the equipotential surfaces due to a uniform charge on a round conductor to be cylindrical and concentric with the conductor. Such is actually true for the case under discussion except in the region near conductor b. The potential of conductor b is that of the equipotential surface intersecting b. Therefore, in determining V_{ab} a path may be followed from conductor a through a region of undistorted equipotential surfaces to the equipotential surface intersecting conductor b. Then, moving along the equipotential surface to b gives no further change in voltage. This path of integration is indicated in Fig. 3.4 together with the direct path. Of course, the potential difference is the same regardless of the path over which the integration of the field intensity is taken.[2] By following the path through the undistorted

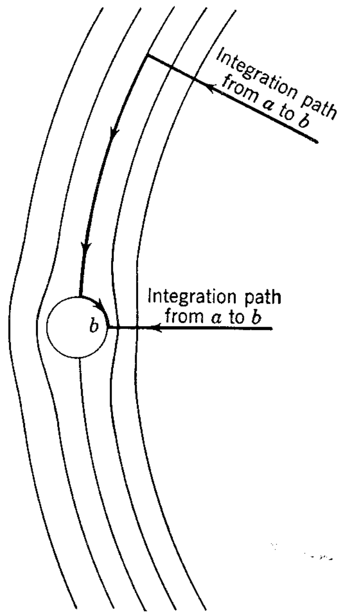


Fig. 3.4 Equipotential surfaces of a portion of the electric field caused by a charged conductor a not shown. Conductor b causes the equipotential surfaces to become distorted. Arrows indicate optional paths of integration between a point on the equipotential surface of conductor b and the conductor a whose charge q_a creates the equipotential surfaces shown.

[2] See W. H. Timbie and V. Bush, "Principles of Electrical Engineering," 4th ed., pp. 519–520, John Wiley & Sons, Inc., New York, 19

region, we see that the distances corresponding to D_2 and D_1 of Eq. (3.3) are D and r_a, respectively in determining V_{ab} due to q_a. Similarly, in determining V_{ab} due to q_b, the distances corresponding to D_2 and D_1 of Eq. (3.3) are r_b and D, respectively. Considering both q_a and q_b, we obtain

$$V_{ab} = \underbrace{\frac{q_a}{2\pi k} \ln \frac{D}{r_a}}_{\text{due to } q_a} + \underbrace{\frac{q_b}{2\pi k} \ln \frac{r_b}{D}}_{\text{due to } q_b} \qquad \text{volts} \tag{3.5}$$

and, since $q_a = -q_b$ for a two-wire line,

$$V_{ab} = \frac{q_a}{2\pi k}\left(\ln \frac{D}{r_a} - \ln \frac{r_b}{D}\right) \qquad \text{volts} \tag{3.6}$$

or, by combining the logarithmic terms,

$$V_{ab} = \frac{q_a}{2\pi k} \ln \left(\frac{D^2}{r_a r_b}\right) \qquad \text{volts} \tag{3.7}$$

The capacitance between conductors is

$$C_{ab} = \frac{q_a}{V_{ab}} = \frac{2\pi k}{\ln (D^2/r_a r_b)} \qquad \text{farads/meter} \tag{3.8}$$

Converting to microfarads per mile, changing the base of the logarithmic term, and assuming a relative permittivity of $k_r = 1$,

$$C_{ab} = \frac{0.0388}{\log (D^2/r_a r_b)} \qquad \mu\text{f/mile} \tag{3.9}$$

If $r_a = r_b$,

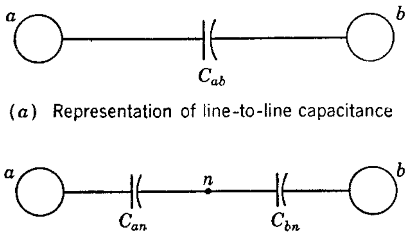
$$C_{ab} = \frac{0.0388}{2 \log D/r} = \frac{0.0194}{\log D/r} \qquad \mu\text{f/mile} \tag{3.10}$$

Equation (3.10) gives the capacitance between the conductors of a two-wire line. Sometimes it is desirable to know the capacitance between one of the conductors and a neutral point between them. For instance, if the line is supplied by a transformer having a grounded center tap, the potential difference between each conductor and the ground is half the potential difference between the two cond[illegible] the *capacitance to ground*, or *capacitance to neutral*, is [illegible] conductor per unit of potential difference between th[illegible] conductor and ground. Thus, the capacitance to neutral for the [illegible]-wire line is twice the *line-to-line capacitance* (capacitance between [illegible]nductors). If the line-to-line capacitance is considered to be comp[illegible] of two equal capaci-tances in series, the voltage across the line divides [illegible]qually between them, and the point between them is at the gro[illegible] potential. Th[illegible] capacitance to neutral is that of one of the two equal series capacitances, or twice the line-to-line capacitance. So,

$$C_n = C_{an} = C_{bn} = \frac{0.0388}{\log D/r} \qquad \mu\text{f/mile, to neutral} \qquad (3.11)$$

The concept of capacitance to neutral is illustrated in Fig. 3.5.

Equation (3.11) corresponds to Eq. (2.36) for inductance. One difference between the equations for capacitance and inductance should be noted carefully. The radius in the equation for capacitance is the actual outside radius of the conductor and not the self GMD of the conductor as in the inductance formula. Also, certain approximations have been made in deriving the capacitance equation which did not enter into the derivation of the inductance equation.



(b) Representation of line-to-neutral capacitance

FIG. 3.5 Relationship between the concepts of line-to-line capacitance and line-to-neutral capacitance.

Equation (3.3), from which Eqs. (3.5) to (3.11) were derived, is based on the assumption of uniform charge distribution over the surface of the conductor. When other charges are present the distribution of charge on the surface of the conductor is not uniform, and the equations derived from Eq. (3.3) are not strictly correct. Nonuniformity of charge distribution can be taken into account by considering the conductors as equipotential surfaces, which they are, rather than as uniformly charged conductors. Then, without much difficulty, the following equation is found:

$$C_n = \frac{0.0388}{\log \left(D/2r + \sqrt{D^2/4r^2 - 1}\right)} \qquad \mu\text{f/mile, to neutral}^3 \qquad (3.12)$$

The formula is for capacitance to neutral for a two-wire line only. For any but the simplest configurations of conductors of parallel-circuit or three-phase lines, the derivation of an equation to account for the actual charge distribution becomes too involved to be practical. The assumption of uniform charge distribution leads to very slight errors if

[3] An alternative form of the equation, giving identical results, is

$$C_n = \frac{0.0388 \times 2.303}{\cosh^{-1}(D/2r)} = \frac{0.0894}{\cosh^{-1}(D/2r)} \qquad \mu\text{f/mile, t[illegible]utral}$$

The derivation may be found in texts on electricity and ma[illegible]ism. See for instance J. C. Slater and N. H. Frank, "Electromagnetism," pp. [illegible]-22, McGraw-Hill Book Company, Inc., New York, 1947; W. B. Boast, "Princ[illegible] of Electric and Magnetic Fields," pp. 92–93, Harper & Brothers, New York, 1948 the spacing between conductors is large compared to their diameters, which is the actual case for open-wire power transmission lines. For a single-phase line the amount of error involved when the charge distribution is assumed to be uniform may be seen by comparing the capacitances computed by Eq. (3.11) to those computed by Eq. (3.12) for different ratios of D/r. Table 3.1 shows the error occurring when Eq. (3.11) is used instead of Eq. (3.12).

TABLE 3.1 ERROR CAUSED BY ASSUMING UNIFORM CHARGE DISTRIBUTION IN COMPUTING CAPACITANCE OF A TWO-WIRE LINE

Ratio D/r	Per Cent Error in Eq. (3.11)
10	0.44
20	0.084
50	0.010
100	0.002
200	0.0005

A question arises as to the value to be used in the denominator of the argument of the logarithm in Eq. (3.11) when the conductor is a stranded cable, since the equation was derived for a solid, round conductor. Since electric flux is perpendicular to the surface of a perfect conductor, the electric field at the surface of a stranded conductor is not the same as the field at the surface of a cylindrical conductor. Therefore, the capacitance calculated for a stranded conductor by substituting the outside radius of the conductor for r in Eq. (3.11) will be slightly in error because of the difference between the field in the neighborhood of such a conductor and the field near a solid conductor for which Eq. (3.11) was derived. The error is very small, however, since only the field very close to the surface of the conductor is affected. The outside radius of the stranded conductor is used in calculating the capacitance.

After the capacitance to neutral has been found, the capacitive reactance existing between one conductor and neutral is found as follows:

$$X_C = \frac{1}{2\pi fC} = \frac{4.093}{f} \times 10^6 \log \frac{D}{r} \qquad \text{ohms/mile, to neutral} \qquad (3.13)$$

Some tables list capacitive susceptance at various spacings for the common conductors or give the outside diameter from which capacitive reactance and its reciprocal, capacitive susceptance, can be calculated. Other tables, as suggested by W. A. Lewis, list *capacitive reactance at 1-ft spacing* for the common conductors. Such tables are used in conjunction with tables of *capacitive reactance spacing factor*. If D and r in Eq. (3.13) are in feet, capacitive reactance at 1-ft spacing is the first term, and capacitive reactance spacing factor is the second term when the equation is expanded as follows:

$$X_C = \frac{4.093}{f} \times 10^6 \log \frac{1}{r} + \frac{4.093}{f} \times 10^6 \log D$$

ohms/mile, to neutral (3.14)

The sum of capacitive reactance at 1-ft spacing and capacitive reactance spacing factor, as given by Eq. (3.14), is capacitive reactance to neutral. Tables A.1 and A.2 in the Appendix list capacitive reactance at 1-ft spacing, and Table A.4 lists values of capacitive reactance spacing factor. The use of capacitive reactance tables is similar to that of inductive reactance tables discussed in Chap. 2.

Example 3.1

Find the capacitive susceptance per mile of a two-conductor single-phase line operating at 60 cps. The conductors are each No. 1/0 seven-strand hard-drawn copper wire spaced 18 ft between centers. This is the line described in Example 2.3.

Solution

The outside diameter of the conductor is $3 \times 0.1228 = 0.368$ in.

$$\text{The radius } r = \frac{0.368}{2 \times 12} = 0.0153 \text{ ft}$$

$$X_C = \frac{4.093}{60} \times 10^6 \log \frac{18}{0.0153} = 0.210 \times 10^6 \text{ ohm/mile, to neutral}$$

$$b_C = 1/X_C = 4.76 \times 10^{-6} \text{ mho/mile, to neutral}$$

Tables of capacitive reactance at 1-ft spacing and capacitive reactance spacing factor give

$$\text{Capacitive reactance at 1-ft spacing} = \frac{4.093}{60} \times 10^6 \log \frac{1}{0.0153}$$
$$= 0.124 \times 10^6 \text{ ohm/mile}$$

$$\text{Capacitive reactance spacing factor} = \frac{4.093}{60} \times 10^6 \log 18$$
$$= 0.086 \times 10^6 \text{ ohm/mile}$$

$$\text{Capacitive reactance to neutral} = 10^6(0.124 + 0.086)$$
$$= 0.210 \times 10^6 \text{ ohm/mile}$$

from which

$$b_C = \frac{1}{0.210 \times 10^6} = 4.76 \times 10^{-6} \text{ mho/mile, to neutral}$$

Capacitive reactance and susceptance from line to line

$$X_C = 2 \times 0.210 \times 10^6 = 0.420 \times 10^6 \text{ ohm/m}$$
$$b_C = 4.76 \times 10^{-6}/2 = 2.38 \times 10^{-6} \text{ mho/mile}$$

3.5 Potential Difference between Two Co… of … Charged Conductors. If a number of conduct… arranged so that they are parallel to each other, the voltage between any two of them can be found by applying Eq. (3.3) repeatedly to determine the voltage between the two conductors in question due to the charge on each conductor in the group independently. The voltage drop between the two conductors is the sum of the voltage drops due to each charged conductor. Such a group of conductors is shown in Fig. 3.6. If we assume that there are no other charged surfaces in the vicinity, the sum of the charges on the conductors is zero. If the ground is far enough away to have negligible effect, and if we assume further that the spacing between conductors is large compared to the radius of any one so that the charge distribution over the surface of a conductor will be uniform, repeated application of Eq. (3.3) will yield accurate results. So, from conductor a to conductor b, the voltage drop is

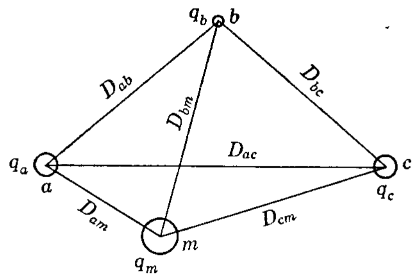


FIG. 3.6 Group of parallel charged conductors.

$$V_{ab} = \frac{1}{2\pi k}\left(q_a \ln \frac{D_{ab}}{r_a} + q_b \ln \frac{r_b}{D_{ba}} + q_c \ln \frac{D_{cb}}{D_{ca}} + \cdots + q_m \ln \frac{D_{mb}}{D_{ma}}\right) \tag{3.15}$$

Each term in Eq. (3.15) is the potential drop from a to b due to the charge on one of the conductors in the group. In a similar manner the voltage drop may be found between other pairs of conductors in the group. For example,

$$V_{ac} = \frac{1}{2\pi k}\left(q_a \ln \frac{D_{ac}}{r_a} + q_b \ln \frac{D_{bc}}{D_{ba}} + q_c \ln \frac{r_c}{D_{ca}} + \cdots + q_m \ln \frac{D_{mc}}{D_{ma}}\right) \tag{3.16}$$

$$V_{am} = \frac{1}{2\pi k}\left(q_a \ln \frac{D_{am}}{r_a} + q_b \ln \frac{D_{bm}}{D_{ba}} + q_c \ln \frac{D_{cm}}{D_{ca}} + \cdots + q_m \ln \frac{r_m}{D_{ma}}\right)$$

If the voltages between conductor a and the other conductors are known, the group of simultaneous equations expressing the voltage drops in terms of charges may be solved for the charges. Since the voltages are sinusoidal and expressed as complex quantities, the charges are sinusoidal and are expressed as complex quantities. Similar equations were found for the complex values of flux linkages where the currents were expressed in complex form. The equations above are seldom solved to find charge but are used in the derivation of capacitance formulas for polyphase circuits.

3.6 Capacitance of a Three-phase Line with Equilateral Spacing. The three identical conductors of radius r of a three-phase line with equilateral spacing are shown in Fig. 3.7. Figure 3.8 is the phasor diagram of voltages for this line. To solve for the capacitance to neutral, we first write the expression for the voltage drops from conductor a

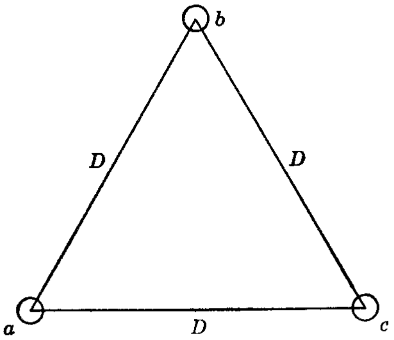


FIG. 3.7 Cross section of a three-phase line with equilateral spacing.

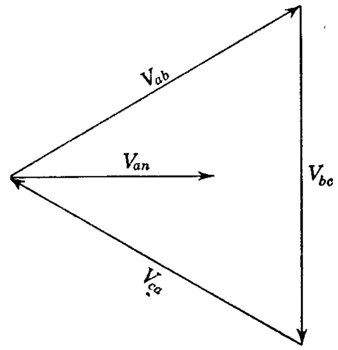


FIG. 3.8 Phasor diagram of the balanced voltages of a three-phase line.

to conductor b and from conductor a to conductor c. Thus, from Eqs. (3.15) and (3.16),

$$V_{ab} = \frac{1}{2\pi k}\left(q_a \ln \frac{D}{r} + q_b \ln \frac{r}{D} + q_c \ln \frac{D}{D}\right) \quad \text{volts} \qquad (3.18)$$

and

$$V_{ac} = \frac{1}{2\pi k}\left(q_a \ln \frac{D}{r} + q_b \ln \frac{D}{D} + q_c \ln \frac{r}{D}\right) \quad \text{volts} \qquad (3.19)$$

Adding Eqs. (3.18) and (3.19) gives

$$V_{ab} + V_{ac} = \frac{1}{2\pi k}\left[2q_a \ln \frac{D}{r} + (q_b + q_c) \ln \frac{r}{D}\right] \quad \text{volts} \qquad (3.20)$$

If we assume there are no other charges in the vicinity, the sum of the charges on the three conductors is zero, and we can substitute $-q_a$ in Eq. (3.20) for $q_b + q_c$ and obtain

$$V_{ab} + V_{ac} = \frac{3q_a}{2\pi k} \ln \frac{D}{r} \quad \text{volts} \qquad (3.21)$$

From the phasor diagram of Fig. 3.8, we obtain the following relations between the line voltages V_{ab} and V_{ac} and the voltage V_{an} from line a

to the neutral of the three-phase circuit:

$$V_{ab} = \sqrt{3}\, V_{an}(0.866 + j0.5) \tag{3.22}$$

$$V_{ac} = -V_{ca} = \sqrt{3}\, V_{an}(0.866 - j0.5) \tag{3.23}$$

Adding Eqs. (3.22) and (3.23) gives

$$V_{ab} + V_{ac} = 3V_{an} \tag{3.24}$$

Substituting $3V_{an}$ for $V_{ab} + V_{ac}$ in Eq. (3.21), we obtain

$$V_{an} = \frac{q_a}{2\pi k} \ln \frac{D}{r} \qquad \text{volts} \tag{3.25}$$

Since capacitance to neutral is the ratio of the charge on a conductor to the voltage between that conductor and neutral,

$$C_n = \frac{q_a}{V_{an}} = \frac{2\pi k}{\ln D/r} \qquad \text{farads/meter, to neutral} \tag{3.26}$$

For a relative permittivity of $k_r = 1$,

$$C_n = \frac{0.0388}{\log D/r} \qquad \mu\text{f/mile, to neutral} \tag{3.27}$$

Comparison of Eqs. (3.27) and (3.11) shows that the two are identical. These equations express the capacitance to neutral for single-phase and equilaterally spaced three-phase lines, respectively. We saw in Chap. 2 that the equations for inductance per conductor were the same for single-phase and equilaterally spaced three-phase lines.

The term *charging current* is applied to the current associated with the capacitance of a line. For a single-phase circuit, the charging current is the product of the line-to-line voltage and the line-to-line susceptance, or

$$I_{chg} = j\omega C_{ab} V_{ab} \tag{3.28}$$

For a three-phase line, the charging current is found by multiplying the voltage to neutral by the capacitive susceptance to neutral. This gives the charging current per phase and is in accord with the calculation of balanced three-phase circuits on the basis of a single phase with neutral return. The charging current in phase a is

$$I_{chg} = j\omega C_n V_{an} \tag{3.29}$$

3.7 Capacitance of a Three-phase Line with Unsymmetrical Spacing.

When the conductors of a three-phase line are not equilaterally spaced, the problem of calculating the capacitance becomes more difficult. If such a line is not transposed, the capacitances of each phase to neutral are unequal, and, if the line is transposed, the capacitance of any one phase to neutral is different for each position occupied by the conductor in the transposition cycle. In the transposed line, however, the *average* capacitance to neutral of any phase for the complete transposition cycle is the same as the average capacitance to neutral of any other phase, since each phase occupies the same position as every other phase over an equal distance. The dissymmetry of the untransposed line is slight for the usual configuration, and capacitance calculations are carried out as though all lines were transposed.

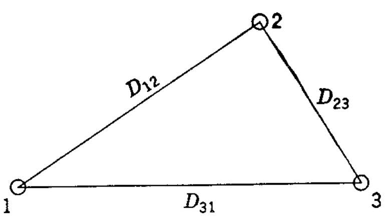


FIG. 3.9 Cross section of a three-phase line with unsymmetrical spacing.

Equation (3.15) may be applied to the line shown in Fig. 3.9 to compute V_{ab} due to the charges on all three conductors. Three equations are found for V_{ab} for the three different parts of the transposition cycle. With phase a in position 1, b in position 2, and c in position 3,

$$V_{ab} = \frac{1}{2\pi k}\left(q_a \ln \frac{D_{12}}{r} + q_b \ln \frac{r}{D_{12}} + q_c \ln \frac{D_{23}}{D_{31}}\right) \quad \text{volts} \quad (3.30)$$

With a in position 2, b in position 3, and c in position 1,

$$V_{ab} = \frac{1}{2\pi k}\left(q_a \ln \frac{D_{23}}{r} + q_b \ln \frac{r}{D_{23}} + q_c \ln \frac{D_{31}}{D_{12}}\right) \quad \text{volts} \quad (3.31)$$

and, with a in position 3, b in position 1, and c in position 2,

$$V_{ab} = \frac{1}{2\pi k}\left(q_a \ln \frac{D_{31}}{r} + q_b \ln \frac{r}{D_{31}} + q_c \ln \frac{D_{12}}{D_{23}}\right) \quad \text{volts} \quad (3.32)$$

Equations (3.30) to (3.32) are similar in form to Eqs. (2.66) to (2.68) for the flux linkages of one conductor of a transposed line. Each of the latter equations gives the value of flux linkages in one part of the transposition cycle. An average value of flux linkages over the complete transposition cycle was found by noting that the current in any phase was the same in any part of the transposition cycle. If the charge per unit length on the conductor of each phase was the same for [illegible] of that phase as for any other position in the transposition cycle, the voltages computed by Eqs. (3.30) to (3.32) would all be different, and an average voltage could be found for the complete cycle. Actually, the capacitance to neutral of a phase in one part of the transposition cycle is in parallel with the capacitances to neutral of that phase in the other parts of the transposition cycle. Therefore, if we disregard voltage drop along the line, the voltage to neutral of a phase in one part of the cycle is equal to the voltage to neutral of that phase in any part of the cycle. Hence, the voltage between any two conductors is the same in one part of the transposition cycle as in other parts between the same conductors. Since the voltage is the same anywhere in the transposition cycle, it follows that the charge on any conductor must be different when the position of the conductor changes with respect to the other conductors. A treatment of Eqs. (3.30) to (3.32) analogous to that of Eqs. (2.66) to (2.68) is not rigorous.

Equations (3.30) to (3.32) have 10 unknowns, the voltage V_{ab} and nine charges, for the charge on each of the three conductors is different in each of the three positions occupied by a conductor in the three parts of the transposition cycle. Thus, a rigorous solution for capacitance in terms of the ratio of charge to potential difference requires six more equations in order to eliminate all the unknowns except one voltage and one charge. Three additional equations, similar to Eqs. (3.30) to (3.32), may be written for the voltage V_{bc}, and the latter voltage may be expressed as $V_{ab}(-0.5 - j0.866)$ if the voltages on the line are assumed to be balanced. The other three equations required may be obtained by equating the sum of the charges in each of the three parts of the transposition cycle to zero.

The rigorous solution for capacitance is too involved to be practical except perhaps for flat spacing with equal distances between adjacent conductors. With the usual spacings and conductors, sufficient accuracy is obtained by assuming the charge per unit length on a conductor to be the same in every part of the transposition cycle. When the above assumption is made with regard to charge, the voltage between a pair of conductors is different for each part of the transposition cycle. Then an average value of voltage between the conductors can be found, and the capacitance calculated from the average voltage. We obtain the average voltage by adding Eqs. (3.30), (3.31), and (3.32) and dividing the result by three. The average voltage between conductors a and b, based on the assumption of the same charge on a conductor regardless of its position in the transposition cycle, is

$$V_{ab} = \frac{1}{6\pi k}\left[q_a \ln\left(\frac{D_{12}D_{23}D_{31}}{r^3}\right) + q_b \ln\left(\frac{r^3}{D_{12}D_{23}D_{31}}\right) + q_c \ln\left(\frac{D_{12}D_{23}D_{31}}{D_{12}D_{23}D_{31}}\right)\right] \quad \text{volts}$$

$$= \frac{1}{2\pi k}\left(q_a \ln\frac{D_{eq}}{r} + q_b \ln\frac{r}{D_{eq}}\right) \quad \text{volts} \tag{3.33}$$

where

$$D_{eq} = \sqrt[3]{D_{12}D_{23}D_{31}} \tag{3.34}$$

Similarly, the average voltage drop from conductor a to conductor c is

$$V_{ac} = \frac{1}{2\pi k}\left(q_a \ln \frac{D_{eq}}{r} + q_c \ln \frac{r}{D_{eq}}\right) \qquad \text{volts} \qquad (3.35)$$

Applying Eq. (3.24) to find the voltage to neutral, we have

$$3V_{an} = V_{ab} + V_{ac} = \frac{1}{2\pi k}\left(2q_a \ln \frac{D_{eq}}{r} + q_b \ln \frac{r}{D_{eq}} + q_c \ln \frac{r}{D_{eq}}\right) \qquad \text{volts} \qquad (3.36)$$

Since $q_a + q_b + q_c = 0$ in a balanced three-phase circuit,

$$3V_{an} = \frac{3}{2\pi k} q_a \ln \frac{D_{eq}}{r} \qquad \text{volts} \qquad (3.37)$$

and

$$C_n = \frac{q_a}{V_{an}} = \frac{2\pi k}{\ln D_{eq}/r} \qquad \text{farads/meter, to neutral} \qquad (3.38)$$

For a relative permittivity of $k_r = 1$,

$$C_n = \frac{0.0388}{\log D_{eq}/r} \qquad \mu\text{f/mile, to neutral} \qquad (3.39)$$

Equation (3.39) for capacitance to neutral of a transposed three-phase line corresponds to Eq. (2.71) for the inductance per phase of a similar line.

Example 3.2

Find the capacitance and capacitive reactance per mile of the line described in Example 2.4. If the line is operated at 22,000 volts, find the charging current per mile.

Solution

$$r = \frac{0.258}{2 \times 12} = 0.01075 \text{ ft}$$

$$D_{eq} = 5.45 \text{ ft}$$

$$C_n = \frac{0.0388}{\log (5.45/0.01075)} = 0.01438\ \mu\text{f/mile, to neutral}$$

$$X_C = \frac{10^6}{2\pi \times 60 \times 0.01438} = 0.185 \times 10^6 \text{ ohm/mile, to neutral}$$

or from tables

Capacitive reactance at 1-ft spacing = 0.1345×10^6

Capacitive reactance spacing factor for 5.45 ft = $0.0503 \times$

Capacitive reactance = 0.1848×1 ohm/mile, to neutral

The magnitude of the charging current is

$$I_{chg} = 2\pi \times 60 \times 0.01438 \times 10^{-6} \times 22{,}000/\sqrt{3}$$
$$= 68.8 \times 10^{-3} \text{ amp/mile}$$

3.8 Effect of Earth on the Capacitance of Three-phase Transmission Lines. Earth affects the capacitance of a transmission line because its presence alters the electric field of the line. If we assume the earth to be a perfect conductor in the form of a horizontal plane of infinite extent, we realize that the electric field of charged conductors above the earth is not the same as it would be if the equipotential surface of the earth were not present. The electric field of the charged conductors is forced to conform to the presence of the earth's equipotential surface.

Consider a circuit consisting of a single overhead conductor with a return path through the earth. In charging the conductor, charges come from the earth to reside on the conductor, and a potential difference exists between the conductor and earth. The earth has a charge equal in magnitude to that on the conductor but of opposite sign. Electric flux from the charges on the conductor to the charges on the surface of the earth is perpendicular to the earth's surface, since the surface is assumed to be a perfect conductor. Let us imagine a fictitious conductor of the same size and shape as the overhead conductor lying directly below the original conductor at a distance equal to twice the distance of the conductor above the plane of the ground. The fictitious conductor is below the surface of the earth by a distance equal to the distance of the overhead conductor above the earth. If the earth is removed and a charge equal and opposite to that on the overhead conductor is assumed on the fictitious conductor, the plane midway between the original conductor and the fictitious conductor is an equipotential surface and occupies the same position as the equipotential surface of the earth. The electric flux between the overhead conductor and this equipotential surface is the same as that which existed between the conductor and the earth. Thus, for purposes of calculation of capacitance, the earth may be replaced by a fictitious charged conductor below the surface of the earth by a distance equal to that of the overhead conductor above the earth. Such a conductor has a charge equal in magnitude and opposite in sign to that of the original conductor and is called the *image conductor*.

The method of calculating capacitance by replacing the earth by the image of an overhead conductor can be extended to more than one conductor. If we locate an image conductor for each overhead conductor, the flux between the original conductors and their images is perpendicular to the plane which replaces the earth, and that plane is an equipotential surface. The flux above the plane is the same as it is when the earth is present instead of the image conductors.

To apply the method of images to the calculation of capacitance for a three-phase line, refer to Fig. 3.10. We shall assume that the line is transposed and that conductors a, b, and c carry the charges q_a, q_b, and q_c and occupy positions 1, 2, and 3, respectively, in the first part of the transposition cycle. The plane of the earth is shown, and below it are the conductors with the image charges $-q_a$, $-q_b$, and $-q_c$. Equations

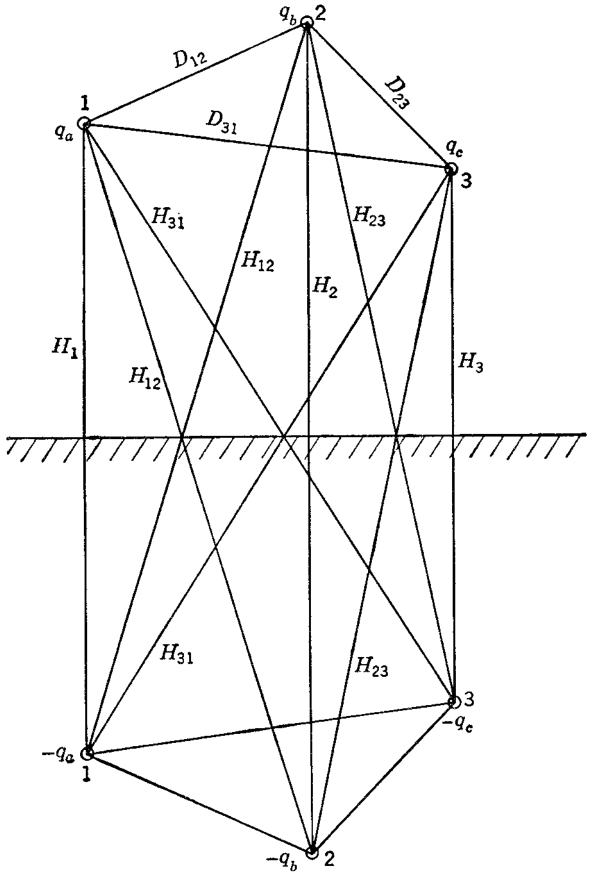


FIG. 3.10 Three-phase line and its image.

for the three parts of the transposition cycle can be written for the voltage drop from conductor a to conductor b as determined by the three charged conductors and their images. With conductor a in position 1, b in position 2, and c in position 3,

$$V_{ab} = \frac{1}{2\pi k}\left[q_a\left(\ln\frac{D_{12}}{r} - \ln\frac{H_{12}}{H_1}\right) + q_b\left(\ln\frac{r}{D_{12}} - \ln\frac{H_2}{H_{12}}\right) + q_c\left(\ln\frac{D_{23}}{D_{31}} - \ln\frac{H_{23}}{H_{31}}\right)\right] \quad (3.40)$$

With conductor a in position 2, b in position 3, and c in position 1,

$$V_{ab} = \frac{1}{2\pi k}\left[q_a\left(\ln\frac{D_{23}}{r} - \ln\frac{H_{23}}{H_2}\right) + q_b\left(\ln\frac{r}{D_{23}} - \ln\frac{H_3}{H_{23}}\right) + q_c\left(\ln\frac{D_{31}}{D_{12}} - \ln\frac{H_{31}}{H_{12}}\right)\right] \quad (3.41)$$

and, with a in position 3, b in position 1, and c in position 2,

$$V_{ab} = \frac{1}{2\pi k}\left[q_a\left(\ln\frac{D_{31}}{r} - \ln\frac{H_{31}}{H_3}\right) + q_b\left(\ln\frac{r}{D_{31}} - \ln\frac{H_1}{H_{31}}\right) + q_c\left(\ln\frac{D_{12}}{D_{23}} - \ln\frac{H_{12}}{H_{23}}\right)\right] \quad (3.42)$$

If the approximately correct assumption of constant charge per unit length of conductor throughout the transposition cycle is made, an average value of V_{ab} for the three parts is

$$V_{ab} = \frac{1}{6\pi k}\left\{q_a\left[\ln\left(\frac{D_{12}D_{23}D_{31}}{r^3}\right) - \ln\left(\frac{H_{12}H_{23}H_{31}}{H_1H_2H_3}\right)\right] + q_b\left[\ln\left(\frac{r^3}{D_{12}D_{23}D_{31}}\right) - \ln\left(\frac{H_1H_2H_3}{H_{12}H_{23}H_{31}}\right)\right]\right\} \quad (3.43)$$

$$V_{ab} = \frac{1}{2\pi k}\left[q_a\left(\ln\frac{D_{eq}}{r} - \ln\frac{\sqrt[3]{H_{12}H_{23}H_{31}}}{\sqrt[3]{H_1H_2H_3}}\right) + q_b\left(\ln\frac{r}{D_{eq}} - \ln\frac{\sqrt[3]{H_1H_2H_3}}{\sqrt[3]{H_{12}H_{23}H_{31}}}\right)\right] \quad (3.44)$$

Similarly,

$$V_{ac} = \frac{1}{2\pi k}\left[q_a\left(\ln\frac{D_{eq}}{r} - \ln\frac{\sqrt[3]{H_{12}H_{23}H_{31}}}{\sqrt[3]{H_1H_2H_3}}\right) + q_c\left(\ln\frac{r}{D_{eq}} - \ln\frac{\sqrt[3]{H_1H_2H_3}}{\sqrt[3]{H_{12}H_{23}H_{31}}}\right)\right] \quad (3.45)$$

and

$$V_{ab} + V_{ac} = 3V_{an} = \frac{1}{2\pi k}\left[2q_a\left(\ln\frac{D_{eq}}{r} - \ln\frac{\sqrt[3]{H_{12}H_{23}H_{31}}}{\sqrt[3]{H_1H_2H_3}}\right) + (q_b + q_c)\left(\ln\frac{r}{D_{eq}} - \ln\frac{\sqrt[3]{H_1H_2H_3}}{\sqrt[3]{H_{12}H_{23}H_{31}}}\right)\right] \quad (3.46)$$

Since $q_a + q_b + q_c = 0$

$$3V_{an} = \frac{3}{2\pi k}\left[q_a\left(\ln\frac{D_{eq}}{r} - \ln\frac{\sqrt[3]{H_{12}H_{23}H_{31}}}{\sqrt[3]{H_1H_2H_3}}\right)\right] \quad (3.47)$$

and

$$C_n = \frac{0.0388}{\log D_{eq}/r - \log\left(\sqrt[3]{H_{12}H_{23}H_{31}}/\sqrt[3]{H_1H_2H_3}\right)} \quad \mu\text{f per mile to neutral} \quad (3.48)$$

Comparison of Eqs. (3.39) and (3.48) shows that the effect of the earth is to increase the capacitance of a line. To account for the earth the denominator of Eq. (3.39) must have subtracted from it the term $\log\left(\sqrt[3]{H_{12}H_{23}H_{31}}/\sqrt[3]{H_1H_2H_3}\right)$. If the conductors are high above ground compared to the distances between them, the diagonal distances in the numerator of the correction term are nearly equal to the vertical distances in the denominator, and the term is very small. This is the usual case, and the effect of ground is generally neglected for three-phase lines except for calculations by symmetrical components when the sum of the three line currents is not zero. Calculations of capacitance for this condition will be considered in Chap. 12.

3.9 Parallel-circuit Three-phase Lines. Let us consider two special arrangements of parallel-circuit lines, the double-circuit line with hexagonal spacing and the double-circuit line with flat, vertical spacing. The equation for capacitance of each of these lines is relatively simple. Many double-circuit lines have flat, vertical spacing, and the spacing of most other double-circuit lines is intermediate between flat spacing and hexagonal spacing.

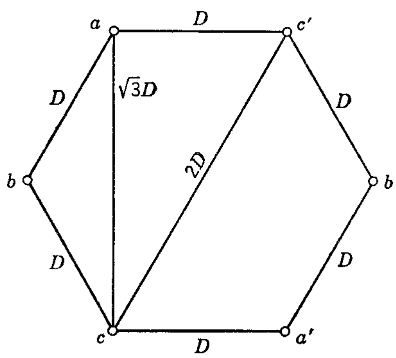


FIG. 3.11 Double-circuit three-phase line with hexagonal spacing.

Consider first the double-circuit line with hexagonal spacing shown in Fig. 3.11. Phase a is composed of conductors a and a', phase b of conductors b and b', and phase c of conductors c and c'. The two conductors of each phase are electrically in parallel and have the same charge. Because of the symmetrical arrangement the phases are balanced, and the conductors of each individual phase are also balanced, if the effect of ground is neglected. Therefore, transposition of the conductors is not necessary to balance the phases. The equations for voltage drop may be written in the usual manner, and the derivation of the expression for capacitance proceeds as follows:

$$V_{ab} = \frac{1}{2\pi k}\left[q_a\left(\ln\frac{D}{r} + \ln\frac{\sqrt{3}}{2}\right) + q_b\left(\ln\frac{r}{D} + \ln\frac{2}{\sqrt{3}}\right)\right.$$
$$\left. + q_c\left(\ln\frac{1}{\sqrt{3}} + \cdots\right)\right] \quad (3.49)$$

$$V_{ab} = \frac{1}{2\pi k}(q_a - q_b)\ln\left(\frac{\sqrt{3}\,D}{2r}\right) \quad \text{volts} \quad (3.50)$$

$$V_{ac} = \frac{1}{2\pi k}(q_a - q_c)\ln\left(\frac{\sqrt{3}\,D}{2r}\right) \quad \text{volts} \quad (3.51)$$

$$V_{ab} + V_{ac} = 3V_{an} = \frac{1}{2\pi k}\,(2q_a - q_b - q_c)\ln\left(\frac{\sqrt{3}\,D}{2r}\right) \qquad \text{volts} \qquad (3.52)$$

and, since $q_a + q_b + q_c = 0$,

$$3V_{an} = \frac{3q_a}{2\pi k}\ln\left(\frac{\sqrt{3}\,D}{2r}\right) \qquad \text{volts} \qquad (3.53)$$

Then

$$C_n = \frac{q_a}{V_{an}} = \frac{2\pi k}{\ln\,(\sqrt{3}\,D/2r)} \text{ farads/meter/conductor, to neutral} \qquad (3.54)$$

$$C_n = \frac{0.0388}{\log\,(\sqrt{3}\,D/2r)} \qquad \mu\text{f/mile/conductor, to neutral} \qquad (3.55)$$

Equations (3.54) and (3.55) give the capacitance from one conductor to neutral, not from one phase to neutral. The expression for capacitance was found by taking the ratio of the charge on only one of the two conductors of a phase to the voltage to neutral. To find the capacitance to neutral per phase, we note that each phase consists of two identically charged conductors in parallel. Therefore, the capacitance to neutral per phase is twice the capacitance to neutral of one conductor, or

$$C_n = 2 \times \frac{0.0388}{\log\,(\sqrt{3}\,D/2r)} \qquad \mu\text{f/mile/phase, to neutral} \qquad (3.56)$$

We recall that the inductance of parallel-circuit three-phase lines was calculated by using the method of GMD. Let us apply a *modified* method of GMD to the calculation of the capacitance of a hexagonally-spaced double-circuit line. In applying the method to capacitance calculations, actual radii of the individual conductors of a phase will be used to obtain the *modified* self GMD of a phase. We speak of the modified self GMD and the modified method of GMD in connection with capacitance calculations because we are not following the mathematical concept of GMD discussed in Chap. 2 when we use the actual radius of a conductor composing one of the circuits instead of the self GMD of that conductor. We must use the radius of a conductor rather than its self GMD b[illegible]use all the charge resides on the surface. The idea of self G[illegible] inductance calculations because of the internal flux [illegible] of a con-ductor. We shall still combine the parallel conduct[illegible] phase by the method of geometric mean distances, and we s[illegible] the method in all other respects. For the hexagonally space[illegible] Fig. 3.11 the modified self GMD for all phases is the same [illegible] differs from the self GMD by the substitution o[illegible] r'. So,

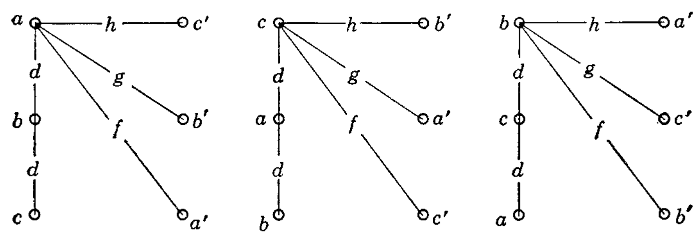
$$D_s = \sqrt{2rD}$$
$$\text{GMD } a \text{ to } b = D_{ab} = \sqrt[4]{3}\, D$$
$$\text{GMD } b \text{ to } c = D_{bc} = \sqrt[4]{3}\, D$$
$$\text{GMD } c \text{ to } a = D_{ca} = \sqrt[4]{3}\, D$$
$$D_{eq} = \sqrt[3]{D_{ab}D_{bc}D_{ca}} = \sqrt[4]{3}\, D$$

and, replacing r in Eq. (3.39) by D_s,

$$C_n = \frac{0.0388}{\log D_{eq}/D_s} = \frac{0.0388}{\log (\sqrt[4]{3}\, D/\sqrt{2rD})} = 2 \times \frac{0.0388}{\log (\sqrt{3}\, D/2r)}$$

μf/mile/phase, to neutral

which is identical to Eq. (3.56) and shows that our modified GMD method is valid in this case. We note that the GMD method always gives *per phase* values, rather than *per conductor* values, because it com-



(*a*) Phase *a* in position 1 (*b*) Phase *a* in position 2 (*c*) Phase *a* in position 3

FIG. 3.12 Arrangement of the conductors of a double-circuit three-phase line in the three parts of a transposition cycle.

bines the conductors electrically in parallel in any one phase in computing distances.

Now consider a double-circuit line with flat, vertical spacing, as shown in Fig. 3.12. Such a line is not balanced without transpositions. Therefore, the derivation will be made for a transposed line, and again charge on a conductor per unit length will be assumed to remain the same throughout the transposition cycle. Since the conductor of each phase of one circuit is in electrical parallel with the conductor of the same phase in the other circuit, a conductor in one circuit has the same charge as the conductor in the other circuit with which it is in parallel. In order to compute the voltage drop from a to b, the charge q_a on conductor a and an identical charge q_a on conductor a' must be considered. Charges on the conductors of the other phases are treated in a similar manner. With phase a in position 1,

$$V_{ab} = \frac{1}{2\pi k}\left[q_a\left(\ln\frac{d}{r} + \ln\frac{g}{f}\right) + q_b\left(\ln\frac{r}{d} + \ln\frac{h}{g}\right) + q_c\left(\ln\frac{d}{2d} + \ln\frac{g}{h}\right)\right] \quad (3.57)$$

With a in position 2,

$$V_{ab} = \frac{1}{2\pi k}\left[q_a\left(\ln\frac{d}{r} + \ln\frac{g}{h}\right) + q_b\left(\ln\frac{r}{d} + \ln\frac{f}{g}\right) + q_c\left(\ln\frac{2d}{d} + \ln\frac{h}{g}\right)\right] \quad (3.58)$$

and, with a in position 3,

$$V_{ab} = \frac{1}{2\pi k}\left[q_a\left(\ln\frac{2d}{r} + \ln\frac{h}{f}\right) + q_b\left(\ln\frac{r}{2d} + \ln\frac{f}{h}\right) + q_c\left(\ln\frac{d}{d} + \ln\frac{g}{g}\right)\right] \quad (3.59)$$

The average value of V_{ab} in the three positions is

$$V_{ab} = \frac{1}{6\pi k}\left[q_a\left(\ln\frac{2d^3}{r^3} + \ln\frac{g^2}{f^2}\right) - q_b\left(\ln\frac{2d^3}{r^3} + \ln\frac{g^2}{f^2}\right)\right] \quad (3.60)$$

Similarly,

$$V_{ac} = \frac{1}{6\pi k}\left[q_a\left(\ln\frac{2d^3}{r^3} + \ln\frac{g^2}{f^2}\right) - q_c\left(\ln\frac{2d^3}{r^3} + \ln\frac{g^2}{f^2}\right)\right] \quad (3.61)$$

and

$$V_{ab} + V_{ac} = 3V_{an} = \frac{3q_a}{6\pi k}\ln\left(\frac{2d^3g^2}{r^3f^2}\right) \quad (3.62)$$

$$V_{an} = \frac{q_a}{2\pi k}\ln\left[\frac{\sqrt[3]{2}\,d}{r}\left(\frac{g}{f}\right)^{2/3}\right] \quad \text{volts} \quad (3.63)$$

$$C_n = \frac{0.0388}{\log\left[\frac{\sqrt[3]{2}\,d}{r}\left(\frac{g}{f}\right)^{2/3}\right]} \quad \mu\text{f/mile, one conductor to neutral} \quad (3.64)$$

The capacitance to neutral per phase (two conductors in parallel) is

$$C_n = 2 \times \frac{0.0388}{\log\left[\frac{\sqrt[3]{2}\,d}{r}\left(\frac{g}{f}\right)^{2/3}\right]} \quad \mu\text{f/mile/phase, to neutral} \quad (3.65)$$

Now let us apply our modified GMD method to the derivation of the expression for capacitance to determine whether the method is valid. By comparison with Eqs. (2.75) and (2.77),

$$D_{eq} = 2^{1/6}d^{1/2}g^{1/3}h^{1/6}$$

and

$$D_s = r^{1/2}f^{1/3}h^{1/6}$$

where r replaces r' since we are dealing with capacitance instead of inductance. Then

$$C_n = \frac{0.0388}{\log D_{eq}/D_s} = \frac{0.0388}{\log\left[2^{1/6}\left(\frac{d}{r}\right)^{1/2}\left(\frac{g}{f}\right)^{1/3}\right]} \tag{3.66}$$

$$C_n = 2 \times \frac{0.0388}{\log\left[\frac{\sqrt[3]{2}\,d}{r}\left(\frac{g}{f}\right)^{2/3}\right]} \quad \mu\text{f/mile/phase, to neutral} \tag{3.67}$$

Since Eqs. (3.65) and (3.67) are identical, the modified GMD method holds to the same close approximation as Eq. (3.65). Both equations are slightly in error because of the assumption of the same charge on a conductor in any position of the transposition cycle, because of the neglect of the effect of earth, and because we have assumed in all our derivations a uniform distribution of charge over the surface of the conductors. All these differences are negligible for the usual overhead line.

Since the modified GMD method has been shown to be valid for hexagonal spacing and for flat, vertical spacing, it is reasonable to assume that it may be used for arrangements intermediate between the two.

Example 3.3

Find the capacitance and the 60-cycle capacitive susceptance to neutral per mile per phase of the double-circuit line described in Example 2.5.

Solution

From Example 2.5, $D_{eq} = 16.1$ ft.

The calculation of the modified D_s is the same as in Example 2.5 except that r is used instead of r'. The outside diameter of 19-strand, 300,000-circular-mil conductor is 0.629 in.

$$r = \frac{0.629}{2 \times 12} = 0.026 \text{ ft}$$

$$D_s = (\sqrt{26.9 \times 0.026}\,\sqrt{21.0 \times 0.026}\,\sqrt{26.9 \times 0.026})^{1/3}$$
$$= \sqrt{0.026}(26.9 \times 21.0 \times 26.9)^{1/6} = 0.803 \text{ ft}$$

$$C_n = \frac{0.0388}{\log (16.1/0.803)} = 0.0299\ \mu\text{f/mile/phase, to neutral}$$

$$b_c = 2\pi fC = 2\pi \times 60 \times 0.0299 = 11.27 \text{ micromhos/mile/phase, to neutral}$$

3.10 Summary. The capacitance of a single-circuit line may be found by Eq. (3.11) if the line is single-phase, or by Eq. (3.39) if the line is three-phase. These two equations are the same except that the numerator of the argument of the logarithm of Eq. (3.39) is the distance of the equivalent equilateral spacing of the line rather than the distance between the two conductors of a single-phase line. In both equations,

r is the outside radius of the conductor. For parallel-circuit lines, a modified method of GMD is used wherein the outside radius of a conductor enters the computations in place of the self GMD of the conductor found in inductance calculations. Several approximations are made in deriving capacitance formulas, but the importance of all of them, including the effect of earth, is usually very slight.

PROBLEMS

3.1 A three-phase transmission line has flat, horizontal spacing with 6 ft between adjacent conductors. At a certain instant the charge on one of the outside conductors is 0.1×10^{-3} coulomb/mile, and the charge on the center conductor and on the other outside conductor is -0.05×10^{-3} coulomb/mile. The radius of each conductor is 0.1 in. Neglect the effect of ground, and find the voltage drop between the two identically charged conductors at the instant specified.

3.2 The 60-cycle capacitive reactance to neutral of a solid conductor, which is one conductor of a three-phase line with an equivalent equilateral spacing of 4 ft, is 186×10^3 ohms/mile. What value of reactance would be specified in a table listing the capacitive reactance of the conductor at 1-ft spacing for 25 cps? What is the cross-sectional area of the conductor in circular mils?

3.3 Derive an equation for the capacitance to neutral per mile of a single-phase line, taking into account the effect of ground. Use the same nomenclature as in the equation derived for the capacitance of a three-phase line where the effect of ground is represented by image charges.

3.4 Calculate the capacitance to neutral per mile of a single-phase line composed of two No. 2 single-strand conductors spaced 10 ft apart and 25 ft above ground. Compare the values obtained by Eqs. (3.11) and (3.12) and by the equation derived in Prob. 3.3.

3.5 Derive a formula for the capacitance between the single inner conductor and the concentric outer sheath of a power cable. Assume that the radius of the inner conductor is a and that the inner radius of the sheath is b.

3.6 A single-conductor power cable has a conductor of No. 2 solid copper. Paper insulation separating the conductor from the concentric lead sheath has a thickness of $\frac{3}{32}$ in. and a relative permittivity of 3.7. The thickness of the lead sheath is $\frac{5}{64}$ in. Find the capacitive reactance per mile between the inner conductor and the lead sheath.

3.7 A three-phase transmission line has two conductors 8 ft apart in a horizontal plane. The third conductor is 3 ft above the plane of the other two and midway between them. The conductors are solid, round wires with a capacitive reactance at 1-ft spacing of 0.1345 megohm/mile at 60 cps. Find the capacitive reactance to neutral per mile of line at 60 cps, and find the radius of the wire.

3.8 A three-phase 60-cycle transmission line has flat, horizontal spaci[illegible] between adjacent conductors. The conductors are No. 2/0 hard-[illegible] strand copper. The voltage of the line is 110 kv. Find the capacitive [illegible] to neutral and the charging current per mile.

3.9 The six conductors of a double-circuit three-phase li[illegible] 19-strand 300,000-circular-mil hard-drawn copper arranged as shown in Fig. 2.20 [illegible] that the vertical spacing is 13 ft, the longer horizontal distance is 28 ft [illegible] the shorter horizontal distances are 22 ft. Find the capacitive reactance to neut[illegible] and the charging current per mile per phase and per conductor at 132 kv and 60 c[illegible]

CHAPTER 4

RESISTANCE AND SKIN EFFECT

4.1 Resistance. The resistance of transmission-line conductors is the most important cause of power loss in a transmission line. The term resistance, unless specifically qualified, means effective resistance. The effective resistance of a conductor is

$$R = \frac{\text{power loss in the conductor}}{I^2} \qquad \text{ohms} \tag{4.1}$$

where the power is in watts and I is the rms current in the conductor in amperes. The effective resistance is equal to the d-c resistance of the conductor only if the distribution of current throughout the conductor is uniform. At frequencies of 60 cps and below, the difference between effective resistance and d-c resistance is less than 1% for the copper conductors of less than 350,000 circular mils in cross section listed in Table A.1. We shall discuss nonuniformity of current distribution and the ratio of effective resistance to d-c resistance after reviewing some fundamental concepts of d-c resistance.

Direct-current resistance is given by the formula

$$R_0 = \frac{\rho l}{A} \qquad \text{ohms} \tag{4.2}$$

where ρ = resistivity of the conductor
l = length
A = cross-sectional area

Any consistent set of units can be used. In power work, l is usually given in feet, A in circular mils, and ρ in ohms per circular-mil foot.

The resistivity of standard annealed copper at 20°C is [illegible] ohms/circular-mil foot, and its conductivity is 100%. Hard-drawn copper, with a tensile strength about 50% greater than that of annealed copper, has a conductivity approximately 3% lower. If the conductivity of hard-drawn copper is not known exactly, it is assumed to be [illegible] with a resistivity of 10.66 ohms/circular-mil foot at 20°C. Average commercial hard-drawn aluminum has a conductivity of 61% and a resistivity of 17.00 ohms/circular-mil foot at 20°C.

The d-c resistance of stranded conductors is greater than the value computed by Eq. (4.1) because spiraling of the strands makes them longer than the conductor itself. For each mile of conductor the current in all strands except the one in the center flows in more than a mile of wire. The increased resistance due to spiraling is estimated as 1% for three-strand conductors and 2% for concentrically stranded conductors.

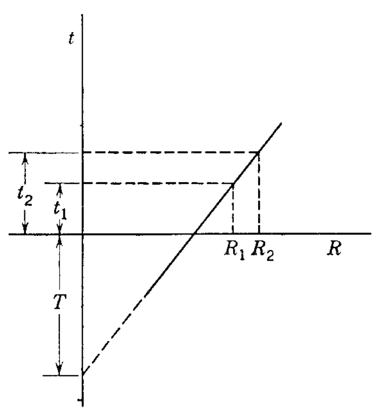


Fig. 4.1 Resistance of a metallic conductor as a function of temperature.

The variation of resistance of metallic conductors with temperature is practically linear over the normal range of operation. If temperature is plotted on the vertical axis and resistance on the horizontal axis, as in Fig. 4.1, extension of the straight-line portion of the graph provides a convenient method of correcting resistance for changes in temperature. The point of intersection of the extended line with the temperature axis at zero resistance is a constant of the material. From the geometry of Fig. 4.1

$$\frac{R_2}{R_1} = \frac{T + t_2}{T + t_1} \tag{4.3}$$

where R_1 and R_2 are the resistances of the conductor at temperatures t_1 and t_2, respectively, in degrees centigrade and T is the constant determined from the graph. Values of the constant T are as follows:

T = 234.5 for annealed copper of 100% conductivity
T = 241 for hard-drawn copper of 97.3% conductivity
T = 228 for hard-drawn aluminum of 61% conductivity

4.2 The Influence of Skin Effect on Resistance. Uniform distribution of current throughout the cross section of a conductor exists only for direct current. As the frequency of alternating current increases, the nonuniformity of distribution becomes more pronounced. An increase in frequency causes more current to be concentrated near the surface of the conductor and less in the interior. This phenomenon is called *skin effect*. We can understand the reason for the crowding of the current toward the surface of a conductor by recalling our discussion of internal flux linkages in Chap. 2. Filaments of conductor on the surface are not linked by internal flux, and the flux linkages of a filament near the surface of a conductor are less than those of a filament in the interior. If the current is unvarying direct current, the magnitude of the current depends only on the applied voltage and the resistance of the conductor and is not affected by inductance. Therefore, if the material of the conductor is uniform, the current will flow equally in all parts of the cross section. If the current is alternating, however, the filaments in the center of the conductor will have a greater voltage drop due to varying flux than will the filaments nearer the surface because those on the interior are linked by more flux. Therefore, the inductive reactance of an interior filament is greater than that of a filament near the surface. The resistance of all the filaments is the same if their areas are equal. The impedance of interior filaments, however, is greater than that of outer filaments. Since all the filaments are electrically in parallel their voltage drops must be equal, and this can be so only if less current flows in the interior filaments of higher impedance. Later in this chapter we shall develop the expression for the magnitude and phase angle of the current density as a function of distance from the center of a wire and find how skin effect alters both resistance and inductance calculated on the assumption of uniform current density.

We can see from a numerical example the reason why nonuniform distribution of current causes an increase in effective resistance. Suppose the three wires in parallel shown in Fig. 4.2*a* carry equal alternating currents of 5 amp. If the resistance of each wire is 10 ohms, the power loss for the three wires with a total current of 15 amp is $3 \times 5^2 \times 10 = 750$ watts. If the impedance of the middle wire of the parallel circuit is increased by increasing its inductance (perhaps by wrapping it with a high-permeability tape, or merely by adding some inductance in series), a higher voltage must be applied to the parallel circuit to obtain a total current of 15 amp for the three branches. The increased voltage causes more current to flow in the outside branches. The current in the middle branch, whose inductance is increased, will be less if the total current is still 15 amp. Suppose that under the new conditions, 5.5 amp flow in each outer branch and 4 amp flow in the middle branch, as indicated in Fig. 4.2*b*. Since the resistances of the branches remain the same, the total power loss for a current of 15 amp is

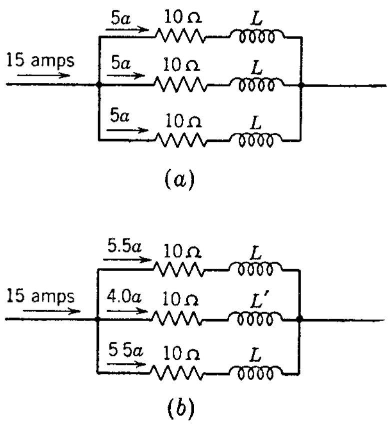


Fig. 4.2 Parallel branches of equal resistance carrying unequal branch currents to illustrate skin effect.

$$2 \times 5.5^2 \times 10 + 4^2 \times 10 = 765 \text{ watts}$$

For the original condition of equal current in the three branches, the effective resistance of the circuit is

$$R = \frac{750}{15^2} = 3.33 \text{ ohms}$$

which is the equivalent resistance of three 10-ohm resistors in parallel. With unequal currents in each resistor, however, the effective resistance of the circuit is

$$R = \frac{765}{15^2} = 3.40 \text{ ohms}$$

The increased inductance in the middle branch causes the current in that branch to be out of phase with the currents in the other two branches.

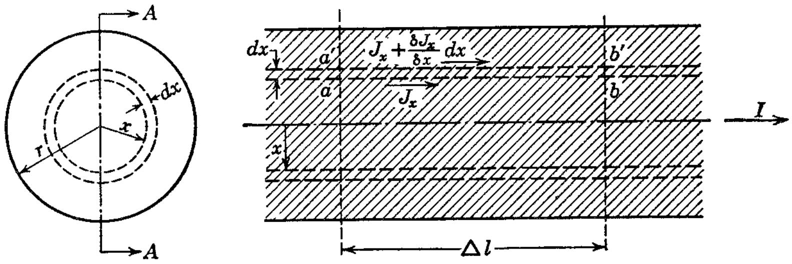


FIG. 4.3 Cross section and longitudinal section of a cylindrical conductor.

To obtain a resultant current of 15 amp requires that the current in the middle branch be somewhat greater than 4.0 amp when the other currents are each 5.5 amp. Therefore, the effective resistance will be even larger than the value computed above.

4.3 Current Density in a Cylindrical Conductor with Skin Effect. We shall approach the problem of determining the effect of nonuniform current density in a conductor by obtaining an expression for current density as a function of distance from the center of the conductor. Consider the conductor whose cross section and longitudinal section are shown in Fig. 4.3. From Eq. (2.11) the magnetic field intensity in the conductor at a distance x from the center is

$$H_x = \frac{I_x}{2\pi x} \tag{4.4}$$

where I_x is the current enclosed by the tubular element of radius x. If I_x is the rms value of the current, H_x is the rms value of the field intensity. We shall find it necessary to deal with instantaneous values, and therefore it is desirable to express Eq. (4.4) in instantaneous form. The instantaneous value of the field intensity is

$$\Re e\,[H_{x,\max}\epsilon^{j\omega t}] = \Re e\left[\frac{I_{x,\max}\epsilon^{j\omega t}}{2\pi x}\right] \tag{4.5}$$

where the symbol $\Re e$ means "real part of."[1] It is customary to omit the symbol $\Re e$, so that

$$H_{x,\max}\epsilon^{j\omega t} = \frac{I_{x,\max}\epsilon^{j\omega t}}{2\pi x} \tag{4.6}$$

and, upon dividing both sides of the equation by $\sqrt{2}$ to convert from maximum to rms values, we have

$$H_x\epsilon^{j\omega t} = \frac{I_x\epsilon^{j\omega t}}{2\pi x} \tag{4.7}$$

which is the expression of Eq. (4.4) in instantaneous form.

To find the current I_x in terms of current density, let J_x be the current density at a distance x from the center of the conductor. Then the current in the walls of the tubular element of radius x and wall thickness dx is $2\pi x J_x\,dx$, and the current enclosed by the tube (that is, in the cylinder of radius x) is

$$I_x = \int_0^x 2\pi x J_x\,dx \tag{4.8}$$

[1] Introduction of the factor $\epsilon^{j\omega t}$ follows an accepted convention of notation. The instantaneous value of a current which varies sinusoidally may be expressed by

$$i = |I_{\max}|\cos\omega t$$

Where $|I_{\max}|$ is the magnitude, or absolute value, of the maximum current. Another way of expressing the same current is

$$i = \Re e\,[|I_{\max}|\epsilon^{j\omega t}]$$

By Euler's formula,

$$\epsilon^{j\omega t} = \cos\omega t + j\sin\omega t$$

and

$$\Re e\,[\epsilon^{j\omega t}] = \cos\omega t$$

Therefore,

$$\Re e\,[|I_{\max}|\epsilon^{j\omega t}] = |I_{\max}|\cos\omega t$$

If

$$i = |I_{\max}|\cos(\omega t + \alpha)$$

we can say

$$i = \Re e\,[|I_{\max}|\epsilon^{j(\omega t+\alpha)}] = \Re e\,[|I_{\max}|\epsilon^{j\alpha}\epsilon^{j\omega t}]$$

and, letting

$$I_{\max} = |I_{\max}|\,\epsilon^{j\alpha} = |I_{\max}|\underline{/\alpha}$$

we have

$$i = \Re e\,[I_{\max}\epsilon^{j\omega t}]$$

where $I_{\max}$ is complex.

Expressing Eq. (4.4) by this notation yields Eq. (4.5).

For a further discussion of the subject, see E. A. Guillemin, "Communication Networks," vol. I, pp. 70–75, John Wiley & Sons, Inc., New York, 1931.

Substituting in Eq. (4.7), we obtain

$$2\pi x H_x \epsilon^{j\omega t} = \int_0^x 2\pi x J_x \epsilon^{j\omega t}\, dx \tag{4.9}$$

Taking the partial derivative with respect to x yields

$$2\pi \left[x \frac{\partial}{\partial x} (H_x \epsilon^{j\omega t}) + H_x \epsilon^{j\omega t} \right] = 2\pi x J_x \epsilon^{j\omega t} \tag{4.10}$$

or, since $\epsilon^{j\omega t}$ is independent of x, we have

$$2\pi x \epsilon^{j\omega t} \frac{\partial H_x}{\partial x} + 2\pi H_x \epsilon^{j\omega t} = 2\pi x J_x \epsilon^{j\omega t} \tag{4.11}$$

Dividing by $2\pi x \epsilon^{j\omega t}$, we obtain

$$\frac{\partial H_x}{\partial x} + \frac{1}{x} H_x = J_x \tag{4.12}$$

Equation (4.12) contains both H_x and J_x as variables dependent upon x. If we can find another relation between H_x and J_x, we can eliminate H_x and obtain a differential equation having J_x as the only variable dependent on x. Such a relation can be found by applying Kirchhoff's voltage law to the voltage drops around the closed path $a'b'ba$ shown in Fig. 4.3*b*. The voltage drops consist of ohmic drops on the paths ab and $a'b'$ and of a voltage drop caused by the changing flux linking the closed loop $a'b'ba$. The instantaneous ohmic drop from a to b is $J_{x,\max}\epsilon^{j\omega t}\rho\, \Delta l$ where $J_{x,\max}\epsilon^{j\omega t}$ is the instantaneous current density on the path ab and ρ is the resistivity of the conductor.[2] Similarly, from a' to b' the ohmic drop is

$$\left[J_{x,\max}\epsilon^{j\omega t} + \frac{\partial}{\partial x} (J_{x,\max}\epsilon^{j\omega t})\, dx \right] \rho\, \Delta l \tag{4.13}$$

since $\frac{\partial}{\partial x} (J_{x,\max}\epsilon^{j\omega t})\, dx$ is the difference in current density between the two paths. Around the loop $a'b'ba$ the total ohmic drop is

[2] The product of current density J, resistivity ρ, and length Δl is the ohmic voltage drop because in a cross-sectional area A

$$I = JA$$

and

$$R_0 = \frac{\rho\, \Delta l}{A}$$

and, therefore, ohmic drop in the area A for a length Δl is

$$IR_0 = JA \left(\frac{\rho\, \Delta l}{A} \right) = J\rho\, \Delta l$$

$$\rho\,\Delta l\left[J_{x,\max}\epsilon^{j\omega t} + \frac{\partial}{\partial x}\,(J_{x,\max}\epsilon^{j\omega t})\,dx\right] - \rho\,\Delta l J_{x,\max}\epsilon^{j\omega t}$$
$$= \rho\,\Delta l\,\frac{\partial}{\partial x}\,(J_{x,\max}\epsilon^{j\omega t})\,dx \qquad (4.14)$$

The voltage *drop* clockwise around the loop $a'b'ba$ due to the changing flux in the loop is $-\partial\phi/\partial t$. The negative sign is necessary because increasing flux due to current in the direction shown induces a voltage *rise* in a clockwise direction. The voltage drop is the negative of the voltage rise. Equating the voltage drops around the loop to zero, in accordance with Kirchhoff's law, gives

$$\rho\,\Delta l\,\frac{\partial}{\partial x}\,(J_{x,\max}\epsilon^{j\omega t})\,dx - \frac{\partial\phi}{\partial t} = 0 \qquad (4.15)$$

The flux ϕ linking the path $a'b'ba$ is in the tubular element of thickness dx and is concentric with the tube. It is a function of both time t and distance x from the center of the conductor. Equating the instantaneous flux to the product of instantaneous flux density and area gives

$$\phi = B_{x,\max}\epsilon^{j\omega t}\,\Delta l\,dx = \mu H_{x,\max}\epsilon^{j\omega t}\,\Delta l\,dx \qquad (4.16)$$

Therefore, upon substitution of Eq. (4.16) in Eq. (4.15), we obtain

$$\rho\,\Delta l\,\frac{\partial}{\partial x}\,(J_{x,\max}\epsilon^{j\omega t})\,dx - \frac{\partial}{\partial t}\,(\mu H_{x,\max}\epsilon^{j\omega t}\,\Delta l)\,dx = 0 \qquad (4.17)$$

Then converting to rms values, assuming constant permeability, expanding the partial derivative with respect to t, and noting that $H_{x,\max}$ is not a function of t and that $\epsilon^{j\omega t}$ is not a function of x, we have

$$\rho\,\Delta l\epsilon^{j\omega t}\,\frac{\partial J_{x,\max}}{\partial x} - j\omega\,\Delta l\mu\epsilon^{j\omega t}H_{x,\max} = 0 \qquad (4.18)$$

In interpreting Eq. (4.18) we must remember that H_x and J_x are complex and that in taking the partial derivative with respect to t we have omitted the symbol $\mathcal{R}e$.[3]

[3] The student can show, by replacing $\epsilon^{j\omega t}$ by $\cos\omega t + j\sin\omega t$, that

$$\frac{\partial}{\partial t}\,\mathcal{R}e\,[I_m\epsilon^{j\omega t}] = \mathcal{R}e\left[\frac{\partial}{\partial t}\,(I_m\epsilon^{j\omega t})\right]$$

from which it follows that

$$\frac{\partial}{\partial t}\,\mathcal{R}e\,[I_m\epsilon^{j\omega t}] = \mathcal{R}e\,[j\omega I_m\epsilon^{j\omega t}]$$

and, omitting $\mathcal{R}e$,

$$\frac{\partial}{\partial t}\,(I_m\epsilon^{j\omega t}) = j\omega I_m\epsilon^{j\omega t}$$

which is the expected result for the derivative of the whole expression including real and imaginary components of $I_m\epsilon^{j\omega t}$. We must remember, however, that the [illegible]taneous value of $\partial i/\partial t$ is the horizontal projection (real part) of $j\omega I_m\epsilon^{j\omega t}$.

Dividing Eq. (4.18) by $\epsilon^{j\omega t}$ and by $\sqrt{2}$ to obtain rms values, we have

$$H_x = \frac{-j\rho}{\omega\mu} \frac{\partial J_x}{\partial x} \tag{4.19}$$

Substituting H_x from Eq. (4.19) in Eq. (4.12), we obtain

$$-\frac{j\rho}{\omega\mu} \frac{\partial^2 J_x}{\partial x} - \frac{j\rho}{\omega\mu x} \frac{\partial J_x}{\partial x} = J_x \tag{4.20}$$

or, multiplying by $j\omega\mu/\rho$ and noting that J_x (a complex number and function of x) is not a function of t,

$$\frac{d^2 J_x}{dx} + \frac{1}{x} \frac{dJ_x}{dx} - \frac{j\omega\mu}{\rho} J_x = 0 \tag{4.21}$$

The change from partial derivatives to total derivatives is possible in Eq. (4.21) because the only independent variable is x. Eq. (4.21) is the second-order differential equation relating the rms value of the current density to the distance from the center of the conductor.

Equation (4.21) is a special form of the long-recognized Bessel equation.[4] It may be written in more concise form as follows:

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + k^2 y = 0 \tag{4.22}$$

To solve Eq. (4.22) assume a solution in the form of an infinite series, or

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \tag{4.23}$$

Then

$$\frac{d^2 y}{dx^2} = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots \tag{4.24}$$

$$\frac{1}{x} \frac{dy}{dx} = \frac{a_1}{x} + 2a_2 + 3a_3 x + 4a_4 x^2 + 5a_5 x^3 + 6a_6 x^4 + \cdots \tag{4.25}$$

and

$$k^2 y = k^2 a_0 + k^2 a_1 x + k^2 a_2 x^2 + k^2 a_3 x^3 + k^2 a_4 x^4 + \cdots \tag{4.26}$$

[4] The solutions to Eq. (4.21) are called Bessel functions of zero order. The Bessel equation having solutions of the nth order is

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(k^2 - \frac{n^2}{x^2}\right) y = 0$$

The solutions are of zero order when n is zero. There are two indepen[illegible] called Bessel functions of the first and second kind. We are con[illegible] with the solution of the first kind since the solution of the second kind indi[illegible]s infinite current density at the center of the conductor, an impossible condit[illegible] A general discussion of Bessel functions can be found in N. W. McLachlan, "B[illegible] Functions for Engineers," Oxford University Press, London, 1934. For com[illegible]ensive tables of Bessel functions, see Jahnke and Emde, "Tables of Function[illegible] Dover Publications, New York, 1943.

In order to satisfy Eq. (4.22), the sum of the coefficients of each power of x, when the above equations (4.24) to (4.26) are added, must equal zero. Thus

$$\begin{aligned} a_1 &= 0 \\ 2a_2 + 2a_2 + k^2a_0 &= 0 \\ 6a_3 + 3a_3 + k^2a_1 &= 0 \\ 12a_4 + 4a_4 + k^2a_2 &= 0 \\ 20a_5 + 5a_5 + k^2a_3 &= 0 \\ 30a_6 + 6a_6 + k^2a_4 &= 0 \end{aligned}$$

All the odd coefficients are zero since they depend on a_1. The even coefficients depend on a_0. In terms of a_0 they are

$$\begin{aligned} a_2 &= -\frac{k^2a_0}{2^2} \\ a_4 &= \frac{k^4a_0}{2^2 \times 4^2} \\ a_6 &= -\frac{k^6a_0}{2^2 \times 4^2 \times 6^2} \end{aligned}$$

Substituting these coefficients in Eq. (4.23) gives the following series solution:

$$y = a_0\left[1 - \frac{(kx)^2}{2^2} + \frac{(kx)^4}{2^2 \times 4^2} - \frac{(kx)^6}{2^2 \times 4^2 \times 6^2} + \cdots\right] \tag{4.27}$$

If k is real, the series of Eq. (4.27) is known as the Bessel function of the first kind, zero order, and is designated by the symbol $J_0(kx)$, where J_0 is a mathematical symbol not to be confused with our symbol for current density. For Eq. (4.21)

$$k^2 = -\frac{j\omega\mu}{\rho} \tag{4.28}$$

and the solution for current density at radius x is

$$J_x = a_0\left[1 + \frac{j\omega\mu}{\rho}\frac{x^2}{2^2} - \left(\frac{\omega\mu}{\rho}\right)^2 \frac{x^4}{2^2 \times 4^2} - j\left(\frac{\omega\mu}{\rho}\right)^3 \frac{x^6}{2^2 \times 4^2 \times 6^2} + \cdots\right] \tag{4.29}$$

This particular form of the Bessel function has both real and imaginary terms. Separating the real terms and the imaginary terms and substituting

$$m = \sqrt{\frac{\omega\mu}{\rho}} \tag{4.30}$$

we obtain

$$J_x = a_0\left[1 - \frac{(mx)^4}{2^2 \times 4^2} + \frac{(mx)^8}{2^2 \times 4^2 \times 6^2 \times 8^2} - \cdots\right]$$
$$+ ja_0\left[\frac{(mx)^2}{2^2} - \frac{(mx)^6}{2^2 \times 4^2 \times 6^2} + \frac{(mx)^{10}}{2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2} - \cdots\right] \tag{4.31}$$

$$J_x = a_0(\text{ber } mx + j \text{ bei } mx) \tag{4.32}$$

where

$$\text{ber } mx = 1 - \frac{(mx)^4}{2^2 \times 4^2} + \frac{(mx)^8}{2^2 \times 4^2 \times 6^2 \times 8^2} - \cdots \tag{4.33}$$

and

$$\text{bei } mx = \frac{(mx)^2}{2^2} - \frac{(mx)^6}{2^2 \times 4^2 \times 6^2} + \frac{(mx)^{10}}{2^2 \times 4^2 \times 6^2 \times 8^2 \times 10^2} - \cdots \tag{4.34}$$

The terms "ber" and "bei" are abbreviations for "Bessel real" and "Bessel imaginary."[5]

The coefficient a_0 can be determined if the current density J_r at the surface of the conductor is known, since

$$J_r = a_0(\text{ber } mr + j \text{ bei } mr) \tag{4.35}$$

Solving for a_0 and substituting in Eq. (4.32), we obtain

$$J_x = J_r \frac{\text{ber } mx + j \text{ bei } mx}{\text{ber } mr + j \text{ bei } mr} \tag{4.36}$$

Equation (4.36) expresses the current density anywhere in the conductor in terms of the current density at the surface.

4.4 The Internal Impedance of a Cylindrical Conductor. We are interested in the expression for the current density in a cylindrical conductor as a step toward determining the internal impedance of the conductor when the current is not uniformly distributed throughout the cross section. The internal impedance of a conductor is that part of the impedance of the circuit due to the resistance of the conductor and the flux linkages produced by flux inside the conductor only.

The voltage drop in a filament at the surface of a conductor is caused by current flowing in the resistance of the filament and by the cha[illegible]e in flux linkages external to the wire. It is unaffected by int[illegible] Therefore, if external flux linkages are excluded from co[illegible], the voltage drop in a filament on the surface of a conduct[illegible] the ohmic

[5] Sets of tables are available giving the values of ber and be[illegible] v[illegible]us arguments. The work by N. W. McLachlan, cited in footnote 4 of t[illegible], contains such a set of tables. For another set of tables, see H. B. Dwig[illegible] pp. 214–221, McGraw-Hill Book Company, Inc., New Yo[illegible].

drop ρJ_r volts per unit length only. Filaments not on the surface of the conductor have the same flux linkages due to external flux as filaments on the surface, but filaments below the surface have additional flux linkages caused by internal flux. Since all the filaments are in parallel electrically, the voltage drop in any filament is the same as in a filament on the surface. The decreased current density and resulting decreased ohmic drop in an internal filament are balanced by an increased drop due to internal flux linkages. The voltage drop V_i per unit length in any filament, excluding the drop caused by flux linkages external to the conductor, is

$$V_i = \rho J_r \qquad \text{volts/meter} \tag{4.37}$$

and the internal impedance per unit length is

$$Z_i = \frac{V_i}{I} = \frac{\rho J_r}{I} \qquad \text{ohms/meter} \tag{4.38}$$

where I is the current in the conductor.

As determined from Eq. (2.11), the current I is related to the field intensity at the surface of the wire by

$$I = 2\pi r H_r \tag{4.39}$$

From Eqs. (4.19) and (4.30)

$$H_r = -\frac{j}{m^2}\left(\frac{dJ_x}{dx}\right)_{x=r} \tag{4.40}$$

and, substituting J_x from Eq. (4.36) in Eq. (4.40), we obtain

$$H_r = -\frac{j}{m^2}\,\frac{J_r}{\text{ber } mr + j \text{ bei } mr}\left[\frac{d}{dx}\,(\text{ber } mx + j \text{ bei } mx)\right]_{x=r} \tag{4.41}$$

To simplify the notation, let

$$\text{ber}' \, mx = \frac{d}{d(mx)}\,(\text{ber } mx) = \frac{1}{m}\,\frac{d}{dx}\,(\text{ber } mx) \tag{4.42}$$

and

$$\text{bei}' \, mx = \frac{d}{d(mx)}\,(\text{bei } mx) = \frac{1}{m}\,\frac{d}{dx}\,(\text{bei } mx) \tag{4.43}$$

Then, from Eqs. (4.39) and (4.41), with the notation as specified in Eqs. (4.42) and (4.43), the current is

$$I = \frac{2\pi r J_r}{m}\left(\frac{\text{bei}' \, mr - j \text{ ber}' \, mr}{\text{ber } mr + j \text{ bei } mr}\right) \tag{4.44}$$

The terms ber′ mr and bei′ mr are evaluated as indicated in Eqs. (4.42) and (4.43) by dividing by m the derivatives of ber mx and bei mx with respect to x and letting $x = r$. Upon solving Eq. (4.44) for J_r and substituting in Eq. (4.38), we obtain for the internal impedance

$$Z_i = \frac{\rho m}{2\pi r}\left(\frac{\text{ber } mr + j \text{ bei } mr}{\text{bei}' \, mr - j \text{ ber}' \, mr}\right) \qquad \text{ohms/meter} \qquad (4.45)$$

Thus the internal impedance of a wire can be found at any frequency if its radius, resistivity, and permeability are known. To be consistent with the rationalized mks system of units, resistivity must be expressed in ohm-meters (sometimes called ohms per meter cube), and permeability as $4\pi \times 10^{-7}$ times relative permeability (see footnote 4 of Chap. 2).

4.5 Skin-effect Resistance Ratio. The internal impedance of a conductor is composed of resistance and inductive reactance. The real part of the complex impedance is the effective resistance. We can find the effective resistance of a wire by rationalizing the expression for internal impedance given by Eq. (4.45) and separating the real and imaginary parts. Thus the effective resistance is

$$R = \frac{\rho m}{2\pi r} \frac{\text{ber } mr \text{ bei}' \, mr - \text{bei } mr \text{ ber}' \, mr}{(\text{bei}' \, mr)^2 + (\text{ber}' \, mr)^2} \qquad \text{ohms/meter} \qquad (4.46)$$

It can be shown that as the frequency approaches zero, the effective resistance given by Eq. (4.46) approaches the d-c resistance given by Eq. (4.2). At low frequencies the current distribution becomes more uniform. The low-frequency or d-c resistance is

$$R_0 = \frac{\rho}{A} = \frac{\rho}{\pi r^2} \qquad \text{ohms/meter} \qquad (4.47)$$

for ρ in ohm-meters and r in meters. The ratio of effective resistance to d-c resistance is

$$\frac{R}{R_0} = \frac{mr}{2} \frac{\text{ber } mr \text{ bei}' \, mr - \text{bei } mr \text{ ber}' \, mr}{(\text{bei}' \, mr)^2 + (\text{ber}' \, mr)^2} \qquad (4.48)$$

Equation (4.48) gives the ratio of effective resistance to d-c resistance as a function of mr.

The factor mr is the product of the radius in meters and the value of m calculated from Eq. (4.30) with ρ in ohm-meters. It may be more convenient to compute mr from the d-c resistance of the wire and the relative permeability μ_r. From Eq. (4.30)

$$mr = r\sqrt{\frac{\omega\mu}{\rho}} = r\sqrt{\frac{2\pi f \times 4\pi \times 10^{-7}\mu_r}{\rho}}$$

$$mr = \sqrt{\frac{2f \times 4\pi \times 10^{-7}\mu_r}{\rho/\pi r^2}} \qquad (4.50)$$

The d-c resistance of a wire per unit length is

$$R_0 = \frac{\rho}{\pi r^2} \qquad \text{ohms/meter} \qquad (4.51)$$

or

$$R_0 = \frac{\rho}{\pi r^2} \times 1{,}609 \qquad \text{ohms/mile} \tag{4.52}$$

Substituting $R_0/1{,}609$ for $\rho/\pi r^2$ in Eq. (4.51) gives

$$mr = 0.0636 \sqrt{\frac{\mu_r f}{R_0}} \tag{4.53}$$

where μ_r = relative permeability of the wire
f = frequency, cps
R_0 = d-c resistance of the wire, ohms/mile

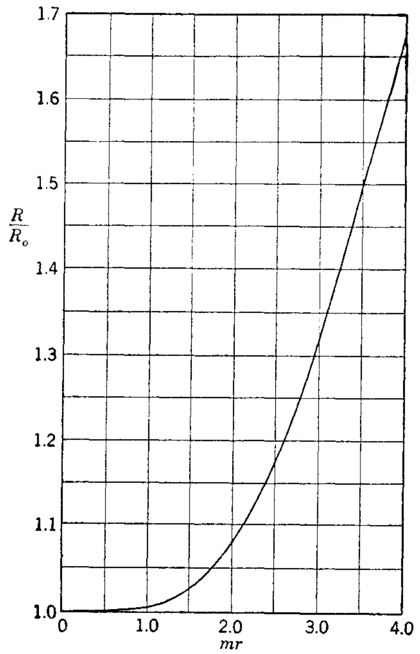


FIG. 4.4 Ratio of a-c resistance to d-c resistance for a cylindrical conductor having a uniform magnetic field around the periphery. The ratio is plotted as a function of mr, where

$$mr = 0.0636 \sqrt{\mu_r f / R_0}$$

and R_0 is the d-c resistance in ohms per mile.

Tabulated values of the ratio of effective resistance to d-c resistance calculated from Eq. (4.48) have been published by the U.S. Bureau of Standards.[6] The resistance ratios plotted in Fig. (4.4) are from this source, which lists the ratios for values to $mr = 100$. At frequencies of 60 cps or less, stranding has negligible effect on the ratio of effective to d-c resistance of concentrically stranded conductors, and effective resistance may be found by multiplying the d-c resistance of the stranded conductor by the ratio read from Fig. (4.4) for a wire.

4.6 Resistance from Tables of Conductor Characteristics. Some of the factors considered in the discussion of resistance and skin effect can be verified by referring to the tables of conductor characteristics in the Appendix. The increase of d-c resistance caused by stranding is illustrated by a hard-drawn [illegible]p-per conductor with a [illegible] tional area of 66,[illegible] [illegible]ular mils and composed of 1, 3, or 7 strands. The d-c resistan[illegible] at 25°C for a single strand is listed in Table A.1 as 0.864 ohm/[illegible]. For 3 strands

[6] See E. B. Rosa and F. W. Grover, "Formulas and Tab[illegible] for the Calculation of Mutual and Self Inductance," Scientific Paper 169, *Bull.* [illegible] *Standards*, vol. 8, no. 1, Table XXII, 1912.

the d-c resistance is 0.873 ohm/mile, and for 7 strands the value is 0.881 ohm/mile. Note that the resistances for the 3- and 7-strand conductors are 1% and 2%, respectively, above the resistance of the solid conductor. This is in agreement with the principle stated in Sec. 4.1.

The values of 0.864 and 0.945 ohm/mile at 25 and 50°C, respectively, given in Table A.1 for the 66,370-circular-mil copper conductor, are verified by Eqs. (4.2) and (4.3). The d-c resistance at 20°C of a solid conductor is

$$R_0 = \frac{10.66 \times 5{,}280}{66{,}370} = 0.848 \text{ ohm/mile}$$

and correcting to 25°C

$$R_0 = 0.848 \frac{241 + 25}{241 + 20} = 0.864 \text{ ohm/mile}$$

or at 50°C

$$R_0 = 0.848 \frac{241 + 50}{241 + 20} = 0.945 \text{ ohm/mile}$$

Examination of the tables shows that skin effect at frequencies up to 60 cps is negligible for the smaller conductors. The 60-cycle resistance of the 66,370-circular-mil conductor is equal to the d-c resistance. Skin effect becomes appreciable, however, at power frequency for the large conductors. For instance, the d-c resistance of a 500,000-circular-mil hard-drawn copper conductor with either 19 or 37 strands is 0.1280 ohm/mile at 50°C, but the 60-cycle effective resistance is 0.1303 ohm/mile. For this conductor, stranding does not appreciably alter the ratio of effective to d-c resistance computed from Fig. (4.4). From Eq. (4.53)

$$mr = 0.0636 \sqrt{\frac{60}{0.128}} = 1.38$$

and from Fig. (4.4) the resistance ratio is 1.02. Then the 60-cycle resistance is

$$R = 1.02 \times 0.1280 = 0.1305 \text{ ohm/mile}$$

4.7 Skin-effect Inductance Ratio. The imaginary component of the internal impedance of a conductor is the inductive reactance due to internal flux linkages. Rationalizing the expression for internal impedance given by Eq. (4.45) and discarding the real part gives the following expression for internal inductive reactance:

$$\omega L_i = \frac{\rho m}{2\pi r} \frac{\text{bei } mr \text{ bei}' \, mr + \text{ber } mr \text{ ber}' \, mr}{(\text{bei}' \, mr)^2 + (\text{ber}' \, mr)^2} \qquad \text{ohms per meter} \quad (4.54)$$

Equation (2.18) gives the internal flux linkages of a wire having uniform current density. Dividing Eq. (2.18) by the current I gives L_{i0} which is the internal inductance at frequencies so low that the assumption of uniform current is valid. Thus

$$L_{i0} = \frac{\mu}{8\pi} \qquad \text{henrys/meter} \tag{4.55}$$

and from Eq. (4.54) the ratio of internal inductance of a wire at any frequency to internal inductance at zero frequency is

$$\frac{L_i}{L_{i0}} = \frac{4}{mr}\left[\frac{\text{bei } mr \text{ bei}' \, mr + \text{ber } mr \text{ ber}' \, mr}{(\text{bei}' \, mr)^2 + (\text{ber}' \, mr)^2}\right] \tag{4.56}$$

The ratio approaches unity as the frequency approaches zero and substantiates the use of the principle of partial flux linkages in deriving Eq. (2.18). As frequency increases the ratio becomes smaller, for skin effect causes the current to crowd toward the surface of the wire and thereby reduces the number of internal flux linkages. Tabulated values of the skin-effect inductance ratio are available in the paper by Rosa and Grover cited in footnote 6 of this chapter. The values are plotted in Fig. 4.5. Internal inductance calculated for uniform current density should be corrected by the ratio read from Fig. 4.5.

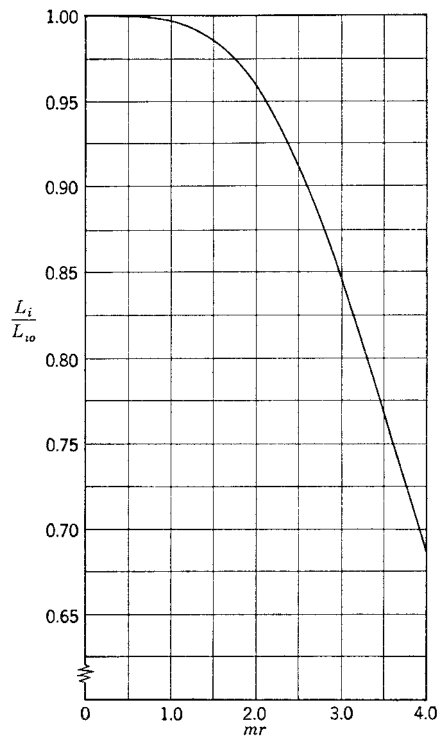


Fig. 4.5 Ratio of actual internal inductance to the low-frequency internal inductance of a cylindrical conductor having a uniform magnetic field around the periphery. The ratio is plotted as a function of mr, where

$$mr = 0.0636\sqrt{\mu_r f/R_0}$$

and R_0 is the d-c resistance in ohms per mile.

If the self GMD (GMR) of a conductor is used to compute inductance by Eq. (2.57) or similar equations, the value used for D_s may be adjusted to account for skin effect. If this is done the frequency for which the value of D_s is listed must be specified. In Tables A.1 and A.2, for instance, the values listed for self GMD are for 60 cycles and when used in Eq. (2.57) give inductance values which include skin effect. The values for inductive reactance at 1-ft spacing listed in the tables are corrected to take skin effect into account at the frequency for which they are specified.

4.8 Other Losses. The high electric field intensity surrounding high-voltage power lines accounts for an additional energy loss in the transmission of power. The high voltage gradient at the surface of a wire sometimes accelerates electrons in the air sufficiently to ionize air molecules by collision. If the voltage gradient at the wire exceeds a certain critical value, the process of ionization becomes cumulative and results in appreciable loss of energy. The ionization is characterized by a faint glow surrounding the wire and is called *corona*. The critical voltage depends on wire size and spacing and on atmospheric conditions. Corona is most likely to occur when the diameter of the conductor is small compared to the distance between wires. High voltage, small wires, and close spacing contribute to a high voltage gradient which may induce corona. Damp weather increases the loss from corona, and a rough or dirty surface on a conductor increases the probability of the occurrence of corona.

Empirical methods for the calculation of corona loss are available in the literature.[7] When a line is designed, the effects of corona are considered, and the design is modified, if necessary, to reduce corona loss to a minimum, usually below 2 kw/mile for a three-phase line under normal conditions. Radio influence due to corona must also be considered and may be more important than line losses. As discussed in Sec. 1.2, tests on the 500-kv Tidd experimental line included an extensive investigation of corona as a source of loss and as a radio influence factor.[8]

Another loss occurring on transmission lines is caused by the leakage of current at the insulators which support the lines at the towers. It differs from leakage through the insulation of cables because it is lumped at the insulators and not uniformly distributed along the line. Even so it would be computed as though uniformly distributed by representing it as a conductance, but in overhead lines the leakage is negligible. Since leakage at insulators of overhead lines is negligible and corona loss is

[7] The empirical equations for corona loss considered most accurate are those given by Peterson and by Carroll and Rockwell. See W. S. Peterson, discussion of the paper by J. S. Carroll and B. Cozzens, "Corona Loss Measurements for the Design of Transmission Lines to Operate at Voltages between 220 Kv. and 330 Kv.," *Trans. AIEE*, vol. 52, pp. 62–63, March 1933; J. S. Carroll and M. M. Rockwell, "Empirical Method of Calculating Corona Loss from High-voltage Transmission Lines," *Trans. AIEE*, vol. 56, pp. 558–565, May, 1937.

[8] See I. W. Gross, C. F. Wagner, O. Naef, and R. L. Tremaine, "Corona Investigation on Extra-high-voltage Lines—500-kv Test Project of the Ameri[illegible] Ga[illegible] Electric Company," *Trans. AIEE*, vol. 70, pp. 75–91, 1951; see G. D. L[illegible] E. Pakala, S. C. Bartlett, and C. D. Fahrnkopf, "Radio Influence [illegible] Field and Laboratory—500-kv Test Project of the American Gas and [illegible] Company," *Trans. AIEE*, vol. 70, pp. 251–264, 1951. For the minimum [illegible] of conductors having tolerable radio interference at the higher transmiss[illegible] voltages, see H. L. Rorden and R. S. Gens, "Investigation of Radio Noise [illegible] Pertains to the Design of High-voltage Transmission Lines," *Trans. AIEE*, vol[illegible], part II, pp. [illegible], 1952.

usually small in a line which is properly designed, the conductance between conductors of an overhead line is assumed to be zero.

In a transmission line there is a nonuniformity of current distribution in addition to that caused by skin effect. In a two-wire line, slightly fewer lines of flux link the elements nearest each other on opposite sides of the line than link the elements farther apart. Therefore elements in the near sides have lower inductance than elements on the far sides. The result is a higher current density in the elements of adjacent conductors nearest each other than in the elements farther apart. The effective resistance is increased by the nonuniformity of current distribution. The phenomenon is known as proximity effect. The increase in resistance depends on the frequency, distance between conductors, conductor size, and permeability. Proximity effect is present for three-phase as well as single-phase circuits. Even at very high frequencies if the ratio of spacing between wires to the radius of the wires of a two-wire line is greater than 15 to 1, the increase of resistance due to proximity effect is only 1%. For the usual spacing of overhead lines at 60 cycles, the proximity effect is much less than the probable error in determining the resistance and is neglected.

PROBLEMS

4.1 Compute ber 1.8 and bei 1.8.

4.2 Compute ber′ 1.8 and bei′ 1.8.

4.3 Calculate R/R_0 and L_i/L_{i0} for $mr = 1.8$.

4.4 Find the d-c resistance per mile for a wire having $mr = 1.8$ at 60 cps.

4.5 Compute the 60-cycle resistance at 50°C and the 60-cycle inductive reactance at 1-ft spacing for round hard-drawn copper wire having a cross-sectional area of 800,000 circular mils. Compare the result with the values given in Table A.1 for the 37-strand copper conductor of the same area. Explain the reasons for any discrepancy. Use Figs. 4.4 and 4.5.

4.6 Specify the two steel-reinforced aluminum cables which have approximately the same 60-cycle resistance at 25°C as a 19-strand, 300,000-circular-mil hard-drawn copper conductor. Would the copper conductor or one of the ACSR conductors be expected to have the lowest corona loss for the same spacing between conductors? Give reasons.

4.7 Calculate the d-c resistance per mile at 25 and 50°C for a solid 400,000-circular-mil hard-drawn copper conductor. Compare the result with the d-c resistance listed in Table A.1 for a 19-strand conductor, and explain any discrepancy.

4.8 Calculate the 60-cycle resistance per mile at 25°C for a solid 400,000-circular-mil hard-drawn copper conductor. Account for the difference between the result and the value listed in Table A.1 for a 19-strand conductor. Does the tabulated value of resistance for the 19-strand conductor agree with the value computed by increasing the d-c resistance of the solid conductor by 2% and multiplying by the resistance ratio used in determining the 60-cycle resistance of the solid conductor?

CHAPTER 5

CURRENT AND VOLTAGE RELATIONS ON A TRANSMISSION LINE

5.1 Introduction. An important problem to be considered in the design of a transmission line and in its operation is the maintenance of the voltage within specified limits at various points in the system. Load studies were described in Chap. 1 as essential in planning the operation of a system under existing or contemplated conditions and in determining the voltage at points throughout the system.

In this chapter we shall develop formulas by which we can calculate the voltage, current, and power factor at any point on a transmission line provided we know these values at one point. Loads are usually specified by their voltage, power, and power factor, from which current can be calculated for use in the equations.

Even when load studies are made on a calculating board or from data obtained during operation, the formulas which we are about to derive are important because they indicate the effect of the various parameters of a transmission line on the voltage drop along the line for various loads. The equations will also be useful in calculating the efficiency of transmission and later in calculating the limits of power flow over the line under both steady-state and transient conditions.

5.2 Representation of Lines. Normally, transmission lines are operated with balanced three-phase loads. Although the lines are not spaced equilaterally and may not be transposed, the resulting dissymmetry is slight, and the phases are considered to be balanced. Figure 5.1 shows a Y-connected generator supplying a balanced-Y load through a transmission line. The equivalent circuit of the transmission line has been simplified by including only the series resistance and inductive reactance, which are shown as concentrated, or *lumped*, parameters and not uniformly distributed along the line. It makes no difference, as far as measurements at the ends of the line are concerned, whether the parameters are lumped or uniformly distributed if the shunt admittance is neglected, for the current is the same throughout the line in that case.

The generator is represented by an impedance connected in series with the generated emf of each phase.

Reviewing the theory of polyphase circuits indicates that no current can flow in the connection between the neutral *o* of the generator and the neutral *n* of the load in a balanced system since the sum of the currents flowing toward *n* in the three phases is zero. Thus points *o* and *n* are at the same potential, no current flows in the neutral connection, and omission of the neutral connection causes no change in the circuit,

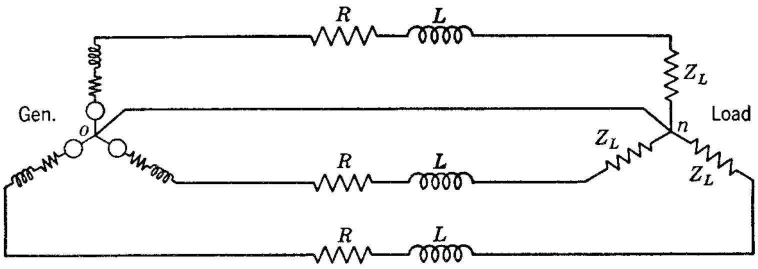


FIG. 5.1 Generator supplying a balanced-Y load through a transmission line.

provided the circuit is balanced. To solve the circuit, a neutral connection is assumed to be present and to carry the sum of the three phase currents, which is zero, however, for balanced conditions. The circuit is solved by applying Kirchhoff's voltage law around a closed path involving one phase and the neutral. Such a closed path is shown in Fig. 5.2. Calculations made for this path are extended to the whole three-phase circuit by remembering that the currents for the other two phases are equal in magnitude to the current of the phase calculated and

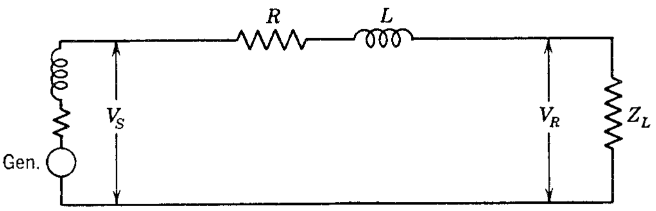


FIG. 5.2 Single-phase equivalent of the circuit of Fig. 5.1.

are displaced 120° and 240° in phase. It is immaterial whether the load, specified by its voltage, power, and power factor, is Δ- or Y-connected since the Δ can always be replaced by its equivalent Y for the purpose of calculation.

A transmission line has four parameters—resistance and inductance, which make up the series impedance of the line, and capacitance and conductance, which determine the shunt admittance from line to line or from line to neutral. Resistance, inductance, and capacitance have been considered in detail in preceding chapters. Inductance was computed for one phase of a balanced three-phase line, and capacitance was computed from line to neutral, so that each would be applicable to the solution of a three-phase line as a single line with a neutral return of zero impedance as shown in Fig. 5.2. Shunt conductance, as was mentioned in Chap. 4, is almost always neglected in power transmission lines when calculating voltage and current.

The classification of power transmission lines according to length depends upon what approximations are justified in treating the parameters of the line. Resistance, inductance, and capacitance are uniformly distributed along the line, and exact calculations of long lines must recognize this fact. For lines of medium length, however, half of the shunt capacitance may be considered to be lumped at each end of the line without causing appreciable error in calculating the voltage and current

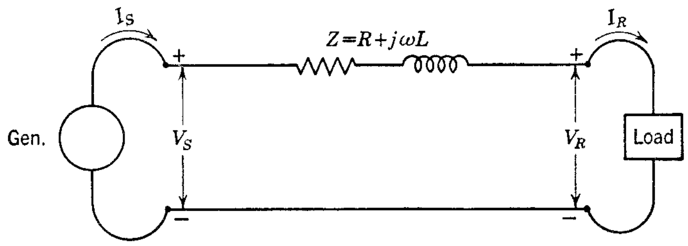


FIG. 5.3 Equivalent circuit of a short transmission line.

at the terminals. For short lines, the total capacitive susceptance is so small that it may be omitted. In so far as the handling of capacitance is concerned, open-wire 60-cycle lines less than about 50 miles long are short lines. Medium length lines are roughly between 50 and 100 miles long. Lines more than 100 miles long require calculation in terms of distributed constants if a high degree of accuracy is required, although for some purposes the nominal π can be used for lines up to 200 miles long.

In order to distinguish between the total series impedance of a line and the series impedance per unit length, the following nomenclature is adopted:

z = series impedance per unit length, per phase
y = shunt admittance per unit length, per phase to neutral
l = length of the line
$Z = zl$ = total series impedance per phase
$Y = yl$ = total shunt admittance per phase to neutral

5.3 The Short Transmission Line. The equivalent circuit of a short transmission line is shown in Fig. 5.3, where I_S and I_R are the sending- and receiving-end currents and V_S and V_R are the sending- and receiving-end line-to-neutral voltages. When the instantaneous current is flowing in the direction of the arrows marked on the circuit diagram, the current is assumed to be positive. A half cycle later, when the current is flowing in the opposite direction, it is negative. Polarity marks perform a similar function by showing the assumed positive direction of voltage drop at the ends of the line. The instantaneous value of the voltage to neutral is assumed to be positive when the terminal marked + is at a higher potential than the terminal marked −. When the terminal marked + is at a lower potential than the other terminal, the instantaneous voltage to neutral is negative.

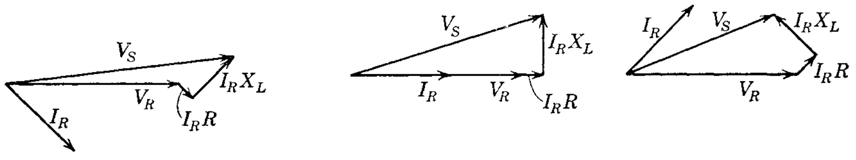
The circuit is solved as a simple series a-c circuit. Since there are no shunt arms, the current is the same at the sending and receiving ends of the line, and

$$I_S = I_R \tag{5.1}$$

The voltage at the sending end is

$$V_S = V_R + I_R Z \tag{5.2}$$

where Z is zl, the total series impedance of the line.



(*a*) Load p. f. = 70% lag (*b*) Load p. f. = 100% (*c*) Load p. f. = 70% lead

FIG. 5.4 Phasor diagrams of a short transmission line. All diagrams are drawn for the same magnitudes of V_R and I_R.

The effect of the variation of the power factor of the load on the *voltage regulation* of a line is most easily understood for the short line and, therefore, will be considered at this time. Voltage regulation of a transmission line is the rise in voltage at the receiving end, expressed in per cent of full load voltage when full load at a specified power factor is removed while the sending-end voltage is held constant. In the form of an equation

$$\text{Per cent regulation} = \frac{|V_{NL}| - |V_{FL}|}{|V_{FL}|} \times 100 \tag{5.3}$$

where $|V_{NL}|$ = magnitude of the receiving-end voltage at no load
$|V_{FL}|$ = magnitude of the receiving-end voltage at full load[1]

After the load on a short transmission line, represented by the circuit of Fig. 5.3, is removed, the voltage at the receiving end is equal to the voltage at the sending end. In Fig. 5.3, with the load connected, the receiving-end voltage is designated by V_R, and $|V_R| = |V_{FL}|$. The sending-end voltage is V_S, and $|V_S| = |V_{NL}|$. The phasor diagrams of Fig. 5.4 are drawn for the same magnitudes of receiving-end voltage

[1] In this book symbols between vertical bars represent the magnitudes, or absolute values, of the enclosed quantities.

and current and show that a larger value of sending-end voltage is required to maintain a given receiving-end voltage when the receiving-end current is lagging the voltage than when the same current and voltage are in phase. A still smaller sending-end voltage is required to maintain the given receiving-end voltage when the receiving-end current leads the voltage. The voltage drop is the same in the series impedance of the line

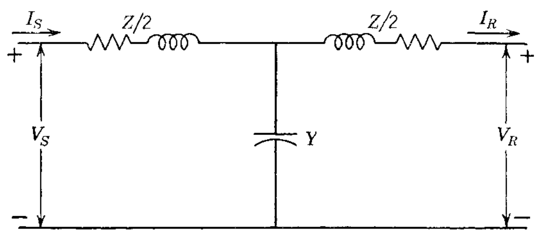


FIG. 5.5 Nominal-T circuit of a medium-length transmission line.

in all cases, but because of the different power factors it is added to the receiving-end voltage at a different angle in each case. The regulation is greatest for lagging power factors and least, or even negative, for leading power factors. The inductive reactance of a transmission line is larger than the resistance, and the principle of regulation illustrated in Fig. 5.4 is true for any load supplied by a predominantly inductive circuit. The relation between power factor and regulation for long lines is similar to that for short lines but is not visualized so easily.

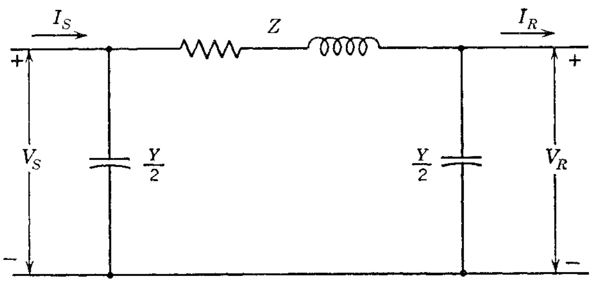


FIG. 5.6 Nominal-π circuit of a medium-length transmission line.

5.4 The Medium-length Line. The shunt admittance, generally pure capacitance, is included in the calculations for a line of medium length. If all the shunt admittance is lumped at the middle of the circuit representing the line, the circuit is called a nominal T. Such a circuit is shown in Fig. 5.5, where Z is zl, the total series impedance per phase of the line, and Y is yl, the total shunt admittance per phase to neutral. The nominal-π circuit, shown in Fig. 5.6, is more often used to represent medium-length lines than is the nominal T. In the nominal-π circuit the total shunt admittance is divided into two equal parts placed at the sending and receiving ends of the line. The equation for V_S in the nominal π may be derived by noting that the current in the capacitance at the receiving end is $V_RY/2$ and the current in the series arm is $I_R + V_RY/2$. Then

$$V_S = \left(V_R \frac{Y}{2} + I_R\right) Z + V_R \tag{5.4}$$

$$V_S = \left(\frac{ZY}{2} + 1\right) V_R + ZI_R \tag{5.5}$$

To derive I_S we note that the current in the shunt capacitance at the sending end is $V_SY/2$, which when added to the current in the series arm gives

$$I_S = V_S \frac{Y}{2} + V_R \frac{Y}{2} + I_R \tag{5.6}$$

and substituting V_S as given by Eq. (5.5) in Eq. (5.6) gives

$$I_S = V_RY\left(1 + \frac{ZY}{4}\right) + \left(\frac{ZY}{2} + 1\right) I_R \tag{5.7}$$

Corresponding equations may be derived for the nominal T. Comparison of Eqs. (5.5) and (5.7) with Eqs. (5.1) and (5.2) shows the effect of including the shunt admittance Y in the computations. If the line is short, the total admittance Y is small, and as Y decreases the equations for the medium-length line approach those of the short line.

Neither the nominal T nor the nominal π exactly represents the actual line, and in cases of doubt about the length of line for which they are sufficiently accurate it is best to use the equivalent circuit discussed in Sec. 5.8, which represents the line exactly. The nominal T and nominal π are not equivalent to each other, as may be seen by application of the Y-Δ transformation equations to either one. The nominal-T and nominal-π circuits are more nearly equal to each other and to the equivalent circuit of the line if the line is split into two or more sections, each represented by its nominal T or π, but the resulting work is more cumbersome where numerical calculations are made than is the use of the equivalent circuit in the first place.

5.5 The Long Transmission Line—Solution of the Differential Equations. The exact solution of any transmission line and the one required for a high degree of accuracy in calculating 60-cycle lines longer than approximately 100 miles long must consider the fact that the parameters of the line are not lumped but are distributed uniformly throughout the length of the line.

Figure 5.7 shows one phase and the neutral connection of a three-phase line. Lumped parameters are not shown because we are ready to consider the solution of the line with the impedance and admittance uniformly distributed. The same diagram also represents a single-phase line if the series impedance of the line is the loop series impedance of the single-phase line instead of the series impedance per phase of the three-phase line and if the shunt admittance is the line-to-line shunt admittance of the single-phase line instead of the shunt admittance to neutral of the three-phase line.

Let us consider a very small element in the line and calculate the difference in voltage and the difference in current between the ends of the element. We will let x be the distance measured from the *receiving end* of the line to the small element of line, and we will let the length of the element be dx. Then $z\,dx$ is the series impedance of the elemental length of the line, and $y\,dx$ is its shunt admittance. The voltage to

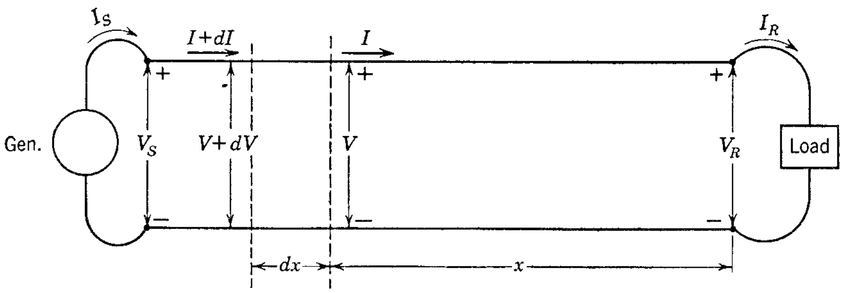


FIG. 5.7 Schematic diagram of a transmission line showing one phase and the neutral return. Nomenclature for the line and the elemental length is indicated.

neutral at the end of the element toward the load is V, and V is the complex expression of the rms voltage, whose magnitude and phase vary with distance along the line. The voltage at the end of the element toward the generator is $V + dV$. The rise in voltage over the elemental length of line in the direction of increasing x is dV, which is the voltage at the end toward the generator minus the voltage at the end toward the load. The rise in voltage in the direction of increasing x is also the product of the current in the element flowing opposite to the direction of increasing x and the impedance of the element, or $Iz\,dx$. Thus

$$dV = Iz\,dx$$

or

$$\frac{dV}{dx} = Iz \tag{5.8}$$

Similarly, the current flowing out of the element toward the load is I. The magnitude and phase of the current I vary with distance along the line because of the distributed shunt admittance along the line. The current flowing into the element from the generator is $I + dI$. The current entering the element from the generator end is higher than the current flowing away from the element in the direction of the load by the amount dI. This difference in current is the current $Vy\,dx$ flowing in the shunt admittance of the element. Thus

$$dI = Vy\,dx$$

or

$$\frac{dI}{dx} = Vy \tag{5.9}$$

Let us differentiate Eqs. (5.8) and (5.9) with respect to x, and obtain

$$\frac{d^2V}{dx^2} = z\frac{dI}{dx} \tag{5.10}$$

and

$$\frac{d^2I}{dx^2} = y\frac{dV}{dx} \tag{5.11}$$

If we substitute the values of dI/dx and dV/dx from Eqs. (5.9) and (5.8) in Eqs. (5.10) and (5.11), respectively, we obtain

$$\frac{d^2V}{dx^2} = yzV \tag{5.12}$$

and

$$\frac{d^2I}{dx^2} = yzI \tag{5.13}$$

Now we have an equation (5.12) in which the only variables are V and x, and another equation (5.13) in which the only variables are I and x. The solutions of Eqs. (5.12) and (5.13) for V and I, respectively, must be expressions which when differentiated twice with respect to x yield the original expression times the constant yz. For instance, the solution for V when differentiated twice with respect to x must yield yzV. This suggests an exponential form of solution. Assume the solution of Eq. (5.12) to be

$$V = A_1\epsilon^{\sqrt{yz}\,x} + A_2\epsilon^{-\sqrt{yz}\,x} \tag{5.14}$$

Taking the second derivative of V with respect to x in Eq. (5.14) yields

$$\frac{d^2V}{dx^2} = yz(A_1\epsilon^{\sqrt{yz}\,x} + A_2\epsilon^{-\sqrt{yz}\,x}) \tag{5.15}$$

which is yz times the assumed solution for V. Therefore Eq. (5.14) is the solution of Eq. (5.12). When we substitute in Eq. (5.8) the value for V given by Eq. (5.14), we obtain

$$I = \frac{1}{\sqrt{z/y}}A_1\epsilon^{\sqrt{yz}\,x} - \frac{1}{\sqrt{z/y}}A_2\epsilon^{-\sqrt{yz}\,x} \tag{5.16}$$

The constants A_1 and A_2 may be evaluated by using the conditions at the receiving end of the line, namely, when $x = 0$, $V = V_R$ and $I = I_R$. Substitution of these values in Eqs. (5.14) and (5.16) yields

$$V_R = A_1 + A_2$$

and

$$I_R = \frac{1}{\sqrt{z/y}}(A_1 - A_2)$$

Substituting $Z_c = \sqrt{z/y}$ and solving for A_1 gives

$$A_1 = \frac{V_R + I_R Z_c}{2}$$

and

$$A_2 = \frac{V_R - I_R Z_c}{2}$$

Then, substituting the values found for A_1 and A_2 in Eqs. (5.14) and (5.16) and letting $\gamma = \sqrt{yz}$, we obtain

$$V = \frac{V_R + I_R Z_c}{2}\epsilon^{\gamma x} + \frac{V_R - I_R Z_c}{2}\epsilon^{-\gamma x} \tag{5.17}$$

$$I = \frac{V_R/Z_c + I_R}{2}\epsilon^{\gamma x} - \frac{V_R/Z_c - I_R}{2}\epsilon^{-\gamma x} \tag{5.18}$$

where $Z_c = \sqrt{z/y}$ and is called the *characteristic impedance* of the line, and $\gamma = \sqrt{yz}$ and is called the *propagation constant*.

Equations (5.17) and (5.18) give the rms values of V and I and their phase angles at any specified point along the line in terms of the distance x from the receiving end to the specified point, provided V_R, I_R, and the parameters of the line are known.

5.6 The Long Transmission Line—Interpretation of the Equations. Both γ and Z_c are complex quantities. The real part of the propagation constant γ is called the *attenuation constant* α and is measured in nepers per unit length. The quadrature part of γ is called the *phase constant* β and is measured in radians per unit length. Thus

$$\gamma = \alpha + j\beta \tag{5.19}$$

and Eqs. (5.17) and (5.18) become

$$V = \frac{V_R + I_R Z_c}{2}\epsilon^{\alpha x}\epsilon^{j\beta x} + \frac{V_R - I_R Z_c}{2}\epsilon^{-\alpha x}\epsilon^{-j\beta x} \tag{5.20}$$

and

$$I = \frac{V_R/Z_c + I_R}{2}\epsilon^{\alpha x}\epsilon^{j\beta x} - \frac{V_R/Z_c - I_R}{2}\epsilon^{-\alpha x}\epsilon^{-j\beta x} \tag{5.21}$$

The properties of $\epsilon^{\alpha x}$ and $\epsilon^{j\beta x}$ help to explain the variation of the voltage and current at any instant with distance along the line. The term $\epsilon^{\alpha x}$ changes in magnitude as x changes, but $\epsilon^{j\beta x}$, which is identical to $\cos \beta x + j \sin \beta x$, always has a magnitude of one and causes a shift in phase of β radians per unit length of the line.

The first term in Eq. (5.20), $[(V_R + I_R Z_c)/2]\epsilon^{\alpha x}\epsilon^{j\beta x}$, increases in magnitude and advances in phase as distance from the receiving end increases. Conversely, as progress along the line from the sending end toward the receiving end is considered, the term diminishes in magnitude and is retarded in phase. This is the characteristic of a traveling wave and is similar to the behavior of a wave in water, which varies in magnitude with time at any point, while its phase is retarded and its maximum value diminishes with distance from the origin. The variation in instantaneous value is not expressed in the term but is understood since V_R and I_R are phasors. Inclusion of the time factor will be discussed later in obtaining an expression for the instantaneous current and voltage anywhere along the line. The first term in Eq. (5.20) is called the *incident* voltage.

The second term in Eq. (5.20), $[(V_R - I_R Z_c)/2]\epsilon^{-\alpha x}\epsilon^{-j\beta x}$, diminishes in magnitude and is retarded in phase from the receiving end toward the sending end. It is called the *reflected* voltage. At any point along the line, the voltage is the sum of the component incident and reflected voltages at that point.

Since the equation for current is similar to the equation for voltage, the current may be considered to be composed of incident and reflected currents.

It is important to realize that Eqs. (5.20) and (5.21) yield complex values of current and voltage and that the coefficients of the exponential terms are phasors which represent voltages and currents whose instantaneous values vary sinusoidally with time. At any point along the line, the maximum value of the incident voltage is $\sqrt{2}\,|(V_R + I_R Z_c)/2|\epsilon^{\alpha x}$. The maximum value of the incident voltage at the receiving end in exponential form is $\sqrt{2}\,|(V_R + I_R Z_c)/2|\epsilon^{j\phi}$ where ϕ is the phase angle of this voltage with respect to the reference voltage or current. Introduction of the operator $\epsilon^{j\beta x}$ accounts for the phase shift with distance along the line. If we desire to specify the phase shift with respect to time, we introduce the operator $\epsilon^{j\omega t}$, which has a magnitu and causes a phase shift at an angular velocity of ω rad second since it is equal to $\cos \omega t + j \sin \omega t$. The exponential expression for the maximum value of the incident voltage at any time and any distance x is $\sqrt{2}\,|(V_R + I_R Z_c)/2|\epsilon^{\alpha x}\epsilon^{j\phi}\epsilon^{j\beta x}\epsilon^{j\omega t}$. The equation for the instantaneous voltage is the real part of the exponential expres[2] Thus

[2] See Chap. 4, footnote 1.

$$
\begin{aligned}
v &= \operatorname{\mathcal{R}e}\left[\sqrt{2}\left|\frac{V_R + I_R Z_c}{2}\right| \epsilon^{\alpha x}\epsilon^{j\beta x}\epsilon^{j\omega t}\epsilon^{j\phi}\right] \\
&= \operatorname{\mathcal{R}e}\left[\sqrt{2}\left|\frac{V_R + I_R Z_c}{2}\right| \epsilon^{\alpha x}\epsilon^{j(\beta x+\omega t+\phi)}\right] \\
&= \sqrt{2}\left|\frac{V_R + I_R Z_c}{2}\right| \epsilon^{\alpha x}\cos\,(\omega t + \beta x + \phi) \qquad (5.22)
\end{aligned}
$$

Figure 5.8 shows the variation of the instantaneous value of incident voltage with distance along the line for two different instants of time.

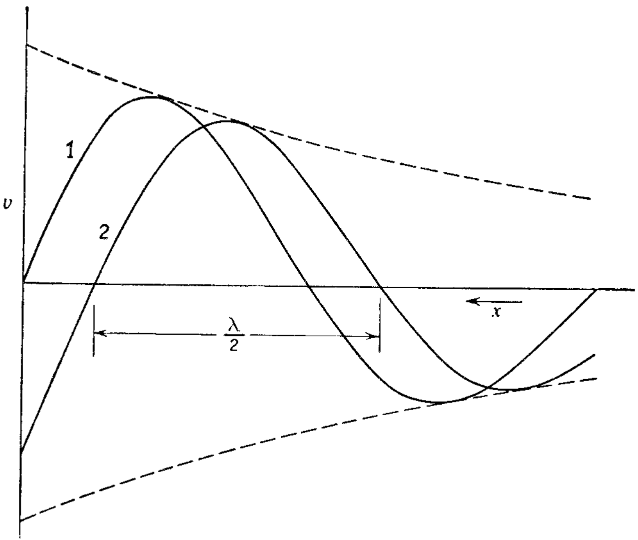


Fig. 5.8 Graph of instantaneous value of incident voltage versus distance along a transmission line for two instants of time. Curve 2 is for a time later than that for curve 1.

The broken lines represent the variation in the maximum value of the incident voltage with distance. The solid curves are instantaneous values of incident voltage plotted against distance for the two instants of time. Equation (5.22) shows that the maximum voltage occurs when $\omega t + \beta x + \phi = 0$. If t increases, x must decrease to maintain the relation $\omega t + \beta x + \phi = 0$. Therefore, at a later time the positive maximum of the incident voltage must occur at a point nearer the [illegible] curve 2 in Fig. 5.8 represents the values of the incident voltage along the line at a later instant of time than that for which curve 1 is drawn. Study of Fig. 5.8 shows the analogy between a voltage [illegible] varying with time and distance and a wave in water, which was mentioned previously. Equation (5.22) is the equation of a wave, and we may speak of the incident and reflected *waves* of voltage and current.

The reflected voltage is similar to the incident voltage except that it diminishes in magnitude and is retarded in phase from the receiving end toward the sending end. Addition of the incident and reflected voltages at any point gives the resultant voltage at the point. To an observer stationed at any point along the line, the voltage appears only as a sinusoidal variation having an rms value determined by the resultant of the two waves.

If a line is terminated in its characteristic impedance Z_c, the receiving-end voltage V_R is equal to $I_R Z_c$, and there is no reflected wave of either voltage or current, as may be seen by substituting $I_R Z_c$ for V_R in Eqs. (5.20) and (5.21). A line terminated in its characteristic impedance is called a flat line or an infinite line. The latter term arises from the fact that a line of infinite length can not have a reflected wave. Usually power lines are not terminated in their characteristic impedance, but communication lines are frequently so terminated in order to eliminate the reflected wave. A typical value of Z_c is 400 ohms for a single-circuit line and 200 ohms for two circuits in parallel. The phase angle of Z_c is usually between 0° and −15°.

In power system work, characteristic impedance is sometimes called *surge impedance*. The term surge impedance, however, is usually reserved for the special case of a lossless line. If a line is lossless, its resistance and conductance are zero, and the characteristic impedance reduces to $\sqrt{L/C}$, a pure resistance. When dealing with high frequencies or with surges due to lightning, losses are often neglected, and the surge impedance becomes important. Surge-impedance loading (SIL) of a line is the power delivered by a line to a purely resistive load equal to its surge impedance. Power system engineers sometimes find it convenient to express the power transmitted by a line in terms of per unit of SIL—that is, as the ratio of the power transmitted to the surge-impedance loading. For instance, the permissible loading of a transmission line may be expressed as a fraction of its SIL.[3]

A *wavelength* λ is the distance along the line between two points where the difference in phase is 360°, or 2π radians, between the voltages or currents measured at the two points. If β is the phase shift in radians per mile, the wavelength in miles is

$$\lambda = \frac{2\pi}{\beta}$$

A distance of a half wavelength is indicated in Fi[illegible] as the distance between adjacent points where the voltage is zero a[illegible] given time. At a

[3] See Central Station Engineers of the Westinghou[illegible]ctric Co[illegible] "Electrical Transmission and Distribution Reference Book," 4th e[illegible] pp. 479–482, East Pitts-burgh, Pa., 1950.

frequency of 60 cps, a wavelength is approximately 3,000 miles. The velocity of propagation of a wave in miles per second is the product of the wavelength in miles and the frequency in cycles per second, or

$$\text{Velocity} = f\lambda \tag{5.24}$$

If there is no load on a line, I_R is equal to zero, and, as determined by Eqs. (5.20) and (5.21), the incident and reflected voltages are equal in magnitude and in phase at the receiving end. In this case the incident and reflected currents are equal in magnitude but 180° out of phase at the receiving end. Thus, the incident and reflected currents cancel each other at the receiving end of an open line but not at any other point unless the line is entirely lossless so that the attenuation α is zero.

Example 5.1

A single-circuit 60-cycle transmission line is 225 miles long. The load is 125,000 kw at 200 kv with 100% power factor. Evaluate the incident and reflected voltages at the receiving end of the line and at the sending end. Determine the line voltage at the sending end from the incident and reflected voltages. Compute the wavelength and velocity of propagation. The parameters of the line are

$$\begin{aligned} R &= 0.172 \text{ ohm/mile} \\ L &= 2.18 \text{ millihenrys/mile} \\ C &= 0.0136\mu\text{f/mile} \\ G &= 0 \end{aligned}$$

Solution

$$\begin{aligned} z &= 0.172 + j2\pi \times 60 \times 2.18 \times 10^{-3} \\ &= 0.172 + j0.822 = 0.841\underline{/78.2^\circ} \text{ ohm/mile} \\ y &= 0 + j2\pi \times 60 \times 0.0136 \times 10^{-6} \\ &= 0 + j5.13 \times 10^{-6} = 5.13 \times 10^{-6}\underline{/90^\circ} \text{ mho/mile} \end{aligned}$$

$$\begin{aligned} \gamma l &= \sqrt{yz}\, l = 225 \sqrt{0.841 \times 5.13 \times 10^{-6}}\ \underline{\Big/ \frac{78.2^\circ + 90^\circ}{2}} \\ &= 0.467\underline{/84.1^\circ} = 0.0481 + j0.465 \end{aligned}$$

$$Z_c = \sqrt{\frac{z}{y}} = \sqrt{\frac{0.841}{5.13 \times 10^{-6}}}\ \underline{\Big/ \frac{78.2^\circ - 90^\circ}{2}} = 405\underline{/-5.9^\circ} \text{ ohms}$$

$$V_R = \frac{200{,}000}{\sqrt{3}} = 115{,}200\underline{/0^\circ} \text{ volts to neutral}$$

$$I_R = \frac{125{,}000{,}000}{\sqrt{3} \times 200{,}000} = 361\underline{/0^\circ} \text{ amp}$$

Designate the incident voltage as V^+ and the reflected voltage as V^-. Then, at the receiving end, where $x = 0$,

$$
\begin{aligned}
V_R^+ &= \frac{V_R + I_R Z_c}{2} = \frac{115{,}200\underline{/0^\circ} + 361\underline{/0^\circ} \times 405\underline{/-5.9^\circ}}{2} \\
&= 57{,}600 + 72{,}500 - j7{,}500 = 130{,}100 - j7{,}500 \\
&= 130{,}100\underline{/-3.3^\circ} \text{ volts} \\
V_R^- &= \frac{V_R - I_R Z_c}{2} = \frac{115{,}200\underline{/0^\circ} - 361\underline{/0^\circ} \times 405\underline{/-5.9^\circ}}{2} \\
&= 57{,}600 - 72{,}500 - j7{,}500 = -14{,}900 + j7{,}500 \\
&= 16{,}700\underline{/153.3^\circ} \text{ volts}
\end{aligned}
$$

At the sending end, where $x = l$,

$$
\begin{aligned}
V_S^+ &= \frac{V_R + I_R Z_c}{2} \epsilon^{\alpha l} \epsilon^{j\beta l} = 130{,}100\underline{/-3.3^\circ}\ \epsilon^{0.0481} \epsilon^{j0.465} \\
&= 130{,}100\underline{/-3.3^\circ} \times 1.049\underline{/26.6^\circ} = 136{,}500\underline{/23.3^\circ} \text{ volts} \\
V_S^- &= \frac{V_R - I_R Z_c}{2} \epsilon^{-\alpha l} \epsilon^{-j\beta l} = 16{,}700\underline{/153.3^\circ} \left(\frac{1}{1.049} \underline{/-26.6^\circ} \right) \\
&= 15{,}900\underline{/126.7^\circ} \text{ volts}
\end{aligned}
$$

The line-to-neutral voltage at the sending end is

$$
\begin{aligned}
V_S &= 136{,}500\underline{/23.3^\circ} + 15{,}900\underline{/126.7^\circ} \\
&= 125{,}300 + j54{,}000 - 9{,}500 + j12{,}750 \\
&= 115{,}800 + j66{,}750 = 133{,}800\underline{/30^\circ} \text{ volts}
\end{aligned}
$$

The line voltage at the sending end is

$$V_S = \sqrt{3} \times 133.8 = 232 \text{ kv}$$

The wavelength and velocity of propagation are computed as follows:

$$
\begin{aligned}
\beta &= \frac{0.465}{225} = 0.002065 \text{ radian/mile} \\
\lambda &= \frac{2\pi}{\beta} = \frac{2\pi}{0.002065} = 3{,}040 \text{ miles} \\
\text{Velocity} &= f\lambda = 60 \times 3{,}040 = 182{,}400 \text{ miles/sec}
\end{aligned}
$$

5.7 The Long Transmission Line—Hyperbolic Form of the Equations. The incident and reflected waves of voltage are seldom found when calculating the voltage of a power line. The reason for discussing the voltage and current of a line in terms of the incident and reflected components is that such an analysis is helpful in obtaining a fuller understanding of some of the phenomena of transmission lines. A more convenient form of the equations for computing current and voltage of a power line is found by introducing hyperbolic functions. Hyperbolic functions are defined in exponential form as follows:

$$\sinh \theta = \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2} \tag{5.25}$$

$$\cosh \theta = \frac{\epsilon^{\theta} + \epsilon^{-\theta}}{2} \tag{5.26}$$

By rearranging Eqs. (5.17) and (5.18) and substituting hyperbolic functions for the exponential terms, a new set of equations is found. The new equations, giving voltage and current anywhere along the line, are

$$V = V_R \cosh \gamma x + I_R Z_c \sinh \gamma x \tag{5.27}$$

and

$$I = I_R \cosh \gamma x + \frac{V_R}{Z_c} \sinh \gamma x \tag{5.28}$$

Letting $x = l$ to obtain the voltage and current at the sending end, we have

$$V_S = V_R \cosh \gamma l + I_R Z_c \sinh \gamma l \tag{5.29}$$

and

$$I_S = I_R \cosh \gamma l + \frac{V_R}{Z_c} \sinh \gamma l \tag{5.30}$$

Equations (5.29) and (5.30) may be solved for V_R and I_R in terms of V_S and I_S to give

$$V_R = V_S \cosh \gamma l - I_S Z_c \sinh \gamma l \tag{5.31}$$

and

$$I_R = I_S \cosh \gamma l - \frac{V_S}{Z_c} \sinh \gamma l \tag{5.32}$$

Equations (5.29) to (5.32) are the fundamental equations of a transmission line. For balanced three-phase lines the current is the line current, and the voltage is the line-to-neutral voltage—that is, the line voltage divided by $\sqrt{3}$. In order to solve the equations, the hyperbolic functions must be evaluated. Since γl is usually complex, the hyperbolic functions are also complex and cannot be found directly from ordinary tables. A very convenient set of charts for evaluating the hyperbolic functions of complex arguments usually encountered in power transmission lines has been published by Woodruff.[4] In Chap. 6 of this book are charts from which $\cosh \gamma l$ may be read in rectangular form. These charts also give values which when multiplied by $|Z_c|$ yield the complex expression for $Z_c \sinh \gamma l$ and values which when divided by $|Z_c|$ yield $(1/Z_c) \sinh \gamma l$. The charts are explained in detail in Chap. 6.

[4] L. F. Woodruff, "Complex Hyperbolic Function Charts," *El*[illegible] vol. 54, pp. 550–554, May, 1935. The Woodruff charts cover the complex h[illegible]bolic functions found in the solution of 60-cycle lines up to about 300 mile[illegible] For more general charts, see A. E. Kennelly, "Chart Atlas of Complex Hyper[illegible] and Circular Functions," Harvard University Press, Cambridge, Mass., 1924. For tables of functions of complex arguments, see A. E. Kennelly, "Tables [illegible] omplex [illegible] and Circular Functions," Harvard University Press, Cambridg[illegible] Mass., 1927.

If special charts are not available, hyperbolic functions of complex arguments may be evaluated in several other ways. The following equations give the expansions of hyperbolic sines and cosines of complex arguments in terms of circular and hyperbolic functions of real arguments:

$$\cosh(\alpha l + j\beta l) = \cosh \alpha l \cos \beta l + j \sinh \alpha l \sin \beta l \qquad (5.33)$$

$$\sinh(\alpha l + j\beta l) = \sinh \alpha l \cos \beta l + j \cosh \alpha l \sin \beta l \qquad (5.34)$$

Equations (5.33) and (5.34) make possible the computation of hyperbolic functions of complex arguments from the tables of circular and hyperbolic functions of real arguments available in various handbooks. The correct mathematical unit for βl is the radian, and the radian is the unit found for βl by computing the quadrature component of γl. Equations (5.33) and (5.34) may be verified by substituting in them the exponential forms of the hyperbolic functions and the similar exponential forms of the circular functions.

Another convenient method of evaluating a hyperbolic function is to expand it in a power series. Expansion by Maclaurin's series yields

$$\cosh \theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots \qquad (5.35)$$

and

$$\sinh \theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \cdots \qquad (5.36)$$

The series converge rapidly for the values of γl usually found for power lines, and sufficient accuracy may be found by evaluating only the first few terms.

A third method of evaluating complex hyperbolic functions is suggested by Eqs. (5.25) and (5.26). Substituting $\alpha + j\beta$ for θ, we obtain

$$\cosh(\alpha + j\beta) = \frac{\epsilon^{\alpha}\epsilon^{j\beta} + \epsilon^{-\alpha}\epsilon^{-j\beta}}{2} = \tfrac{1}{2}(\epsilon^{\alpha}\underline{/\beta} + \epsilon^{-\alpha}\underline{/-\beta}) \qquad (5.37)$$

and

$$\sinh(\alpha + j\beta) = \frac{\epsilon^{\alpha}\epsilon^{j\beta} - \epsilon^{-\alpha}\epsilon^{-j\beta}}{2} = \tfrac{1}{2}(\epsilon^{\alpha}\underline{/\beta} - \epsilon^{-\alpha}\underline{/-\beta}) \qquad (5.38)$$

Example 5.2

Find the voltage, current, and power at the sending end of the line described in Example 5.1.

Solution

From the solution of Example 5.1

$Z_c = 405\underline{/-5.9^\circ}$ ohms
$\gamma l = 0.0481 + j0.465$
$V_R = 115{,}200\underline{/0^\circ}$ volts to n[illegible]l
$I_R = 361\underline{/0^\circ}$ amp

From Eqs. (5.33) and (5.34)

$$
\begin{aligned}
\cosh \gamma l &= \cosh 0.0481 \cos 0.465 + j \sinh 0.0481 \sin 0.465 \\
&\quad (\text{Note: } 0.465 \text{ radian} = 26.6°) \\
\cosh \gamma l &= 1.0012 \times 0.894 + j0.0481 \times 0.448 = 0.895 + j0.0215 \\
&= 0.895\underline{/1.38°} \\
\sinh \gamma l &= \sinh 0.0481 \cos 0.465 + j \cosh 0.0481 \sin 0.465 \\
&= 0.048 \times 0.894 + j1.0012 \times 0.448 = 0.0429 + j0.449 \\
&= 0.449\underline{/84.5°}
\end{aligned}
$$

Then from Eq. (5.29)

$$
\begin{aligned}
V_S &= 115{,}200 \times 0.895\underline{/1.38°} + 361 \times 405\underline{/-5.9°} \times 0.449\underline{/84.5°} \\
&= 103{,}000\underline{/1.38°} + 65{,}600\underline{/78.6°} \\
&= 103{,}000 + j2{,}480 + 13{,}000 + j64{,}400 \\
&= 116{,}000 + j66{,}880 = 133{,}800\underline{/30.0°} \text{ volts}
\end{aligned}
$$

and from Eq. (5.30)

$$
\begin{aligned}
I_S &= 361 \times 0.895\underline{/1.38°} + \frac{115{,}200}{405\underline{/-5.9°}} \times 0.449\underline{/84.5°} \\
&= 323\underline{/1.38°} + 128\underline{/90.4°} = 323 + j7.8 - 0.9 + j128 \\
&= 322 + j136 = 350\underline{/22.9°} \text{ amp}
\end{aligned}
$$

At the sending end

$$
\begin{aligned}
\text{Line voltage} &= \sqrt{3} \times 133.8 = 232 \text{ kv} \\
\text{Line current} &= 350 \text{ amp} \\
\text{Power factor} &= \cos (30.0° - 22.9°) = 0.9923 \\
\text{Power} &= \sqrt{3} \times 232 \times 350 \times 0.9923 = 140{,}000 \text{ kw}
\end{aligned}
$$

5.8 The Equivalent Circuit of a Long Line. The nominal-T and nominal-π circuits, as is pointed out in Sec. 5.4, do not represent a transmission line exactly because they do not account for the parameters of the line being uniformly distributed. The discrepancy between the nominal T and π and the actual line becomes larger as the length of line increases. It is possible, however, to find the equivalent circuit of a long transmission line and to represent the line accurately, in so far as measurements at the ends of the line are concerned, by a network of lumped parameters. Let us assume that a π circuit similar to that of Fig. 5.6 is the equivalent circuit of a long line, but let us call the series arm of our equivalent-π circuit Z' and the shunt arms $Y'/2$ to distinguish them from the arms of the nominal-π circuit. Equation (5.5) gives the sending-end voltage of a symmetrical-π circuit in terms of its series and shunt arms and the voltage and current at the receiving end. By substituting Z' and $Y'/2$ for Z and $Y/2$ in Eq. (5.5), we obtain the sending-end voltage of our equivalent circuit in terms of its series and shunt arms and the voltage and current at the receiving end. So

$$V_S = \left(\frac{Z'Y'}{2} + 1\right) V_R + Z' I_R \tag{5.39}$$

For our circuit to be equivalent to the long transmission line, the coefficients of V_R and I_R in Eq. (5.39) must be identical, respectively, to the coefficients of V_R and I_R in Eq. (5.29). Equating the coefficients of I_R in the two equations yields

$$Z' = Z_c \sinh \gamma l \tag{5.40}$$

$$Z' = \sqrt{\frac{z}{y}} \sinh \gamma l = zl \frac{\sinh \gamma l}{\sqrt{zy}\, l}$$

$$Z' = Z \frac{\sinh \gamma l}{\gamma l} \tag{5.41}$$

where Z is equal to zl, the total series impedance of the line. The term $(\sinh \gamma l)/\gamma l$ is the factor by which the series impedance of the nominal π must be multiplied to convert the nominal π to the equivalent π. For small values of γl, $\sinh \gamma l$ and γl are almost identical, and this fact shows that the nominal π represents the medium-length transmission line quite accurately, in so far as the series arm is concerned.

To investigate the shunt arms of the equivalent-π circuit, we equate the coefficients of V_R in Eqs. (5.29) and (5.39) and obtain

$$\frac{Z'Y'}{2} + 1 = \cosh \gamma l \tag{5.42}$$

Substituting $Z_c \sinh \gamma l$ for Z' gives

$$\frac{Y' Z_c \sinh \gamma l}{2} + 1 = \cosh \gamma l \tag{5.43}$$

and

$$\frac{Y'}{2} = \frac{1}{Z_c} \frac{\cosh \gamma l - 1}{\sinh \gamma l} \tag{5.44}$$

Another form of the expression for the shunt admittance of the equivalent circuit may be found by substituting in Eq. (5.44) the following identity:

$$\tanh \frac{\gamma l}{2} = \frac{\cosh \gamma l - 1}{\sinh \gamma l} \tag{5.45}$$

The identity may be verified by substituting the exponential forms of Eqs. (5.25) and (5.26) for the hyperbolic functions and by recalling that $\tanh \theta = \sinh \theta / \cosh \theta$. Now

$$\frac{Y'}{2} = \frac{1}{Z_c} \tanh\left(\frac{\gamma l}{2}\right) \tag{5.46}$$

$$\frac{Y'}{2} = \frac{Y}{2} \frac{\tanh (\gamma l/2)}{\gamma l/2} \tag{5.47}$$

where Y is equal to yl, the total shunt admittance of the line. Equation (5.47) shows the correction factor used to convert the admittance of the shunt arms of the nominal π to that of the equivalent π. Since $\tanh (\gamma l/2)$ and $\gamma l/2$ are very nearly equal for small values of γl, the nominal π represents the medium-length transmission line quite accurately, for we have seen previously that the correction factor for the series arm is negligible for medium-length lines. The equivalent-π circuit is shown in Fig. 5.9. An equivalent-T circuit may also be found for a transmission line.

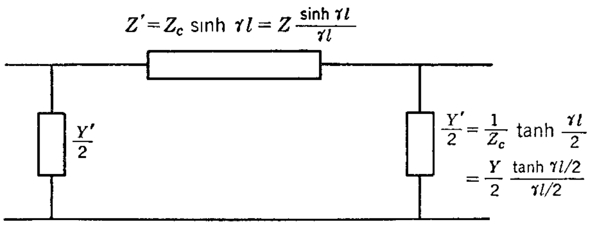


FIG. 5.9 Equivalent-π circuit of a transmission line.

With some charts of complex hyperbolic functions, Eqs. (5.41) and (5.47) are the most desirable. When charts are not available, it may be easier to use Eqs. (5.40) and (5.44).[5]

Example 5.3

Find the equivalent-π circuit for the line described in Example 5.1 and compare it to the nominal π.

Solution

Since $\sinh \gamma l$ and $\cosh \gamma l$ are already known from Example 5.2, Eqs. (5.40) and (5.44) will be used.

$$Z' = 405\underline{/-5.9^\circ} \times 0.449\underline{/84.5^\circ} = 182\underline{/78.6^\circ} \text{ ohms in the series arm}$$

$$\frac{Y'}{2} = \frac{0.895 + j0.0215 - 1}{182\underline{/78.6^\circ}} = \frac{-0.105 + j0.0215}{182\underline{/78.6^\circ}} = \frac{0.107\underline{/168.4^\circ}}{182\underline{/78.6^\circ}}$$

$$= 0.000588\underline{/89.8^\circ} \text{ mho in each shunt arm}$$

[5] The General Electric Company publication entitled "A-C Network Analyzer Manual," GET-1285a, General Electric Company, Schenectady, N.Y., 1950, lists correction factors to convert the series impedance of a line to the value required in the equivalent π and gives capacitance values for the shunt arms of the equivalent π.

The nominal-π circuit has a series impedance of

$$Z = 225 \times 0.841\underline{/78.2^\circ} = 189\underline{/78.2^\circ} \text{ ohms}$$

and equal shunt arms of

$$\frac{Y}{2} = \frac{5.12 \times 10^{-6}\underline{/90^\circ}}{2} \times 225 = 0.000575\underline{/90^\circ} \text{ mho}$$

Comparison of the values found for the nominal-π and equivalent-π circuits in Example 5.3 shows that the difference is slight for a typical line of 225 miles, from which we conclude that the nominal π may represent long lines sufficiently well if a high degree of accuracy is not required.

5.9 Summary. Any transmission line may be represented accurately, as far as conditions at its terminals are concerned, by its equivalent-π circuit. The results of calculations made on the equivalent circuit will be identical with those found by the equations for a long transmission line. Medium-length lines are represented very accurately by the nominal π, and very short lines may be solved by considering them as series impedances. Solutions made from an equivalent circuit are preferred to solutions from transmission-line equations when it is desired to include terminal equipment with the line in making calculations. As will be described later, nominal-π or equivalent-π circuits represent transmission lines on a-c calculating boards.

PROBLEMS

5.1 A 10-mile 60-cycle single-circuit three-phase line is composed of No. 4/0 19-strand hard-drawn copper conductors equilaterally spaced with 5 ft between centers. It delivers 2,500 kw at 11,000 volts to a balanced load. What must be the sending-end voltage when the power factor is 80% lagging, when the power factor is unity, and when the power factor is 90% leading? Assume a wire temperature of 50°C.

5.2 Derive the equations for V_S and I_S for the nominal-T circuit of a transmission line in terms of V_R, I_R, and the total series impedance and shunt admittance of the line.

5.3 A 60-cycle three-phase transmission line has its three conductors arranged in flat, horizontal spacing with 15 ft between adjacent conductors. The conductors are No. 4/0 19-strand hard-drawn copper. The line is 75 miles long and carries a load of 30,000 kw at 115 kv, with 0.8 power factor lagging. Find the voltage, current, power, and power factor at the sending end. What is the efficiency of transmission? Assume a wire temperature of 50°C.

5.4 Derive equations similar to Eqs. (5.40) and (5.44) for the equivalent T of a transmission line.

5.5 A single-circuit 60-cycle three-phase transmission line has the following parameters:

$R = 0.30$ ohm/mile
$L = 2.10$ millihenrys/mile
$C = 0.014$ μf/mile

The voltage at the receiving end is 132 kv. If the line is open at the receiving end, find the rms value and phase angle of the following:

(*a*) The incident voltage to neutral at the receiving end. (Select this voltage as reference in computing the phase angles of the other voltages.)

(*b*) The reflected voltage to neutral at the receiving end.

(*c*) The incident voltage to neutral at 75 miles from the receiving end.

(*d*) The reflected voltage to neutral at 75 miles from the receiving end.

(*e*) The resultant voltage to neutral and the line-to-line voltage at 75 miles from the receiving end.

5.6 Find the incident and reflected current for the open line of Prob. 5.5 at the receiving end and 75 miles from the receiving end.

5.7 If the line of Prob. 5.5 is 75 miles long and delivers 40,000 kw at 132 kv and 80% power factor lagging, determine the sending-end voltage, current, power, and power factor. Compute the efficiency of transmission, characteristic impedance, wavelength, and velocity of propagation.

5.8 Justify Eqs. (5.33) and (5.34) by substituting in them the exponential expressions for the circular and hyperbolic functions.

5.9 Evaluate $\cosh\theta$ and $\sinh\theta$ for $\theta = 0.5\,\underline{/75^\circ}$ by the series expansions and by the formulas involving circular and hyperbolic functions of real arguments.

5.10 Justify Eq. (5.45) by substituting for its hyperbolic functions the equivalent exponential expressions.

5.11 A 60-cycle three-phase transmission line is 175 miles long. It has a total series impedance of $35 + j140$ ohms and a shunt admittance of 930×10^{-6} mho. It delivers 40,000 kw at 220 kv, with 90% power factor lagging. Find the voltage at the sending end by (*a*) the short-line approximation, (*b*) the nominal-π approximation, (*c*) the long-line equation.

5.12 Determine the equivalent-π circuit for the line of Prob. 5.11.

5.13 Determine the voltage regulation for the line described in Prob. 5.11. Assume that the sending-end voltage remains constant.

5.14 A three-phase 60-cycle transmission line is 240 miles long. The voltage at the sending end is 220 kv. The parameters of the line are: $R = 0.2$ ohm/mile, $X = 0.8$ ohm/mile, and $Y = 5.3 \times 10^{-6}$ mho/mile. Find the sending-end current when there is no load on the line.

5.15 If the load on the line described in Prob. 5.14 is 75,000 kw at 220 kv, with unity power factor, calculate the current, voltage, and power at the sending end. Assume that the sending-end voltage is held constant, and calculate the voltage regulation of the line for the load specified above.

CHAPTER 6

GENERALIZED CIRCUIT CONSTANTS

6.1 General Circuit Equations. A three-phase transmission line, as we saw in Chap. 5, can be represented by a circuit consisting of two terminals where power enters the circuit and two terminals where power leaves the circuit. The circuit is said to be *passive, linear,* and *bilateral.* It is passive because it contains no sources of electric energy, linear because the impedances of its elements are independent of the amount of current passing through them, and bilateral because the impedances are independent of the direction of the current. The most general network consisting of a pair of input terminals and a pair of output terminals has an impedance connected between each combination of two of its four terminals and is called a four-terminal network. It can be shown that any linear, passive, and bilateral four-terminal network can be represented by either a T or π circuit which is equivalent to it in so far as measurements at the input or output terminals are concerned.[1] Such a T or π circuit actually is a three-terminal network since one terminal is common to both the sending and receiving ends. We must remember, however, that the T or π circuit is equivalent to the general four-terminal network only for measuring characteristics at the sending and receiving ends and not between one terminal at the sending end and one terminal at the receiving end.

To find the relations between sending- and receiving-end quantities of a four-terminal network, let us determine the voltage and current at the sending end of the unsymmetrical-T circuit of Fig. 6.1 in terms of the voltage and current at the receiving end, since an unsymmetrical T is equivalent to the general four-terminal network for measurements made at the ends of the network with the assumption that the network is passive, linear, and bilateral. The current at the sending end is

$$I_S = I_R + Y(V_R + I_R Z_2) = YV_R + (1 + YZ_2)I_R \tag{6.1}$$

[1] For example, see W. C. Johnson, "Transmission Lines and Networks," pp. 255–257, McGraw-Hill Book Company, Inc., New York, 1950; L. A. Ware and H. R. Reed, "Communication Circuits," 3d ed., pp. 43–46, John Wiley & Sons, Inc., New York, 1949.

The voltage at the sending end is

$$
\begin{aligned}
V_S = V_R + I_R Z_2 + I_S Z_1 &= V_R + I_R Z_2 + Z_1 Y V_R + I_R Z_1 + I_R Y Z_1 Z_2 \\
&= (1 + YZ_1)V_R + (Z_1 + Z_2 + YZ_1Z_2)I_R \qquad (6.2)
\end{aligned}
$$

The above equations are simplified in form by letting

$$
\begin{aligned}
A &= 1 + YZ_1 \qquad & C &= Y \\
B &= Z_1 + Z_2 + YZ_1Z_2 \qquad & D &= 1 + YZ_2
\end{aligned} \qquad (6.3)
$$

Then Eq. (6.2) becomes

$$V_S = AV_R + BI_R \qquad (6.4)$$

and Eq. (6.1) becomes

$$I_S = CV_R + DI_R \qquad (6.5)$$

Since our unsymmetrical-T circuit is valid for measuring the end conditions of any passive, linear, and bilateral four-terminal network, Eqs. (6.4)

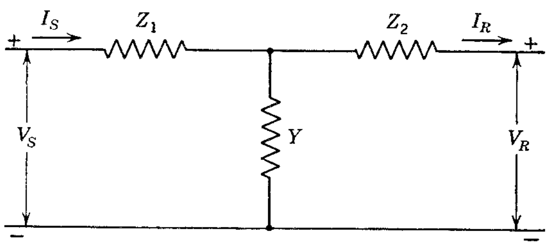


FIG. 6.1 Unsymmetrical-T circuit equivalent to a four-terminal network.

and (6.5) are valid for any such network. The constants A, B, C, and D are called the generalized circuit constants or the $ABCD$ constants of the network, and they can be evaluated for any such four-terminal network.

Solving Eqs. (6.4) and (6.5) for V_R and I_R, we obtain

$$V_R = \frac{DV_S - BI_S}{AD - BC} \qquad (6.6)$$

and

$$I_R = \frac{AI_S - CV_S}{AD - BC} \qquad (6.7)$$

We shall see later that $AD - BC = 1$, and if we accept this relationship between the $ABCD$ constants subject to proof later, Eqs. (6.6) and (6.7) become

$$V_R = DV_S - BI_S \qquad (6.8)$$

and

$$I_R = -CV_S + AI_S \qquad (6.9)$$

Usually the $ABCD$ constants are complex. An interpretation of each constant is found by examination of Eqs. (6.4) and (6.5).

By letting I_R equal zero in Eq. (6.4), we see that the constant A is the ratio of sending-end voltage to receiving-end voltage when the receiving end is open-circuited. The constant A is dimensionless since it is a ratio of two voltages. Unless the voltages are in phase, A is complex and its angle is the phase angle between the voltages.

If we let V_R equal zero in Eq. (6.4), we find that B is the ratio of voltage at the sending end to current at the receiving end with the receiving end short-circuited. Since it is a ratio of voltage to current, the constant B has the dimensions of impedance and is specified in ohms.

Similarly, by letting I_R equal zero in Eq. (6.5), we see that the constant C is the ratio of current at the sending end to voltage at the receiving end with the receiving end open-circuited. Since it is a ratio of current to voltage, the constant C has the dimensions of admittance and is specified in mhos.

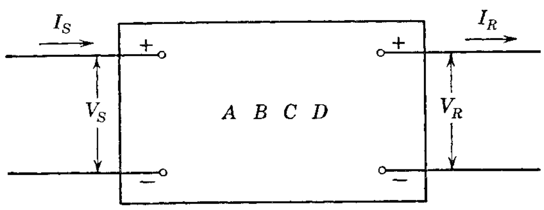


FIG. 6.2 Symbolic diagram representing a four-terminal network.

If we let V_R equal zero in Eq. (6.5), we find that D is the ratio of sending-end current to receiving-end current with the receiving end short-circuited. The constant D is dimensionless since it is a ratio of two currents.

The $ABCD$ constants are widely used in power system work. Some power companies prepare sheets tabulating the $ABCD$ constants of all their transmission lines. A general four-terminal network is often indicated by a diagram similar to Fig. 6.2, where a rectangle encloses the two pairs of terminals and letters symbolizing the generalized circuit constants are placed inside the rectangle. The voltages and currents appearing in equations with the $ABCD$ constants are identified on the diagram.

6.2 Relations between the Generalized Circuit Constants. To prove the relation $AD - BC = 1$, let us connect a generator having an internal voltage E and negligible impedance to the sending end of a general four-terminal network, and let us short-circuit the receiving end. The diagram of the circuit is shown in Fig. 6.3, where the receiving-end current is designated I_2. Applying Eq. (6.4), we obtain

$$E = 0 + BI_2 \tag{6.10}$$

or

$$I_2 = \frac{E}{B} \tag{6.11}$$

Now let us short-circuit the sending end of our network and connect the generator at the receiving end, as shown in Fig. 6.4. The directions that are assumed to be positive for the flow of sending- and receiving-end currents in the equations involving *ABCD* constants are indicated.

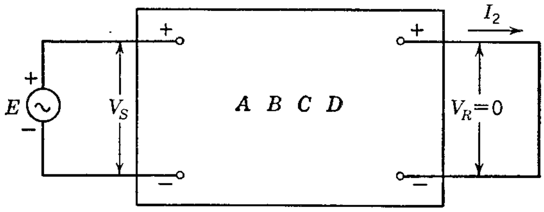


FIG. 6.3 Four-terminal network with a short circuit at the receiving end and $V_S = E$.

Comparison of Figs. 6.3 and 6.4 shows us that we can apply the reciprocity theorem, which states that the current in one branch of a linear, bilateral network due to an electromotive force in a second branch is equal to the current that the electromotive force when transferred to the first branch causes in the second branch, provided that a short circuit

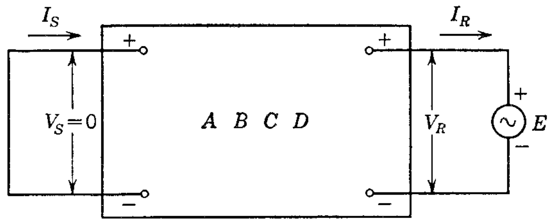


FIG. 6.4 Four-terminal network with a short circuit at the sending end and $V_R = E$.

replaces the electromotive force removed from the second branch.[2] This means that the current flowing out of the network of Fig. 6.4 at the upper terminal of the sending end, in the direction *opposite* to that shown for I_S, is equal to I_2 of Fig. 6.3. Therefore, I_S of Fig. 6.4 is equal to $-I_2$ of Fig. 6.3. For Fig. 6.4, by Eqs. (6.4) and (6.5),

$$0 = AE + BI_R \tag{6.12}$$

and

$$I_S = -I_2 = CE + DI_R$$

[2] The reciprocity theorem is proved in most texts on elementary circuit theory. See for instance R. H. Frazier, "Elementary Electric-circuit Theory," pp. 105–107, McGraw-Hill Book Company, Inc., 1945; K. Y. Tang, "Alternating-current Circuits," 2d ed., pp. 214–215, International Textbook Company, Scranton, Pa., 1951; W. R. LePage, "Analysis of Alternating-current Circuits," pp. [illegible], McGraw-Hill Book Company, Inc., 1952.

Eliminating I_R from Eqs. (6.12) and (6.13), we obtain

$$-I_2 = CE + D\left(\frac{-AE}{B}\right) \tag{6.14}$$

and substituting the value of I_2 from Eq. (6.11) into Eq. (6.14), we obtain

$$-\frac{E}{B} = CE - \frac{ADE}{B} \tag{6.15}$$

which, upon dividing by $-E/B$, gives the relation which we set out to prove, namely,

$$AD - BC = 1 \tag{6.16}$$

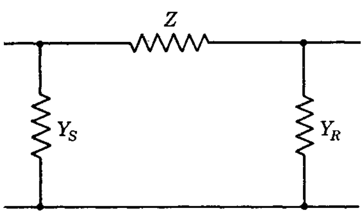


FIG. 6.5 Unsymmetrical-π circuit.

When the $ABCD$ constants are evaluated independently, Eq. (6.16) serves as a partial check on the calculations.

6.3 Generalized Constants of Simple Networks. We have already found the $ABCD$ constants of an unsymmetrical-T network, and the results are given in Eqs. (6.3). The unsymmetrical-π circuit shown in Fig. 6.5 may be analyzed in a similar manner to give the following values for the $ABCD$ constants:

$$\begin{aligned} A &= 1 + Y_R Z \qquad & C &= Y_S + Y_R + ZY_SY_R \\ B &= Z & D &= 1 + Y_S Z \end{aligned} \tag{6.17}$$

A series circuit with shunt admittance equal to zero, which represents short transmission lines and in some cases transformers, has $ABCD$ constants which may be determined by inspection of Eqs. (5.1) and (5.2). They are

$$\begin{aligned} A &= 1 \qquad & C &= 0 \\ B &= Z & D &= 1 \end{aligned} \tag{6.18}$$

Sometimes the $ABCD$ constants of a network are known, and the equivalent circuit must be found to represent the circuit on a calculating board. The equivalent-π circuit is found by solving Eqs. (6.17) for the values of the series and shunt arms, which are

$$\begin{aligned} Z &= B \\ Y_R &= \frac{A - 1}{B} \\ Y_S &= \frac{D - 1}{B} \end{aligned} \tag{6.19}$$

The constants of the symmetrical-π circuit of Fig. 6.6 are found by letting $Y_S = Y_R = Y/2$ in Eqs. (6.17), or by examination of Eqs. (5.5) and (5.7). By either approach

$$\begin{aligned} A &= 1 + \frac{ZY}{2} & C &= Y + \frac{ZY^2}{4} \\ B &= Z & D &= 1 + \frac{ZY}{2} \end{aligned} \tag{6.20}$$

Equations (6.20) may be used to compute the $ABCD$ constants of long transmission lines from the impedance and admittance values of the equivalent π of the long line. Since hyperbolic functions are needed to compute the values for the equivalent π, it is more direct to calculate the $ABCD$ constants by the following equations, which can be verified by inspection of Eqs. (5.29) and (5.30):

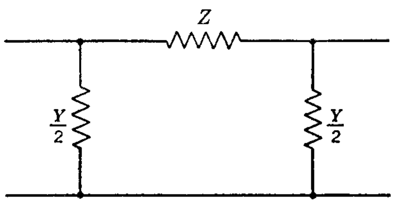


FIG. 6.6 Symmetrical-π circuit.

$$\begin{aligned} A &= \cosh \gamma l & C &= \frac{\sinh \gamma l}{Z_c} \\ B &= Z_c \sinh \gamma l & D &= \cosh \gamma l \end{aligned} \tag{6.21}$$

In all four-terminal networks which are symmetrical—that is, networks which are the same when viewed from either end—the constants A and D are always equal. Whenever $ABCD$ constants are calculated, the results should be checked by the relation $AD - BC = 1$. The verification of this relationship between the $ABCD$ constants found for the networks discussed above is left to the reader.

6.4 Charts of Transmission-line Constants. The $ABCD$ constants of long transmission lines involve hyperbolic functions of γl, where γl is complex. Evaluation of the hyperbolic functions is tedious and time consuming unless charts such as those mentioned in Chap. 5 are available. Povejsil and Johnson[3] have published a set of charts from which the real and quadrature components of the complex $ABCD$ constants can be read, directly for the A constant and in terms of $|Z_c|$ for the B and C constants. Only the length of the line and the ratio of series resistance to inductive reactance need be known to read the charts.

The method of obtaining the curves for the charts is based on the fact that the product of L and C is practically constant for overhead lines of stranded copper or ACSR. The same principle is the basis of curves presented by Edith Clarke[4] for determining complex hyperbolic

[3] D. J. Povejsil and A. A. Johnson, "A Per-unit Interpretation of Transmission Line Constants," *Trans. AIEE*, vol. 70, pp. 194–200, 1951.

[4] E. Clarke, "Simplified Transmission Line Calculations," *Gen. Elec. Rev.*, vol. 29, pp. 321–329, May, 1926.

functions. If the product LC is identical for all lines, γl at a given frequency is a function only of R/X and the length of the line l, where R is the resistance of the line in ohms per mile and X is the inductive reactance of the line in ohms per mile, since

$$\gamma l = \sqrt{yz}\, l$$

and in terms of L and C

$$\gamma l = \sqrt{(R + j\omega L)j\omega C}\, l = \sqrt{\omega^2 LC} \sqrt{-1 + j \frac{R}{\omega L}}\, l$$

$$= \sqrt{\omega^2 LC} \sqrt{-1 + j \frac{R}{X}}\, l \quad (6.22)$$

which is a function only of R/X and l, if LC is constant for all lines.

That LC is practically identical for all overhead lines, regardless of the type of conductor and configuration, may be seen by evaluating $\sqrt{\omega^2 LC}$ in terms of the dimensions of the line, as follows:

$$\sqrt{\omega^2 LC} = \sqrt{\omega^2 0.7411 \times 10^{-3} \log \frac{D_m}{r'} \times \frac{0.0388 \times 10^{-6}}{\log D_m/r}} \quad (6.23)$$

and, since $r' = 0.7788r$ for solid conductors,

$$\sqrt{\omega^2 LC} = \sqrt{\omega^2 28.2 \times 10^{-12} \left(\log \frac{D_m}{r} + \log \frac{1}{0.7788}\right) \frac{1}{\log D_m/r}} \quad (6.24)$$

For $f = 60$ cps,

$$\sqrt{\omega^2 LC} = 10^{-3} \sqrt{4.09 + \frac{0.444}{\log D_m/r}} \quad (6.25)$$

which is very nearly independent of D_m and r. The expression is, therefore, very nearly constant for all lines, since variations of the second term under the radical have little effect for the usual range of values of D_m and r. Although Eq. (6.25) applies only to a solid conductor, the second term under the radical is the only one affected by the type of conductor, and it is affected only slightly. As stated above, variations of the second term have small effect on the result. Povejsil and Johnson point out that an examination of a considerable number of lines showed that the factor $\sqrt{\omega^2 LC}$ was always between 2.05×10^{-3} and [illegible] $\times 10^{-3}$ for 60-cycle overhead lines of stranded copper and ACSR. The charts are based on a value of 2.06×10^{-3} for this quantity.

The $ABCD$ constants in the charts are expressed in terms of the magnitude of the characteristic impedance of the line; that is, in per unit of $|Z_c|$. If $Z_c = |Z_c|/\underline{\zeta}$, the $ABCD$ constants of a transmission line are

$$
\begin{aligned}
A &= \cosh \gamma l \\
B &= |Z_c| \sinh \gamma l \angle \zeta \qquad \text{ohms} \\
C &= \frac{\sinh \gamma l}{|Z_c|} \angle -\zeta \qquad \text{mhos} \\
D &= A
\end{aligned}
\tag{6.26}
$$

The constants A and D are dimensionless and independent of Z_c. When the $ABCD$ constants are expressed in per unit of $|Z_c|$, they become

$$
\begin{aligned}
A \text{ in per unit} &= A = \cosh \gamma l \\
B \text{ in per unit} &= \frac{B \text{ in ohms}}{|Z_c|} = \sinh \gamma l \angle \zeta \\
C \text{ in per unit} &= |Z_c| \times C \text{ in ohms} = \sinh \gamma l \angle -\zeta \\
D \text{ in per unit} &= D = \cosh \gamma l
\end{aligned}
\tag{6.27}
$$

Upon substituting the expression of Eq. (6.22) for γl in Eqs. (6.27) and letting $\sqrt{\omega^2 LC} = 2.06 \times 10^{-3}$, we obtain for the constants in per unit of $|Z_c|$

$$
\begin{aligned}
A &= \cosh\left(2.06 \times 10^{-3} l \sqrt{-1 + j\frac{R}{X}}\right) \qquad \text{per unit} \\
B &= \left[\sinh\left(2.06 \times 10^{-3} l \sqrt{-1 + j\frac{R}{X}}\right)\right] \angle \zeta \qquad \text{per unit} \\
C &= \left[\sinh\left(2.06 \times 10^{-3} l \sqrt{-1 + j\frac{R}{X}}\right)\right] \angle -\zeta \qquad \text{per unit} \\
D &= A
\end{aligned}
\tag{6.28}
$$

The angle ζ associated with the characteristic impedance is also a function of R/X, since

$$
Z_c = \sqrt{\frac{z}{y}} = \sqrt{\frac{R + j\omega L}{j\omega C}} \tag{6.29}
$$

$$
Z_c = |Z_c| \angle \frac{\tan^{-1}(\omega L/R) - 90^\circ}{2} \tag{6.30}
$$

and

$$
\zeta = -\frac{1}{2} \tan^{-1} \frac{R}{X} \tag{6.31}
$$

With ζ expressed in terms of R/X, the $ABCD$ constants in per unit given in Eqs. (6.28) are functions only of the ratio R/X and the length of the line l. The curves plotted in Figs. 6.7 to [illegible] have R/X as a parameter and show the variation of the $ABCD$ constants with line length. Since the constants are complex, the real and quadrature components are plotted separately, and the constants in rectangular form expressed in per unit of $|Z_c|$ are

$$\begin{aligned} A &= A_1 + jA_2 \\ B &= B_1 + jB_2 \\ C &= C_1 + jC_2 \end{aligned} \tag{6.32}$$

Sometimes the constants may be useful in their per-unit form. At other times it may be expedient to convert them to the form required in

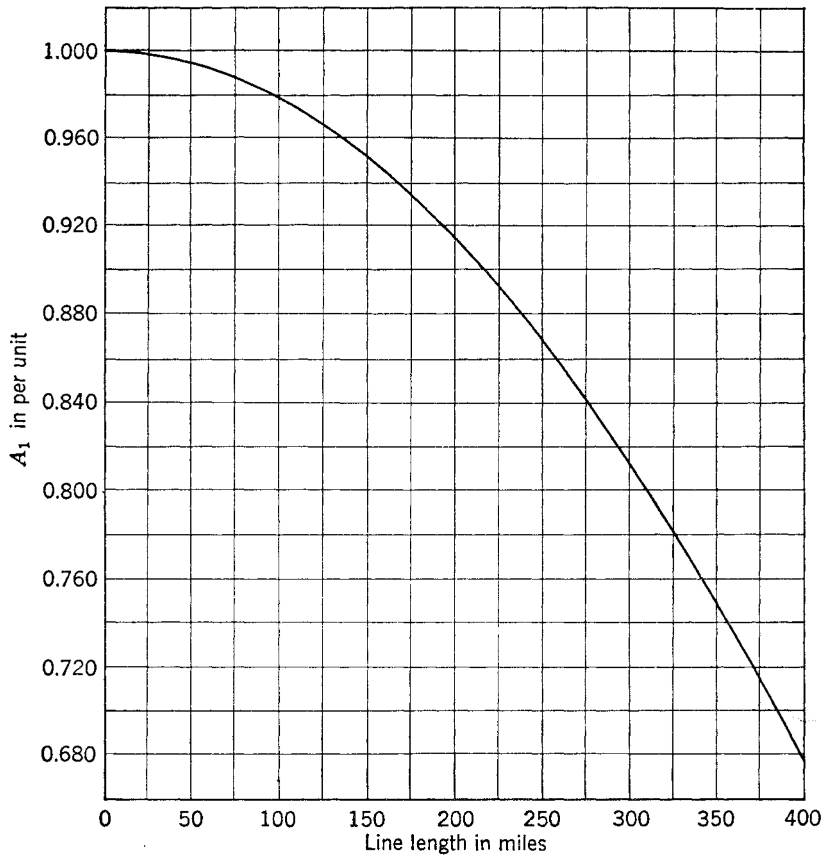


Fig. 6.7 The transmission-line A_1 constant as a function of line length in miles. (From D. J. Povejsil and A. A. Johnson, "A Per-unit Interpretation of Transmission Line Constants," *Trans. AIEE*, vol. 70, pp. 194–200, 1951, by permission.)

equations involving volts and amperes. No conversion is necessary for the constant A. The B and C constants are converted by the operation which is the inverse of that used to obtain Eqs. (6.[illegible]). Thus,

$$\begin{aligned} B \text{ in ohms} &= |Z_c| \times B \text{ in per unit} \\ C \text{ in mhos} &= \frac{C \text{ in per unit}}{|Z_c|} \end{aligned} \tag{6.33}$$

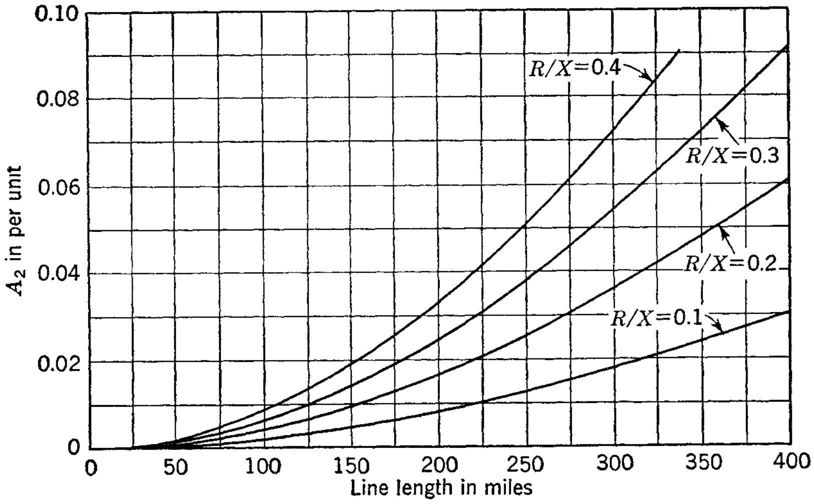


FIG. 6.8 The transmission-line A_2 constant as a function of line length in miles. (From D. J. Povejsil and A. A. Johnson, "A Per-unit Interpretation of Transmission Line Constants," *Trans. AIEE*, vol. 70, pp. 194–200, 1951, by permission.)

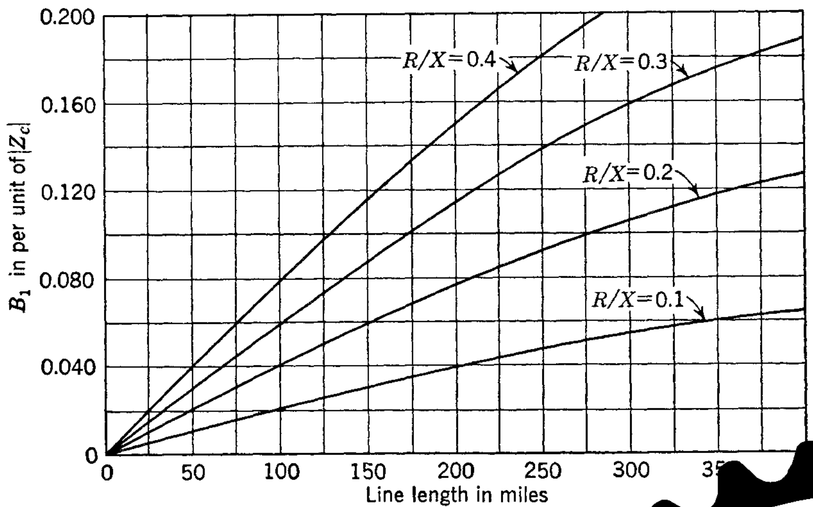


FIG. 6.9 The transmission-line B_1 constant in per unit of $|Z_c|$ a[illegible]tion of line length in miles. (From D. J. Povejsil and A. A. Johnson," A P[illegible]nit Interpreta-tion of Transmission Line Constants," *Trans. AIEE*, vol. [illegible] 194–200, 1951, by permission.)

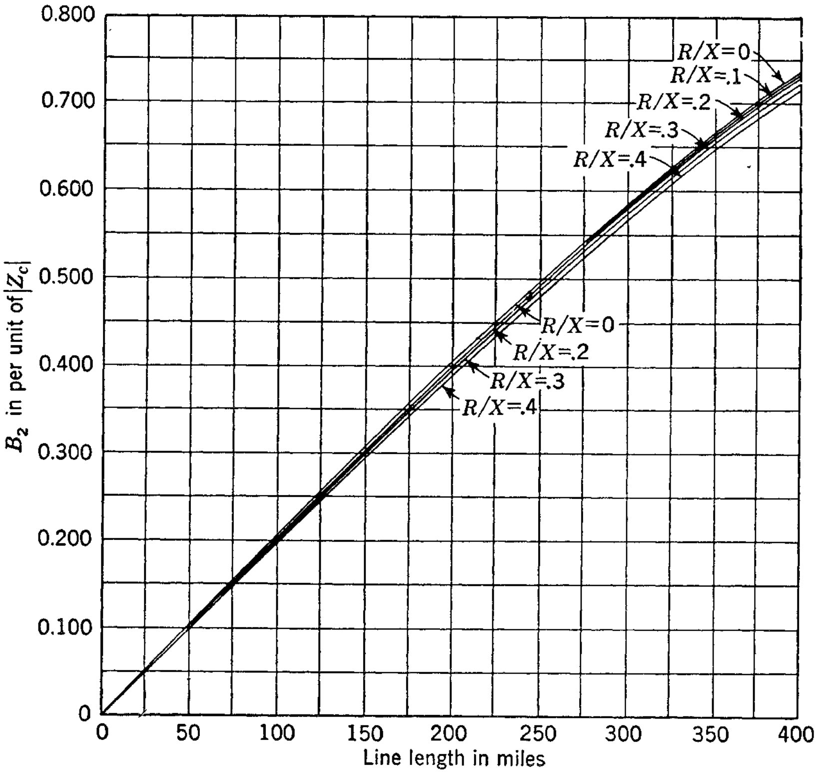


FIG. 6.10 The transmission-line B_2 constant in per unit of $|Z_c|$ as a function of line length in miles. (From D. J. Povejsil and A. A. Johnson, "A Per-unit Interpretation of Transmission Line Constants," *Trans. AIEE*, vol. 70, pp. 194–200, 1951, by permission.)

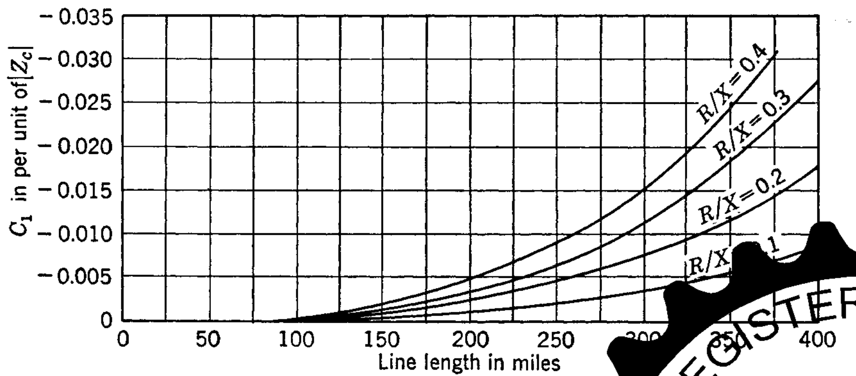


FIG. 6.11 The transmission-line C_1 constant in per unit of $|Z_c|$ as a function of line length in miles. (From D. J. Povejsil and A. A. Johnson, "A Per-unit Interpretation of Transmission Line Constants," *Trans. AIEE*, vol. 70, pp. 194–200, 1951, by permission.)

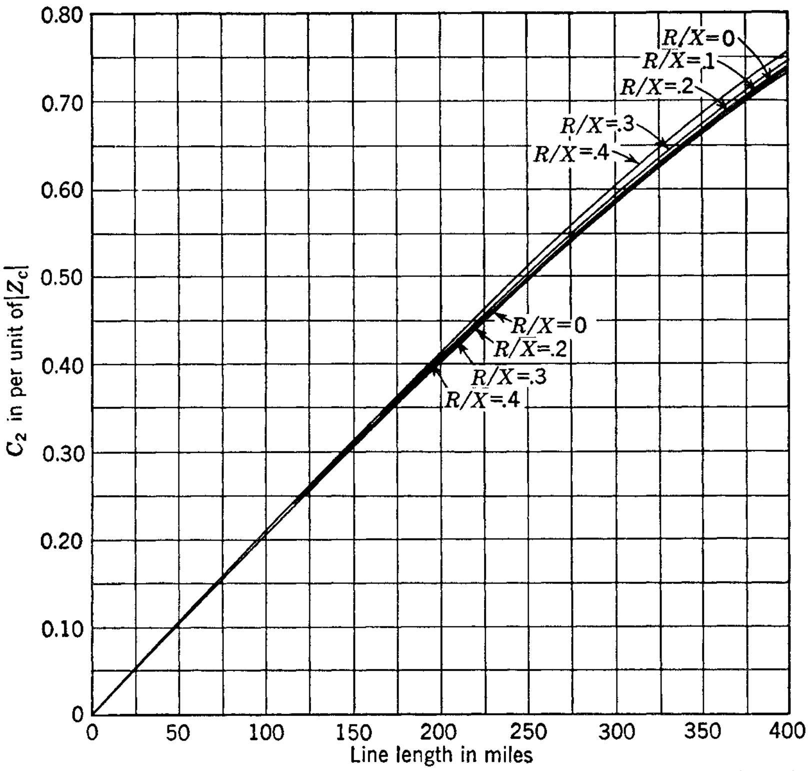


FIG. 6.12 The transmission-line C_2 constant in per unit of $|Z_c|$ as a function of line length in miles. (From D. J. Povejsil and A. A. Johnson, "A Per-unit Interpretation of Transmission Line Constants," *Trans. AIEE*, vol. 70, pp. 194–200, 1951, by permission.)

If the product of L and C is the same for all lines, the magnitude of the characteristic impedance of any line can be read from a chart, for if we assume that

$$\sqrt{\omega^2 LC} = 2.06 \times 10^{-3} \tag{6.34}$$

we can solve for ωC to obtain

$$\omega C = \frac{4.24 \times 10^{-6}}{\omega L} \tag{6.35}$$

Upon substituting this value of ωC in Eq. (6.29), we obt

$$Z_c = \sqrt{\frac{R + j\omega L}{j4.24} \times 10^6 \omega L} \tag{6.36}$$

$$Z_c = \frac{10^3 \omega L}{2.06} \sqrt{-j\,\frac{R}{\omega L} + 1} \tag{6.37}$$

Equation (6.37) shows that Z_c can be expressed in terms of the inductive reactance per mile and the ratio of the resistance to the inductive reactance. Figure 6.13 provides curves from which $|Z_c|$ can be read.

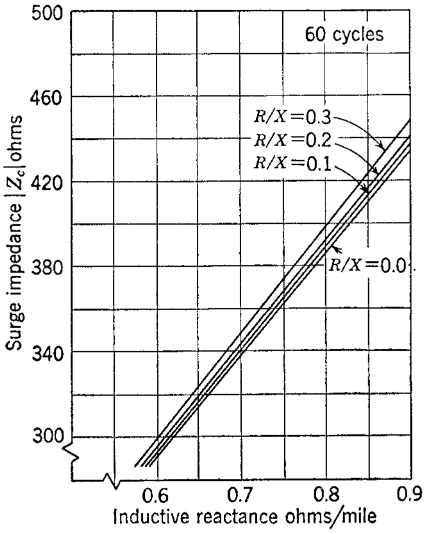


Fig. 6.13 The magnitude of the characteristic impedance of overhead transmission lines as a function of line inductive reactance per mile. (From D. J. Povejsil and A. A. Johnson, "A Per-unit Interpretation of Transmission Line Constants," *Trans. AIEE*, vol. 70, pp. 194–200, 1951, by permission.)

The method of obtaining the $ABCD$ constants from the charts is as follows:

1. Calculate, or obtain from tables, R and X in ohms per mile.
2. Calculate the ratio R/X.
3. Enter Figs. 6.7 to 6.12 with the line length and the ratio R/X, and read the real and quadrature components of each constant in per unit.
4. If B and C in ohms and mhos are desired,
   (a) Read the magnitude of Z_c from Fig. 6.13, or calculate Z_c.
   (b) Obtain the required values of B and C from Eqs. (6.33).

Example 6.1

Obtain the $ABCD$ constants of the line of Example 5.1, and check the sending-end quantities found in Example 5.2.

Solution

From Example 5.1

$$R = 0.172 \text{ ohm/mile}$$
$$X = 0.824 \text{ ohm/mile}$$

Then

$$\frac{R}{X} = \frac{0.172}{0.824} = 0.209$$

From the charts of Figs. 6.7 to 6.12, for a 225-mile line,

$$A_1 = 0.895 \qquad B_1 = 0.089 \qquad C_1 = -0.003$$
$$A_2 = 0.022 \qquad B_2 = 0.440 \qquad C_2 = 0.451$$

and upon combining the real and quadrature components we have

$$
\begin{aligned}
A &= A_1 + jA_2 = 0.895 + j0.022 = 0.895\underline{/1.4^\circ} \\
B &= B_1 + jB_2 = 0.089 + j0.440 = 0.450\underline{/78.6^\circ} \text{ per unit} \\
C &= C_1 + jC_2 = -0.0035 + j0.451 = 0.451\underline{/90.47^\circ} \text{ per unit} \\
D &= A
\end{aligned}
$$

The constants B and C when expressed in per unit are equal in magnitude but not in angle. The difference between the magnitudes of B and C in per unit here is the result of variables which enter in reading the charts and in manipulating the slide rule.

The magnitude of the characteristic impedance can be read from Fig. 6.13 or taken from the solution of Example 5.1. From the example

$$Z_c = 405\underline{/-5.9^\circ} \text{ ohms}$$

Then from Eqs. (6.33)

$$
\begin{aligned}
B &= 405 \times 0.450\underline{/78.6^\circ} = 182.5\underline{/78.6^\circ} \text{ ohms} \\
C &= \frac{0.451}{405}\underline{/90.47^\circ} = 0.000111\underline{/90.47^\circ} \text{ mho}
\end{aligned}
$$

Since

$$
\begin{aligned}
V_R &= 115{,}200\underline{/0^\circ} \text{ volts} \qquad \text{and} \qquad I_R = 361\underline{/0^\circ} \text{ amp} \\
V_S &= 0.895\underline{/1.4^\circ} \times 115{,}200 + 182.5\underline{/78.6^\circ} \times 361\underline{/0^\circ} \\
&= 103{,}000\underline{/1.4^\circ} + 65{,}800\underline{/78.6^\circ} \\
&= 103{,}000 + j2{,}520 + 13{,}000 + j64{,}500 \\
&= 116{,}000 + j67{,}020 = 134{,}000\underline{/30^\circ} \text{ volts}
\end{aligned}
$$

and

$$
\begin{aligned}
I_S &= 0.000111\underline{/90.47^\circ} \times 115{,}200 + 0.895\underline{/1.4^\circ} \times 361 \\
&= 128\underline{/90.47^\circ} + 323\underline{/1.4^\circ} = -1.03 + j128 + 323 + j7.9 \\
&= 322 + j136 = 350\underline{/22.9^\circ} \text{ amp}
\end{aligned}
$$

At the sending end

$$
\begin{aligned}
\text{Line current} &= 350 \text{ amp} \\
\text{Line voltage} &= \sqrt{3} \times 134 = 232 \text{ kv} \\
\text{Power factor} &= \cos(30^\circ - 22.9^\circ) = 0.9923 \\
\text{Power} &= \sqrt{3} \times 232 \times 350 \times 0.9923 = 140{,}[illegible] \text{ kw}
\end{aligned}
$$

Example 6.2

Find the equivalent-π circuit for the line described in Example 6.1 from the $ABCD$ constants obtained from the charts.

Solution

From Eqs. (6.19) and the constants found in Example 6.1

$$Z = B = 182.5\underline{/78.6^\circ} \text{ ohms}$$

$$\frac{Y}{2} = \frac{A-1}{B} = \frac{0.895 + j0.022 - 1}{182.5\underline{/78.6^\circ}} = \frac{-0.105 + j0.022}{182.5\underline{/78.6^\circ}}$$

$$= \frac{0.1072\underline{/168.2^\circ}}{182.5\underline{/78.6^\circ}} = 0.000588\underline{/89.6^\circ} \text{ mho}$$

6.5 Constants of Combined Networks. When a power system consists of series and parallel combinations of circuits whose $ABCD$ constants are known, it is convenient to find the constants of the circuit which is equivalent to the several component networks combined. This is a form of network simplification. In some cases it may consist of the inclusion of the characteristics of the transformers at the terminals of a line in the $ABCD$ constants of the line itself in order to analyze the over-all operation of the line with its terminating transformers. At other times it may be desirable to know the constants of a circuit equivalent to two long lines of different characteristics but operating electrically in parallel.

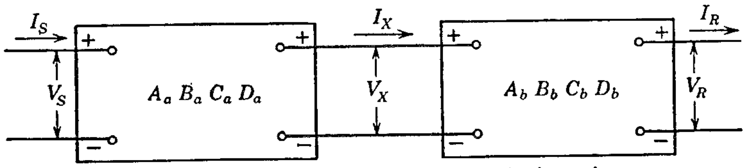


Fig. 6.14 Two four-terminal networks in series.

Consider two circuits in series as shown in Fig. 6.14. The two networks can be combined into a single equivalent network by writing equations for each one separately and eliminating the voltage and current at the junction of the two networks. For circuit a of Fig. 6.14, the voltage and current at the junction are

$$V_X = D_aV_S - B_aI_S \tag{6.38}$$
$$I_X = -C_aV_S + A_aI_S \tag{6.39}$$

and for circuit b the same voltage and current are

$$V_X = A_bV_R + B_bI_R \tag{6.40}$$
$$I_X = C_bV_R + D_bI_R \tag{6.41}$$

Eliminating V_X from Eqs. (6.38) and (6.40) and I_X from Eqs. (6.39) and (6.41), we have

$$D_aV_S - B_aI_S = A_bV_R + B_bI_R \tag{6.42}$$
$$-C_aV_S + A_aI_S = C_bV_R + D_bI_R \tag{6.43}$$

Multiplying Eq. (6.42) by A_a and Eq. (6.43) by B_a and adding the resulting equations, we obtain

$$(A_aD_a - B_aC_a)V_S = (A_aA_b + B_aC_b)V_R + (A_aB_b + B_aD_b)I_R \quad (6.44)$$

Multiplying Eq. (6.42) by C_a and Eq. (6.43) by D_a and adding the resulting equations, we obtain

$$(A_aD_a - B_aC_a)I_S = (A_bC_a + C_bD_a)V_R + (B_bC_a + D_aD_b)I_R \quad (6.45)$$

For the network equivalent to networks a and b in series, from Eqs. (6.44) and (6.45), since $A_aD_a - B_aC_a = 1$,

$$\begin{aligned} A_o &= A_aA_b + B_aC_b \\ B_o &= A_aB_b + B_aD_b \\ C_o &= A_bC_a + C_bD_a \\ D_o &= B_bC_a + D_aD_b \end{aligned} \quad (6.46)$$

If network b is at the sending end and a is at the receiving end, subscripts a and b must be interchanged in Eqs. (6.46).

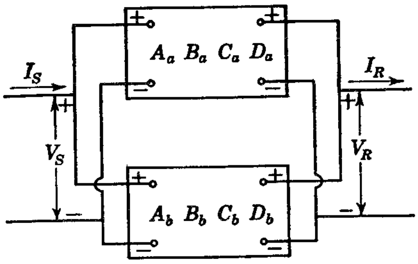


FIG. 6.15 Two four-terminal networks in parallel.

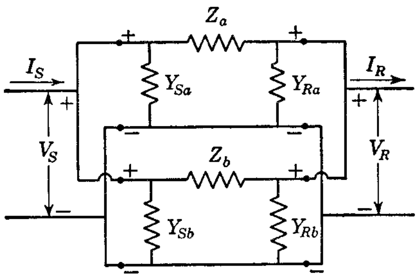


FIG. 6.16 Two equivalent-π circuits in parallel.

If two networks are connected in parallel as shown in Fig. 6.15, a convenient way to derive the $ABCD$ constants of the resultant network is to consider their equivalent-π circuits, shown in Fig. 6.16. The parameters of the resulting single equivalent-π circuit are obtained by adding the admittances which are in parallel and by finding the equivalent impedance of the two impedances in parallel in Fig. 6.16. The parameters of the resulting equivalent-π circuit in terms of the parameters of the circuits equivalent to networks a and b are

$$\begin{aligned} Y_R &= Y_{Ra} + Y_{Rb} \\ Y_S &= Y_{Sa} + Y_{Sb} \\ Z &= \frac{Z_aZ_b}{Z_a + Z_b} \end{aligned} \quad (6.47)$$

Upon substituting in Eqs. (6.47) the appropriate combinations of $ABCD$ constants from Eqs. (6.19) for the admittances and impedances of the π circuits equivalent to networks a and b, we obtain

$$
\begin{aligned}
Y_R &= \frac{A_a - 1}{B_a} + \frac{A_b - 1}{B_b} \\
Y_S &= \frac{D_a - 1}{B_a} + \frac{D_b - 1}{B_b} \\
Z &= \frac{B_a B_b}{B_a + B_b}
\end{aligned}
\tag{6.48}
$$

When the values obtained in Eqs. (6.48) are substituted in Eqs. (6.17), the constants A_o, B_o, and D_o of the resulting equivalent network are found to be

$$
\begin{aligned}
A_o &= \frac{A_a B_b + A_b B_a}{B_a + B_b} \\
B_o &= \frac{B_a B_b}{B_a + B_b} \\
D_o &= \frac{B_b D_a + B_a D_b}{B_a + B_b}
\end{aligned}
\tag{6.49}
$$

The fourth constant C_o may be found by substituting the above expressions for the other three constants in the relation $A_oD_o - B_oC_o = 1$. Thus

$$
C_o = C_a + C_b + \frac{(A_a - A_b)(D_b - D_a)}{B_a + B_b} \tag{6.50}
$$

6.6 Measurement of Generalized Circuit Constants. When a transmission line is being designed, the generalized circuit constants must be determined by methods previously discussed, all of which depend on computing the parameters of the circuit. The accuracy of such computations depends on how closely the assumed data approach actual conditions. If the line is already built, the $ABCD$ constants can be measured by making a few simple tests on the line. In Sec. 6.1 the constants were shown to be ratios of either voltage or current at the sending end of a general four-terminal network to voltage or current at the receiving end of the network with the receiving end open or short-circuited. If the network is a transformer, generator, or some circuit having lumped parameters, measurements of voltage and current at both ends of the line can be made, and the phase angles between sending- and receiving-end quantities can be determined. Thus the $ABCD$ constants [illegible] determined as indicated in Sec. 6.1. It is practicable, [illegible] to measure the magnitudes of the required voltages and currents simultaneously at both ends of a transmission line, but there is no [illegible] way to determine the difference in phase angle between quantities at the two ends of the line. Phase difference is important because the generalized circuit constants are complex. By measuring two impedances at each end of a transmission line, however, the $ABCD$ constants can be computed. The impedances to be measured are

Z_{SO} = sending-end impedance with the receiving end open-circuited
Z_{SS} = sending-end impedance with the receiving end short-circuited
Z_{RO} = receiving-end impedance with the sending end open-circuited
Z_{RS} = receiving-end impedance with the sending end short-circuited

The impedances measured from the sending end can be determined in terms of the $ABCD$ constants from Eqs. (6.4) and (6.5). With $I_R = 0$, the equations give

$$Z_{SO} = \frac{V_S}{I_S} = \frac{A}{C} \tag{6.51}$$

and with $V_R = 0$

$$Z_{SS} = \frac{V_S}{I_S} = \frac{B}{D} \tag{6.52}$$

To find the impedances measured from the receiving end, Eqs. (6.8) and (6.9) must be modified by changing the signs of all the current terms. The change in sign is necessary because, with the voltage applied at the receiving end rather than at the sending end, the direction of current flow assumed to be positive when measuring impedance is opposite to the direction shown in Figs. 6.2 and 6.4 to which Eqs. (6.8) and (6.9) apply. The equations become

$$V_R = DV_S + BI_S \tag{6.53}$$
$$I_R = CV_S + AI_S \tag{6.54}$$

From Eqs. (6.53) and (6.54) with $I_S = 0$

$$Z_{RO} = \frac{V_R}{I_R} = \frac{D}{C} \tag{6.55}$$

and when $V_S = 0$

$$Z_{RS} = \frac{V_R}{I_R} = \frac{B}{A} \tag{6.56}$$

The expression for Z_{RO} and Z_{RS} can be determined by another method. If the positions of Y_S and Y_R are interchanged in the unsymmetrical-π circuit shown in Fig. 6.5, the impedances measured at the sending end of the modified circuit are equal to the receiving-end impedances of the original unsymmetrical π. Equations (6.19) show that Y_S [illegible] terms of the $ABCD$ constants are the same except that [illegible] appears in one where D appears in the other. Therefore the substitution of D for A and of A for D in Eqs. (6.51) and (6.52) gives the receiving-end impedances of Eqs. (6.55) and (6.56).

The values of the $ABCD$ constants in terms of the measured impedances are found as follows:

$$Z_{RO} - Z_{RS} = \frac{AD - BC}{AC} = \frac{1}{AC} \tag{6.57}$$

$$\frac{Z_{RO} - Z_{RS}}{Z_{SO}} = \frac{1}{AC}\frac{C}{A} = \frac{1}{A^2} \tag{6.58}$$

$$A = \sqrt{\frac{Z_{SO}}{Z_{RO} - Z_{RS}}} \tag{6.59}$$

After A is computed by Eqs. (6.59), the other constants may be found by Eqs. (6.51), (6.55), and (6.56). It is good practice to take enough additional data to check the magnitude of each constant by the ratio of a sending-end quantity to a receiving-end quantity, as described in Sec. 6.1.

6.7 Advantages of Generalized Constants. While it may seem unnecessary to introduce generalized circuit constants into a discussion of power systems, the advantages gained by their use and their wide acceptance by the power industry make the understanding of them essential to the engineer. Often they result in more concise expressions for the equations relating voltages and currents, especially where hyperbolic functions are involved.

The greatest advantage to be gained is the increased generality of the expressions that are derived in terms of the $ABCD$ constants. In the next chapter we shall discuss circle diagrams of transmission systems for which $ABCD$ constants may be found. The derivations will be made in terms of the $ABCD$ constants of the equivalent circuit. The $ABCD$ constants may apply to only one piece of apparatus such as a transformer, to a line alone, or to a line plus its terminating transformers and other apparatus. The constants may also apply to any number of series and parallel combinations of lines with their terminal equipment, provided the system resulting from these combinations has only one location for power entering the system and one location for power leaving the system in addition to points where power entering and leaving can be simulated by fixed impedances without emfs and can be included in the $ABCD$ constants.

PROBLEMS

6.1 Find the $ABCD$ constants of a network consisting of a 500-ohm resistor shunted across its sending end, a 1,000-ohm resistor shunted across its receiving end, and 100 ohms of resistance in series between the sending and receiving ends.

6.2 Find the $ABCD$ constants of the T circuit which has 10 ohms of inductive reactance in the series arm nearest the sending end, 20 ohms of inductive reactance in the series arm nearest the receiving end, and 1,000 ohms of capacitive reactance in the shunt arm. What would be the effect of interchanging the two series arms?

6.3 A transmission line has a series impedance of [illegible] + $j0.8$ ohm/mile and a shunt admittance of 5.3×10^{-6} mho/mile. Without referring to Figs. [illegible] to 6.12,

(*a*) Evaluate the $ABCD$ constants if the line is 15 miles long.

(b) Determine the ratio $|V_S|/|V_R|$ if the line is 75 miles long and open at the receiving end.

(c) Determine the ratio $|V_S|/|V_R|$ if the line is 200 miles long and open at the receiving end.

6.4 The A and B constants of a three-phase transmission line are $0.96\underline{/1.0°}$ and $100\underline{/80°}$ ohms, respectively. If the line-to-line voltages at the sending and receiving ends are both 110 kv and the phase angle between them is 30°, find I_R and the power and power factor of the load.

6.5 Calculate the $ABCD$ constants of the nominal-π and equivalent-π circuits of the transmission line of Prob. 5.11. Do not refer to Figs. 6.7 to 6.12.

6.6 Determine the series impedance and shunt admittances of the equivalent-π circuit of the line of Prob. 5.11 from the $ABCD$ constants found in Prob. 6.5.

6.7 Find the $ABCD$ constants of the line of Prob. 5.11 by the charts of Figs. 6.7 to 6.12.

6.8 A 60-cycle three-phase transmission line has an inductive reactance of 0.8 ohm/mile, and the ratio of the resistance to the inductive reactance is 0.20. The line is 150 miles long. Find the $ABCD$ constants from the charts.

6.9 The sending-end voltage of the line described in Prob. 6.8 is 230 kv. Find the open-circuit voltage at the receiving end.

6.10 Find the voltage regulation of the line of Prob. 6.8 for a load of 100,000 kw at a power factor of 0.8 lag in parallel with synchronous condensers of 100,000 kva. Assume that the load voltage is 210 kv for this condition, and that the sending-end voltage is held constant at the value required to maintain 210 kv at the receiving end for the load described.

6.11 Find the $ABCD$ constants of the four-terminal network resulting when a resistance of 10 ohms is connected in series at the sending end of the four-terminal network whose constants are as follows:

$$A = 0.96\underline{/0°}$$
$$B = 40.0\underline{/90°} \text{ ohms}$$
$$C = 0.002945\underline{/90°} \text{ mho}$$
$$D = 0.92\underline{/0°}$$

6.12 Find the series impedance and shunt admittances of the equivalent-π circuit whose $ABCD$ constants are given in Prob. 6.11.

6.13 Measurements on a four-terminal network yield the following values: $Z_{SO} = Z_{RO} = 20$ ohms, pure resistance; $Z_{SS} = Z_{RS} = 5$ ohms, pure resistance. Find the $ABCD$ constants of the network and the parameters of its equivalent π.

6.14 Find the $ABCD$ constants of the transmission line for which $Z_{RO} = Z_{SO} = 1{,}415\underline{/-89.25°}$ ohms and $Z_{RS} = Z_{SS} = 119\underline{/68.95°}$ ohms. Find the parameters of the equivalent π of the circuit.

CHAPTER 7

CIRCLE DIAGRAMS

7.1 Introduction. A graphical analysis of the variation of the voltage, current, or power of a circuit, when some parameter of the circuit is changing, not only saves a great deal of time when the number of points to be calculated is great but also serves to explain some of the results obtained. The locus of the end points of a phasor of voltage or current some place in a circuit is very often a circle when some parameter of the circuit is varied. Thus circles can often be plotted on a set of rectangular coordinates to show the variation of some quantity in a circuit in response to the variation of some other quantity. Such circle diagrams are very helpful in the design and operation of power systems.

As an introduction to the study of circle diagrams of a power circuit, let us consider the equivalent circuit of a short transmission line. Shunt admittance is neglected, and the short line is represented by a series impedance $Z = R + jX$ between the sending and receiving ends of the circuit. Figure 7.1 is the phasor diagram of the circuit and shows the sending-end voltage V_S as the sum of the receiving-end voltage V_R and the voltage drops in the resistance and inductive reactance of the line, IR and jIX, respectively. Now suppose that the magnitudes of the current and of the voltage at the receiving end are held constant while the power factor of the load is varied. Examination of the phasor diagram shows that V_S must vary and that the end points of its phasor must lie on the semicircle prq since the magnitude of the difference between the phasors V_S and V_R is constant. The dotted impedance triangles with hypotenuses op and oq shown on the phasor diagram represent the voltage drops in the line for load power factors of zero lagging and zero leading, respectively. Since the current will not lag or lead V_R by more than 90°,

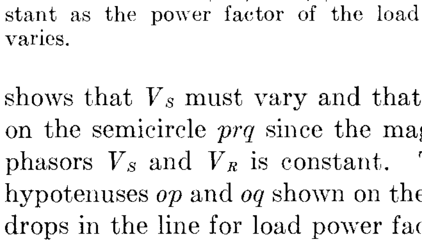


FIG. 7.1 Phasor diagram of a short transmission line. $|V_R|$ and $|I|$ are constant as the power factor of the load varies.

the points p and q are the limiting positions for the end points of the phasor V_S. A diagram such as Fig. 7.1 is easy to construct. If the construction is done with care and to a fairly large scale, many values can be measured quickly on the diagram for the calculation of voltage regulation or to plot a curve of sending-end voltage versus load power factor of a given load voltage and load kva.

7.2 Receiving-end Power Circle Diagram. A versatile form of diagram for a four-terminal network has for its coordinates real power (volt-amperes $\times \cos\theta$) on the horizontal axis and reactive power (volt-amperes $\times \sin\theta$) on the vertical axis. Each load at the receiving end is represented on the chart by a point determined by the real and reactive power of the load. If the points determined by the real power and reactive power at the receiving end of a four-terminal network are plotted on such a set of coordinates for several loads, the points will lie on a circle, provided the voltages at both ends of the network are not allowed to vary in magnitude. If circles are plotted for several values of sending-end voltage and one value of receiving-end voltage, a circle of different radius results for each $|V_S|$, but all such circles are concentric. The circles plotted for several values of receiving-end voltage and one value of sending-end voltage, however, are not concentric. The circles drawn on charts having receiving-end real and reactive power as their horizontal and vertical coordinates are called receiving-end power circle diagrams.

The circle diagram described above is developed from the phasor diagram of a four-terminal network drawn in accord with Eq. (6.4). In order to draw the phasor diagram, let

$$
\begin{aligned}
A &= |A|\underline{/\alpha} \\
B &= |B|\underline{/\beta} \\
D &= |D|\underline{/\Delta}
\end{aligned}
\tag{7.1}
$$

The constant C is not required in the development of circle diagrams, and D is required only for diagrams drawn in terms of sending-end power. Figure 7.2 is the phasor diagram of a four-terminal network with the receiving-end voltage V_R as reference. The phasor AV_R leads V_R by the angle α. If the current I_R is lagging V_R by an angle θ_R, the phasor BI_R leads V_R by the angle $(\beta - \theta_R)$. The sending-end voltage V_S is the sum of AV_R and BI_R, from Eq. (6.4).

All the phasors except I_R on the diagram of Fig. 7.2 a[illegible] Since we desire a diagram of real power and reactive [illegible] we must multiply all the quantities of Fig. 7.2 by current. Since [illegible] interested in power at the receiving end of the network, let u[illegible] V_R/B as the multiplier. V_R/B has the dimensions of current, a[illegible] the product of the voltage BI_R and the current V_R/B is V_RI_R, [illegible] volt-amperes at the receiving end of the network.

Now let us examine the new diagram resulting from our multiplication of all the phasors of Fig. 7.2 by V_R/B. Since V_R is the reference phasor in Fig. 7.2, its phase angle is 0°, and the phasor V_R/B is displaced from the reference phasor by the angle $-\beta$, since

$$\frac{|V_R|/0^\circ}{|B|/\beta} = \frac{|V_R|}{|B|}/-\beta = \frac{|V_R|}{|B|}\epsilon^{-j\beta}$$

Therefore, multiplication of the phasors of voltage on the diagram of Fig. 7.2 by V_R/B shifts all the phasors, and the whole diagram, through the angle $-\beta$. The result is the power diagram of Fig. 7.3. For convenience, the origin of the coordinate system is placed at point O in the new diagram.

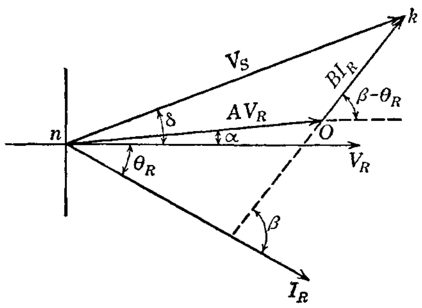


FIG. 7.2 Phasor diagram of a four-terminal network delivering a load current I_R. V_R is the reference phasor.

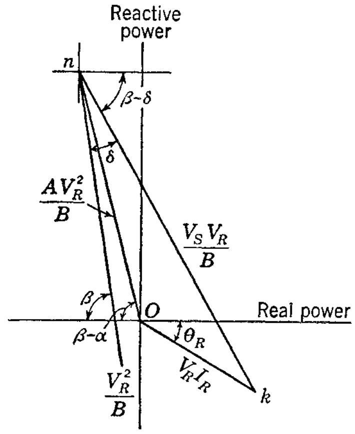


FIG. 7.3 Receiving-end power diagram resulting from multiplying the phasors of Fig. 7.2 by V_R/B. Reactive power drawn by an inductive load is plotted below the horizontal axis.

Now $V_R I_R$ lies at an angle $-\theta_R$, or at an angle θ_R below the horizontal, for $V_R I_R$ is the product of BI_R at an angle $\beta - \theta_R$ and V_R/B at an angle $-\beta$. Since $V_R I_R$ intersects the horizontal axis at the origin at the angle by which the current lags the voltage, the horizontal component of $V_R I_R$ is real power, and the vertical component is reactive power. The coordinate axes may be marked in watts and vars.

In constructing Fig. 7.3, current was taken as lagging the voltage at the load. Thus, the load is inductive, and on the diagram the [illegible] by an inductive load, or flowing into an inductive lo[illegible] negative. Engineers are not entirely in agreement on the sign of reactive power, but most power system engineers use a positive sig[illegible] indicate lagging vars, the vars of an inductive load.[1] With this convention, a capacitor

[1] AIEE Committee, "The Sign of Reactive Power—[illegible] Elec. Eng., vol. 67, pp. 49–53, January, 1948.

receives negative vars from the line. Power system engineers find it convenient to consider a capacitor as *supplying* positive vars rather than *receiving* negative vars. This concept of the action of a capacitor is consistent with the adoption of the positive sign for the vars received by an inductive load. The synchronous condenser is treated as a generator which furnishes the vars required by an inductive load. When a synchronous condenser, or capacitor, is placed at a load center, we can think of the vars required by a lagging load as coming, at least partially, from the condenser or capacitor. Since the lagging vars are not furnished by the transmission line, the line operates at a higher power factor and with lower voltage regulation.

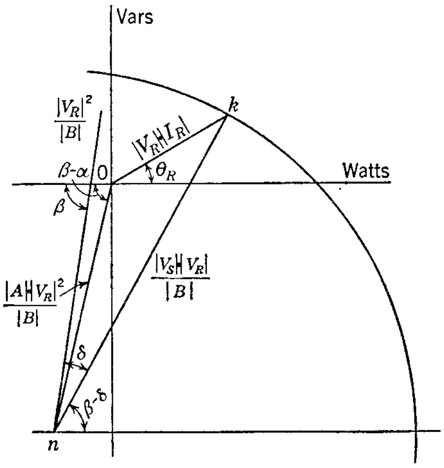


FIG. 7.4 Receiving-end power diagram resulting from rotating the diagram of Fig. 7.3 about the horizontal axis to interchange the points above and below the horizontal axis. Reactive power drawn by an inductive load is plotted above the horizontal axis.

In order to conform to the convention adopted by most power system engineers, this book will use a positive sign to indicate the lagging vars taken by an inductive load. The only alteration required in the power diagram of Fig. 7.3 is the interchanging of points above and below the horizontal axis by rotating the whole diagram about the horizontal axis. Figure 7.4 is the result. Distances on Fig. 7.4 are marked only as magnitudes since they do not have the same angular relation wit[illegible] the horizontal reference axis as the corresponding distances of [illegible] course, the phasor diagrams of current and voltage a[illegible] by the convention adopted for the sign of reactive power.

Now let us determine some points on the pow[illegible]agram of Fig. 7.4 for various loads, with fixed values of $|V_S|$ and [illegible]. First, we notice that the point n is not dependent on the current [illegible] will [illegible] as long as $|V_R|$ is constant. We note further that t[illegible] distance from point n

to point k is constant for fixed values of $|V_S|$ and $|V_R|$. Therefore, as the distance from O to k changes with changing load, the point k, since it must remain at a constant distance from the fixed point n, is constrained to move in a circle whose center is at n. Thus all the points representing loads on a network with fixed values of $|V_S|$ and $|V_R|$ lie on the circle determined by the values of the fixed voltages. If a different value of $|V_S|$ is held constant with the same value for $|V_R|$, the location of point n is unchanged, but a new circle of radius nk is found.

The point n may be located by measuring $|A| \cdot |V_R|^2/|B|$ from the origin at an angle of $\beta - \alpha$ with the horizontal in the third quadrant. Greater accuracy is obtained if the point n is located by computing its horizontal and vertical coordinates. Examination of Fig. 7.4 shows, for the receiving-end diagram,

$$\left.\begin{array}{l} \text{Radius of a receiving-end circle} = \dfrac{|V_S| \cdot |V_R|}{|B|} \qquad \text{volt-amp} \\ \text{Coordinates of the center of a receiving-end circle:} \\ \qquad \text{Horizontal} = -\dfrac{|A|}{|B|} \cdot |V_R|^2 \cos(\beta - \alpha) \qquad \text{watts} \\ \qquad \text{Vertical} = -\dfrac{|A|}{|B|} \cdot |V_R|^2 \sin(\beta - \alpha) \qquad \text{vars} \end{array}\right\} \tag{7.2}$$

Since Eqs. (7.2) and the power diagrams were developed from the constants of a four-terminal network, the voltages are in volts to neutral per phase and the coordinates are watts and vars per phase if the circuit represented by the network is a three-phase circuit. If line-to-line voltages are substituted for the line-to-neutral voltages, each length on the diagram is increased by a factor of 3, since the product of two voltages determines each length and since the line-to-line voltage of a balanced three-phase circuit is $\sqrt{3}$ times the line-to-neutral voltage. Thus the watts and vars on the diagram are total three-phase quantities when line-to-line voltages are used in Eqs. (7.2). In power system work, voltages are specified as kilovolts from line to line, and power quantities are measured in total three-phase kilowatts, kilovars, and kilovolt-amperes, or in megawatts, megavars, and megavolt-amperes. When the voltage terms are *kilovolts from line to line*, Eqs. (7.2) become

$$\left.\begin{array}{l} \text{Radius of a receiving-end circle} = \dfrac{|V_S| \cdot |V_R|}{|B|} \times 10^3 \qquad \text{kva} \\ \text{Coordinates of the center of a receiving-end circle:} \\ \qquad \text{Horizontal} = -\dfrac{|A|}{|B|} \cdot |V_R|^2 \times 10^3 \cos(\beta - \alpha) \qquad \text{kw} \\ \qquad \text{Vertical} = -\dfrac{|A|}{|B|} \cdot |V_R|^2 \times 10^3 \sin(\beta - \alpha) \qquad \text{kvars} \end{array}\right\} \tag{7.3}$$

where the coordinates of the diagram are *total kilowatts and kvars for the three phases.* Omission of the factor 10^3 yields megawatts, megavars, and megavolt-amperes.

The angle between the sending- and receiving-end voltages is called the torque angle and is designated by the symbol δ. As the load changes, the torque angle changes. On the power diagram the torque angle δ is measured between the line $|V_R|^2/|B|$ and the line drawn from point n to the load point k. The line $|V_R|^2/|B|$ is called the reference line since torque angles are measured from it. The reference line and torque angles are important in correlating a receiving-end power diagram with a sending-end power diagram as will be discussed later. Torque angles are also important in studying power-system stability.

If the receiving-end voltage is held constant and receiving-end circles are drawn for different values of sending-end voltage, the resulting circles are concentric because the location of the center of the receiving-end power circles is independent of the sending-end voltage. A family of receiving-end circles is shown in Fig. 7.5 for a constant receiving-end voltage. The radial lines cutting the circles are spaced at 10° intervals from the reference line so that the torque angle can be read for any load. The load line marked on Fig. 7.5 is convenient if the load changes in magnitude while its power factor remains constant. The angle between the load line through the origin and the horizontal axis is the angle whose cosine is the power factor of the load. The load line of Fig. 7.5 is drawn for lagging loads since all the points on the line are in the first quadrant and have positive reactive volt-amperes.

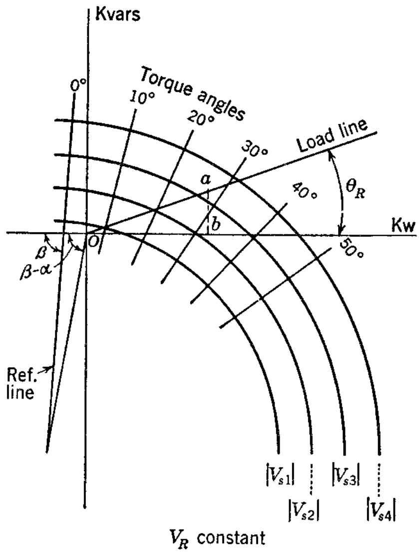


FIG. 7.5 Receiving-end power circles for various values of $|V_S|$ and a constant $|V_R|$.

If the sending-end voltage is fixed, the receiving-end power circles for different receiving-end voltages are not concentric, for [illegible] (7.3) show that the centers of the circles are at different points for each value of $|V_R|$, but the centers all lie on the same line through the origin. The radii of the circles change also as $|V_R|$ changes. Receiving-end power circles for constant $|V_S|$ are shown in Fig. 7.6.

7.3 Sending-end Power Circle Diagram. The sending-end power circle diagram has for its coordinates the real power and reactive power at the sending end of a four-terminal network. The diagram is developed in the same manner as the receiving-end diagram. First, the phasor diagram of voltages is drawn as shown in Fig. 7.7 in accord with Eq. (6.8) with V_S as the reference phasor. The phasor DV_S leads V_S by the

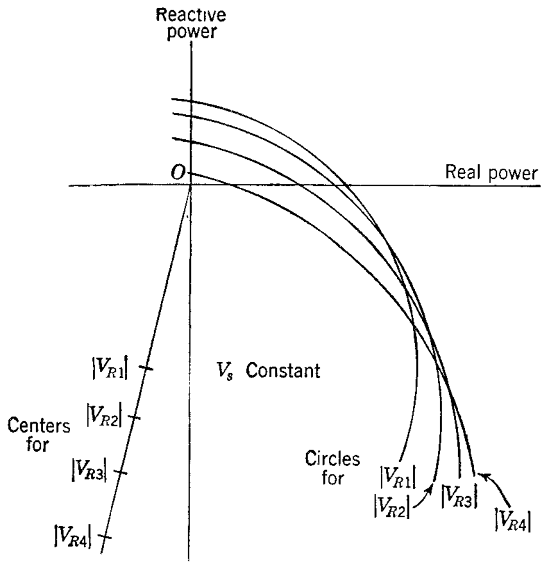


FIG. 7.6 Receiving-end power circles for various values of $|V_R|$ and a constant $|V_S|$.

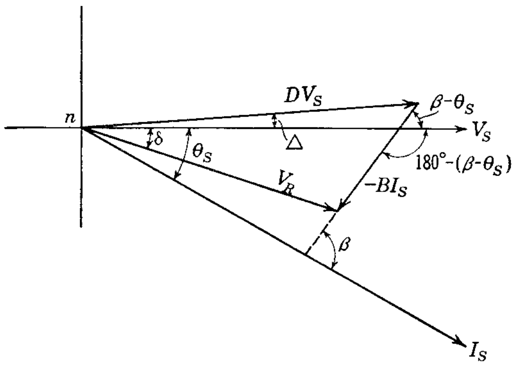


FIG. 7.7 Phasor diagram of a four-terminal network receiving a current I_S at the sending end. V_S is the reference phasor.

angle Δ. If the current I_S is lagging V_S by an angle θ_S, the phasor BI_S leads V_S by the angle $\beta - \theta_S$. The receiving-end voltage V_R is equal to DV_S minus BI_S, from Eq. (6.8).

In order to obtain a power diagram, we multiply all the phasors of Fig. 7.7 by $-V_S/B$, which is equal to $(|V_S|/|B|)\angle 180° - \beta$. The results are the rotation of Fig. 7.7 through the angle $180° - \beta$ and the conversion of all the voltages to volt-amperes. Figure 7.8 is the resulting power diagram with the origin of the coordinates moved to point O. The product of $-BI_S$ and $-V_S/B$ is V_SI_S, the volt-amperes at the sending end. Since V_SI_S intersects the horizontal axis at the origin at an angle $-\theta_S$, the horizontal component of V_SI_S is real power and the vertical component is reactive power. The coordinate axes may be marked in watts and vars. To conform with the convention adopted for the sign of reactive volt-amperes, the diagram must be rotated about the horizontal axis to give Fig. 7.9.

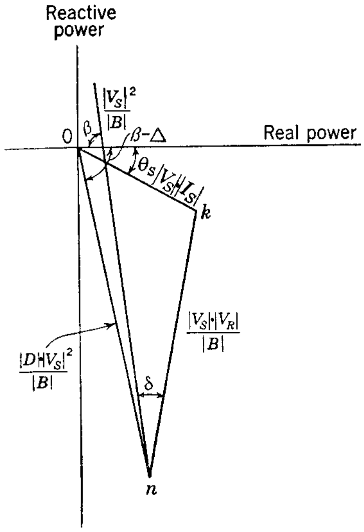


FIG. 7.8 Sending-end power diagram resulting from multiplying the phasors of Fig. 7.7 by $-V_S/B$. Reactive power drawn by an inductive load is plotted below the horizontal axis.

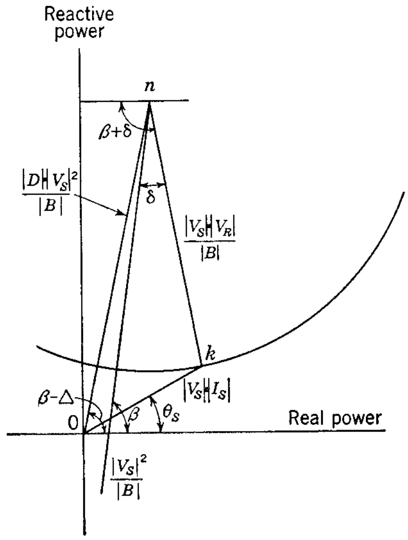


FIG. 7.9 Sending-end power diagram resulting from rotating the diagram of Fig. 7.8 about the horizontal axis to interchange the points above and below the horizontal axis. Reactive power drawn by an inductive load is plotted above the horizontal axis.

If $|V_S|$ and $|V_R|$ are held constant as the power delivered to the network is varied, the location of point n remains fixed, and the distance from point n to point k remains constant. The location of point k, however, varies with changes in the load delivered to the network, and k is constrained to move in a circle since it must remain at a constant distance from the fixed point n. If a different value of $|V_R|$ is held constant with the same constant value of $|V_S|$, the location of point n is unchanged, but the radius of the circle on which point k moves is proportional to $|V_R|$. A family of concentric circles results from several values of $|V_R|$ with a constant $|V_S|$. The circles are not concentric for a constant $|V_R|$ and different values of $|V_S|$, but the centers of the circles all lie on the same straight line making an angle $\beta - \Delta$ with the horizontal. Examination of the sending-end power diagram shows

$$\left.\begin{aligned}&\text{Radius of a sending-end circle} = \frac{|V_S| \cdot |V_R|}{|B|} \qquad \text{volt-amps}\\&\text{Coordinates of the center of a sending-end circle:}\\&\qquad \text{Horizontal} = + \frac{|D|}{|B|} \cdot |V_S|^2 \cos (\beta - \Delta) \qquad \text{watts}\\&\qquad \text{Vertical} = + \frac{|D|}{|B|} \cdot |V_S|^2 \sin (\beta - \Delta) \qquad \text{vars}\end{aligned}\right\} \tag{7.4}$$

where $|V_R|$ and $|V_S|$ are *volts to neutral,* and the coordinates of the diagram are watts and vars per phase. When the voltages are in *line-to-line kilovolts,*

$$\left.\begin{aligned}&\text{Radius of a sending-end circle} = \frac{|V_S| \cdot |V_R|}{|B|} \times 10^3 \qquad \text{kva}\\&\text{Coordinates of the center of a sending-end circle:}\\&\qquad \text{Horizontal} = + \frac{|D|}{|B|} \cdot |V_S|^2 \times 10^3 \cos (\beta - \Delta) \qquad \text{kw}\\&\qquad \text{Vertical} = + \frac{|D|}{|B|} \cdot |V_S|^2 \times 10^3 \sin (\beta - \Delta) \qquad \text{kvars}\end{aligned}\right\} \tag{7.5}$$

where the coordinates of the diagram are *total kilowatts and kvars for all three phases.*

As in the receiving-end diagram, the torque angle is δ. The reference line from which the torque angles are measured on the sending-end diagram is $|V_S|^2/|B|$.

7.4 The Use of Circle Diagrams. Once the circle diagrams have been drawn for a transmission line or any four-terminal network, a great deal of information can be obtained very quickly. Some useful information is the voltage that must be maintained at the sending end for a specified load and voltage at the receiving end. Assume that Fig. 7.5 is the receiving-end power diagram for the value of $|V_R|$ which must be maintained at the load or at the primary terminals of a transformer supplying the load. If the load varies in amount while its power factor remains the same, a load line is drawn through the origin at an angle with the horizontal axis equal to the phase angle of the load. The horizontal coordinate of the point where the load line intersects a circle of constant sending-end voltage is the power at the load for the sending-end voltage of the circle intersected. In this manner information for plotting sending-end voltage versus power at the load for a given load voltage and power factor is obtained rapidly.

Another problem readily solved with the circle diagram is the determination of the amount of reactive power that must be supplied by synchronous condensers at a load in order to increase the power factor, to reduce the amount of voltage regulation, or to maintain constant voltage at the receiving end for a given sending-end voltage. For instance, the circle diagram of Fig. 7.5 may represent the conditions on a transmission line whose receiving-end voltage is to remain constant at the value for which the diagram is drawn, and it may be desirable to operate the line at 100% power factor. If the coordinates of point a on the load line are the kilowatts and kvars of the load, the inductive kvars of the load represented by the vertical line ab must be *supplied* by a synchronous condenser or static capacitors. The line supplies only the real power represented by Ob, and the voltage at the sending end must be slightly greater than $|V_{S2}|$. Another way to consider the load conditions is to think of the combined load and capacitor with the capacitor *drawing* negative, or leading, kvars equal to ab, in which case the combined load consists only of the real power Ob. The same synchronous condenser which supplies positive kvars, or draws negative kvars, may act as an inductance and draw positive kvars, or supply negative kvars, by having its excitation reduced. A problem similar to the one discussed is that of finding the amount of load that could be added at a given power factor to an existing load without making it necessary to increase the sending-end voltage more than a specified amount to keep the receiving-end voltage above a specified minimum value.

Both the sending-end and receiving-end circle diagrams may be needed for the solution of a problem. For instance, we need both diagrams to find the sending-end power for a given receiving-end load. If the load and load voltage are known and the receiving-end power diagram is available for this load voltage, the voltage at the sending end and the torque angle may be read. Then, on a sending-end diagram drawn for the sending-end voltage found from the receiving-end diagram, the sending-end power may be read at the same torque angle and receiving-end voltage. This is an example of the torque angle measured from the reference line of a diagram being required to find corresponding points on the two diagrams.

There is a definite amount of power that may be transmitted through a network at given values of voltage, as may be seen by referring to Fig. 7.4. The load may be increased until the point k is at the intersection of the circle with the horizontal line through the point n. This position of k represents the maximum load that can be rec[eived for the] sending-end and receiving-end voltages for which the ci[rcle diagram is] drawn. In fact, this power, which is called the steady-state st[ability l]imit if the load is a synchronous machine, can be received only [if the l]oad is increased gradually. For a torque angle δ the power rec[eived is], from Fig. 7.4,

$$P_R = \frac{|V_S| \cdot |V_R|}{|B|} \cos(\beta - \delta) - \frac{|A| \cdot |V_R|^2}{|B|} \cos(\beta - \alpha) \qquad (7.6)$$

With the voltages held constant, variation in received power is accompanied by a change in the torque angle δ, the only variable in Eq. (7.6). Maximum power is received when $\delta = \beta$. Thus, the maximum power received by the load is

$$P_{R,\max} = \frac{|V_S| \cdot |V_R|}{|B|} - \frac{|A| \cdot |V_R|^2}{|B|} \cos(\beta - \alpha) \tag{7.7}$$

An equation for the power delivered to the network at the sending end may be written from inspection of the sending-end power circle diagram of Fig. 7.9, from which

$$P_S = -\frac{|V_S| \cdot |V_R|}{|B|} \cos(\beta + \delta) + \frac{|D| \cdot |V_S|^2}{|B|} \cos(\beta - \Delta) \tag{7.8}$$

Maximum power is delivered to the network when $\beta + \delta = 180°$, and

$$P_{S,\max} = \frac{|V_S| \cdot |V_R|}{|B|} + \frac{|D| \cdot |V_S|}{|B|} \cos(\beta - \Delta) \tag{7.9}$$

The maximum power given by Eq. (7.9) cannot be realized practically if the load is a synchronous machine. The angle β is less than 90° if there is any resistance in the network, and, for β less than 90°, δ must be greater than β to realize the maximum power given by Eq. (7.9). Such a condition would give a value of δ greater than that for the steady-state stability limit at the receiving end, which occurs when $\delta = \beta$.

Since B is the series impedance of the equivalent π of the network and is largely inductive reactance for a transmission line, a reduction of the series inductive reactance increases the maximum power which can be received over a transmission line. An important method of reducing the inductive reactance of a transmission line is the addition of series capacitors.[2] An increase in the voltage at the sending or receiving end also increases the maximum power receivable and hence the steady-state stability limit. Further discussion of the important problem of power system stability is reserved for Chap. 15.

Example 7.1

Draw the receiving-end power circle diagram for the line of Example 5.1 for a receiving-end voltage of 200 kv and sending-end voltages of 190 kv, 200 kv, 210 kv, 220 kv, 230 kv, and 240 kv.

1. Check the values of sending-end voltage, current, power, and power factor found in the solutions of Examples 5.1 and 5.2 for a load of 125,000 kw at 200 kv and 100% power factor by drawing the sending-end circle on the same set of coordinates.

[2] A. A. Johnson, J. E. Barkle, and D. J. Povejsil, "Fundamental Effects of Series Capacitors in High-voltage Transmission Lines," *Trans. AIEE*, vol. 70, pp. 526–535, 1951.

2. Find the sending-end voltage for loads of 25,000 kw, 50,000 kw, and 75,000 kw at 90% power factor lagging and at 100% power factor if the receiving-end voltage is 200 kv.

3. For various loads at 90% power factor lagging, find the reactive power supplied by the line and by synchronous condensers in parallel with the loads if the sending-end voltage is maintained at 220 kv and the receiving-end voltage is 200 kv. Determine the power factor at the receiving end of the line.

Solution

The generalized circuit constants, obtained from the solution of Example 6.1, are

$$A = D = 0.895\underline{/1.4^\circ}$$
$$B = 182.5\underline{/78.6^\circ}\text{ ohms}$$

The coordinates of the center of the receiving-end circles are

$$\text{Horizontal} = -\frac{0.895}{182.5} \times (200)^2 \times 10^3 \cos (78.6^\circ - 1.4^\circ) = -43{,}500\text{ kw}$$

$$\text{Vertical} = -\frac{0.895}{182.5} \times (200)^2 \times 10^3 \sin (78.6^\circ - 1.4^\circ)$$
$$= -191{,}000\text{ kvar}$$

The radii of the receiving-end circles $= \dfrac{200}{182.5} \times 10^3 \times |V_S|$ in kva. For the specified sending-end voltages, we obtain the following radii of the receiving-end circles:

$\|V_S\|$, kv	Radius, kva	$\|V_S\|$, kv	Radius, kva
190	208,000	220	241,000
200	219,000	230	252,000
210	230,000	240	263,000

The circles are drawn in Fig. 7.10. The receiving-end reference line is drawn through the center of the circles and makes an angle of $\beta = 78.6^\circ$ with the horizontal axis. Torque-angle lines are drawn for every 5° from the reference line.

1. Since the power factor of the load is 100%, the specified load is located on the horizontal axis at 125,000 kw. The sending-end voltage for this load is found by interpolation between the 230-kv and 2[illegible]-kv circles. The torque-angle line for 30° passes through the [illegible] The readings are

$$|V_S| = 232\text{ kv}$$
$$\delta = 30^\circ$$

The horizontal and vertical coordinates of po[illegible] the receiving-end circles are real and reactive power at the rec[illegible]g-end. [illegible] same set of coordinates is used for the sending-end [illegible]cles, the coordinates must be interpreted as real and reactive pow[illegible] the sending [illegible]

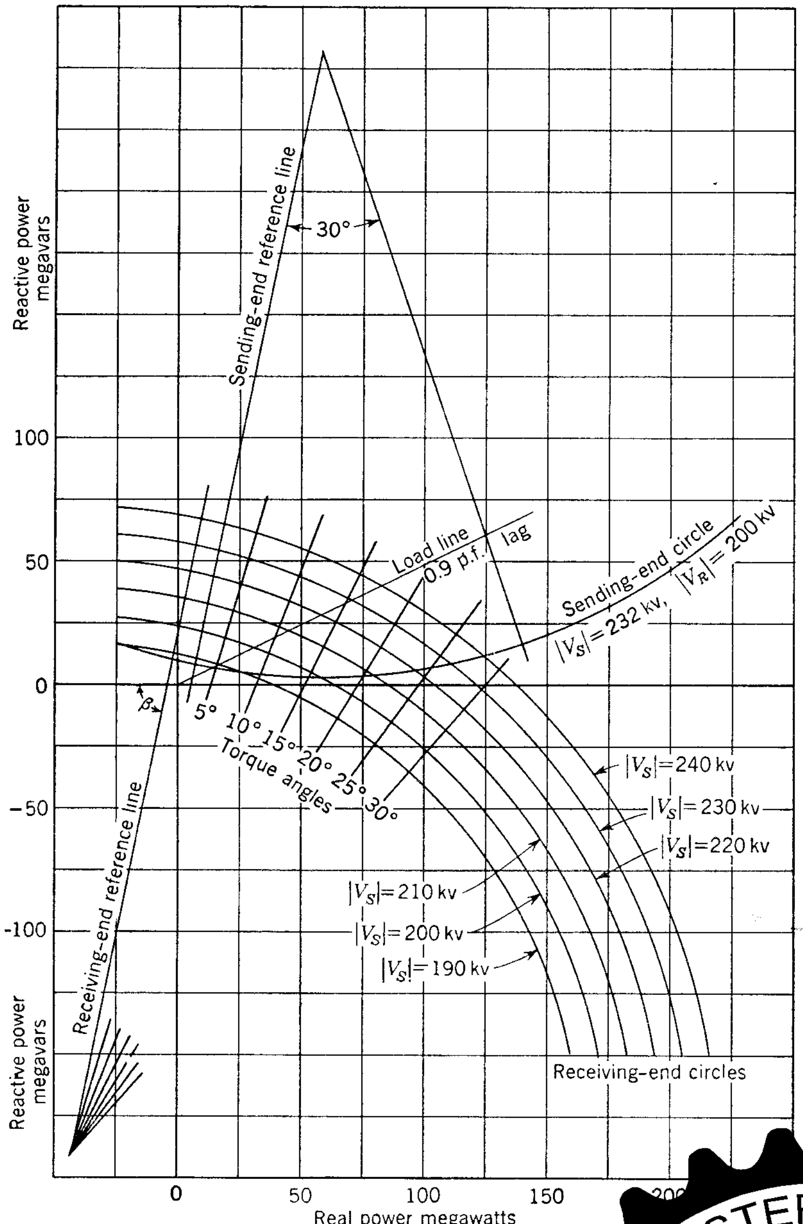


FIG. 7.10 Power circle diagrams for Example 7.1, $|V_R| = 200$ kv, constant.

order to find the power and power factor at the sending end, a sending-end circle is required for $|V_S| = 232$ kv. The coordinates of the center of this circle are

$$\text{Horizontal} = \frac{0.895}{182.5} \times (232)^2 \times 10^3 \cos(78.6° - 1.4°) = 58{,}500 \text{ kw}$$

$$\text{Vertical} = \frac{0.895}{182.5} \times (232)^2 \times 10^3 \sin(78.6° - 1.4°) = 257{,}000 \text{ kvar}$$

and its radius is

$$\frac{200 \times 232}{182.5} \times 10^3 = 254{,}000 \text{ kva}$$

The sending-end circle is shown on the chart together with its reference line and a torque-angle line of 30° for the sending end. The torque-angle line provides the link between the receiving-end circles and the sending-end circle. The load at the sending end is determined by the intersection of the 30°-torque-angle line and the sending-end circle. At this point the readings are

$$\text{Sending-end real power} = 140{,}000 \text{ kw}$$

$$\text{Sending-end reactive power} = 17{,}000 \text{ kvar}$$

$$\tan\theta_S = \frac{17{,}000}{140{,}000}$$

$$\theta_S = 6.93°$$

$$\text{Power factor} = \cos 6.93° = 0.9927$$

$$|I_S| = \frac{140{,}000}{\sqrt{3} \times 232 \times 0.9927} = 351 \text{ amp}$$

All values check the solutions of Examples 5.1 and 5.2 very closely.

2. To determine the sending-end voltages for various loads at 100% power factor, the load points are plotted along the horizontal axis in Fig. 7.10, and the voltage at the sending end for each load is found by interpolating between the circles. A load line is drawn for 90% power factor lagging, and the loads at 90% power factor lagging are plotted along this load line. Again the voltage at the sending end for each load is found by interpolating between the circles. The voltages found from Fig. 7.10 are tabulated below.

Receiving-end power, kw	Sending-end voltage, kv	
	Load p.f. = 0.9	Load p.f. = [illegible]
25,000	196	[illegible]
50,000	213	[illegible]
75,000	233	[illegible]

3. Points for the various loads are plotted along the load line, and the kvars required by each load are read. The kvars supplied by the line with 220 kv at the sending end and 200 kv at the receiving end are determined by reading the kvars at the intersection of the circle of $|V_S| = 220$ kv with the vertical line through the point representing the load. The difference between the kvars required by the load and the kvars supplied by the line must be supplied by the synchronous condensers. The power factor at the receiving end of the line is determined from the real power and reactive power supplied by the line.

For a load of 75,000 kw at 90% power factor lagging, the load line shows that 36,000 kvar are required. The intersection of the 220-kv circle with the ordinate through 75,000 kw shows that the line supplies 20,000 kvar. The remaining 16,000 kvar required by the load must be supplied by the synchronous condensers. Values for a number of loads are tabulated below.

Receiving-end power, kw	Reactive power, kvar			Receiving-end power factor
	Required by the load	Supplied by the line	Supplied by synch. condensers	
0	0	47,000	−47,000	0 lag
25,000	12,000	41,000	−29,000	0.521 lag
50,000	24,000	32,000	− 8,000	0.846 lag
75,000	36,000	20,000	16,000	0.966 lag
100,000	48,000	3,000	45,000	0.999 lag
125,000	60,000	−18,000	78,000	0.990 lead
150,000	72,000	−47,000	119,000	0.950 lead

7.5 A Universal Power Circle Diagram. The power circle diagrams described have several limitations, the most serious of which is that, although a series of concentric receiving-end circles can be drawn for a number of sending-end voltages, the resulting chart is valid for only the one receiving-end voltage for which it is constructed. If several receiving-end voltages are to be investigated, either a new receiving-end chart must be constructed for each new receiving-end voltage, or a new receiving-end chart must be constructed for each sending-end voltage. If the latter course is followed, the resulting circles are not concentric for the receiving-end chart, and the torque-angle lines are drawn from different centers for each circle. If sending-end and receiving-end circles are drawn on one chart with receiving-end real and reactive power as coordinates for the receiving-end circles and sending-end real and reactive power as coordinates for the sending-end circles, the chart must be confined to one receiving-end voltage or to one sending-end voltage, and one set of circles will not be concentric. These limitations of the power circle diagram can be overcome by a modification of the coordinate system.

The modified coordinate system is described by R. D. Goodrich, Jr.[3] To take full advantage of the method, we must express each distance on the modified circle diagram as a ratio of the distance on the original diagram in volt-amperes to a selected reference or base value of volt-amperes equal to $|V|^2/|B|$, where $|B|$ is the generalized circuit constant and $|V|$ is called the reference or base voltage and is chosen arbitrarily. Usually $|V|$ is the nominal line-to-neutral or line-to-line voltage of the system depending on whether the coordinates of the power diagram being modified are per-phase or three-phase quantities. Dimensionless units result from the division of the quantities on the original diagram by $|V|^2/|B|$. Like the dimensionless generalized circuit constants read from the charts in Chap. 6, the dimensionless ratios for our modified circle diagram are called per-unit quantities. Upon performing the specified division on Eqs. (7.2), we obtain

$$\left.\begin{array}{l} \text{Radius of a receiving-end circle} = \dfrac{|V_S|}{|V|} \cdot \dfrac{|V_R|}{|V|} \qquad \text{per unit} \\ \text{Coordinates of the center of a receiving-end circle:} \\ \qquad \text{Horizontal} = -|A| \left(\dfrac{|V_R|}{|V|}\right)^2 \cos(\beta - \alpha) \qquad \text{per unit} \\ \qquad \text{Vertical} = -|A| \left(\dfrac{|V_R|}{|V|}\right)^2 \sin(\beta - \alpha) \qquad \text{per unit} \end{array}\right\} \tag{7.10}$$

The quantities $|V_R|/|V|$ and $|V_S|/|V|$ are ratios of actual voltage to the chosen base voltage and are called the per-unit receiving- and sending-end voltages, respectively. If we consider that $|V_R|$ and $|V_S|$ *specify voltages in per unit* rather than in volts or kilovolts, Eqs. (7.10) become

$$\left.\begin{array}{l} \text{Radius of a receiving-end circle} = |V_S| \cdot |V_R| \qquad \text{per unit} \\ \text{Coordinates of the center of a receiving-end circle:} \\ \qquad \text{Horizontal} = -|A| \cdot |V_R|^2 \cos(\beta - \alpha) \qquad \text{per unit} \\ \qquad \text{Vertical} = -|A| \cdot |V_R|^2 \sin(\beta - \alpha) \qquad \text{per unit} \end{array}\right\} \tag{7.11}$$

Then for a sending-end circle with $|V_S|$ and $|V_R|$ in per unit

$$\left.\begin{array}{l} \text{Radius of a sending-end circle} = |V_S| \cdot |V_R| \qquad \text{per unit} \\ \text{Coordinates of the center of a sending-end circle:} \\ \qquad \text{Horizontal} = |D| \cdot |V_S|^2 \cos(\beta - \Delta) \qquad \text{per unit} \\ \qquad \text{Vertical} = |D| \cdot |V_S|^2 \sin(\beta - \Delta) \qquad \text{per unit} \end{array}\right\} \tag{7.12}$$

[3] See R. D. Goodrich, Jr., "A Universal Power Circle Diagram," *Trans. AIEE*, vol. 70, pp. 2042–2049, 1951. The article contains much valuable material on power circle diagrams with illustrative problems. Circular loci describing system operation with certain restrictions such as constant loss are discussed.

The method described by Goodrich replaces families of circles drawn on one set of coordinates by one family of circles with the location of the origin of the rectangular coordinate system for power quantities determined by the receiving-end voltage for receiving-end quantities and by the sending-end voltage for sending-end quantities.

If we construct a receiving-end circle diagram according to Eqs. (7.11) for various per-unit values of $|V_S|$ and a specified value of $|V_R|$, we obtain the concentric circles shown in Fig. 7.11 with their centers at n and the origin of the coordinate system at O_R. If we choose a larger value of $|V_R|$ and draw circles for the same values of $|V_S| \cdot |V_R|$ as those shown in Fig. 7.11, we obtain a new family of concentric circles having the same radii as those shown in Fig. 7.11 and the same origin O_R. The new circles must be drawn from a different center such as n', as determined by Eq. (7.11) for the new value of $|V_R|$. The center n' lies on the same line through O_R as the point n. Thus far our diagram is identical to the original diagram discussed except for the use of per-unit quantities. If we draw the new set of circles with centers at n' and move the new diagram along the line nO_R so that n and n' coincide without the diagram being rotated, both families of circles coincide. The origin from which power measurements are made for the new value of $|V_R|$ is shifted to O'_R on the extension of the line nO_R.

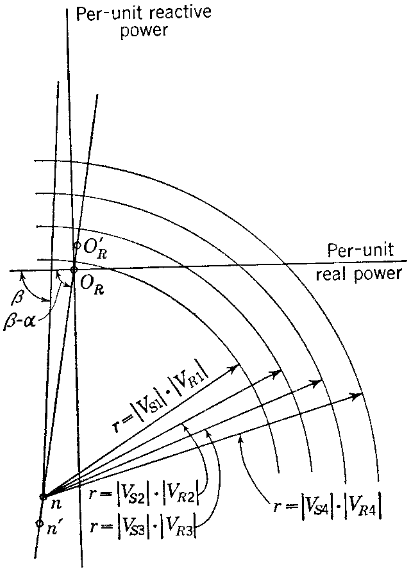


FIG. 7.11 Receiving-end power circle diagram in per-unit quantities.

The distance from the center n to the origin is the square root of the sum of the squares of the horizontal and vertical coordinates given in Eqs. (7.11), which is

$$\text{Distance } n \text{ to } O_R = |A| \cdot |V_R|^2 \qquad \text{per unit} \tag{7.13}$$

The method can now be extended to any number of values [illegible] end voltage. Measurements of power are made from [illegible] origin for each value of $|V_R|$. Polar coordinate paper may [illegible] used, or circles may be drawn on any available set of rectangu[illegible]ordinates. Only the first quadrant need be used. From the comm[illegible]ter of the circles n, the reference line is drawn at an angle β wit[illegible] horizontal just as in the ordinary circle diagram. The line nO_R on which the origins lie is drawn at an angle $\beta - \alpha$ with the horizontal, as in the ordinary diagram. Circles are drawn for convenient per-unit values of $|V_S| \cdot |V_R|$, and origins may be plotted along the line of origins for desired values of $|V_R|$. Torque angles measured from the reference line may be shown. Real power is read to the right of O_R, and positive reactive power is read upward from O_R. Per-unit values measured on the diagram and multiplied by $|V|^2/|B|$ give the three-phase power and reactive volt-amperes if $|V|$ is the line-to-line base voltage from which the per-unit voltages are determined. The circles may be printed or multilithed in advance and used for any problem.

Example 7.2

1. Use a universal circle diagram to check the values of sending-end voltage found in Example 5.1 for a load of 125,000 kw at 200 kv and 100% power factor at the receiving-end of the line. Choose a base of 220 kv.
2. If the line supplies a load of 50,000 kw at 90% power factor lagging and 215 kv, find the sending-end voltage.
3. Determine the voltage regulation for the line for the load of (2).

Solution

The generalized circuit constants obtained from the solution of Example 6.1 are

$$A = D = 0.895\underline{/1.4^\circ}$$
$$B = 182.5\underline{/78.6^\circ}\text{ ohms}$$

from which $\beta - \alpha = 77.2°$. Figure 7.12 shows circles drawn for radii of $|V_S| \cdot |V_R|$ equal to 0.8, 0.9, 1.0, 1.1, and 1.2. The reference line and line of origins are drawn from n at 78.6° and 77.2° with the horizontal, respectively.

1. In per unit on a base of 220 kv

$$|V_R| = {}^{200}\!/_{220} = 0.91 \text{ per unit}$$
$$\text{Distance } n \text{ to } O_{R1} = 0.895(0.91)^2 = 0.740 \text{ per unit}$$

The value of $|V|^2/|B|$ is

$$\frac{(220 \times 10^3)^2}{182.5} = 266 \times 10^6 \text{ volt-amp} = 266{,}000 \text{ kva}$$

The power of the load is

$$\frac{125{,}000}{266{,}000} = 0.470 \text{ per unit}$$

Since the power factor is 1.0, we locate the load at 0.470 per unit to the right of O_{R1} at the point P_1 shown on Fig. 7.12. The point P_1 is found to be at a radius of 0.96 per unit from n. Therefore

$$|V_S| = \frac{0.96}{0.91} = 1.055 \text{ per unit}$$

and converting to kilovolts, we obtain

$$|V_S| = 1.055 \times 220 = 232 \text{ kv}$$

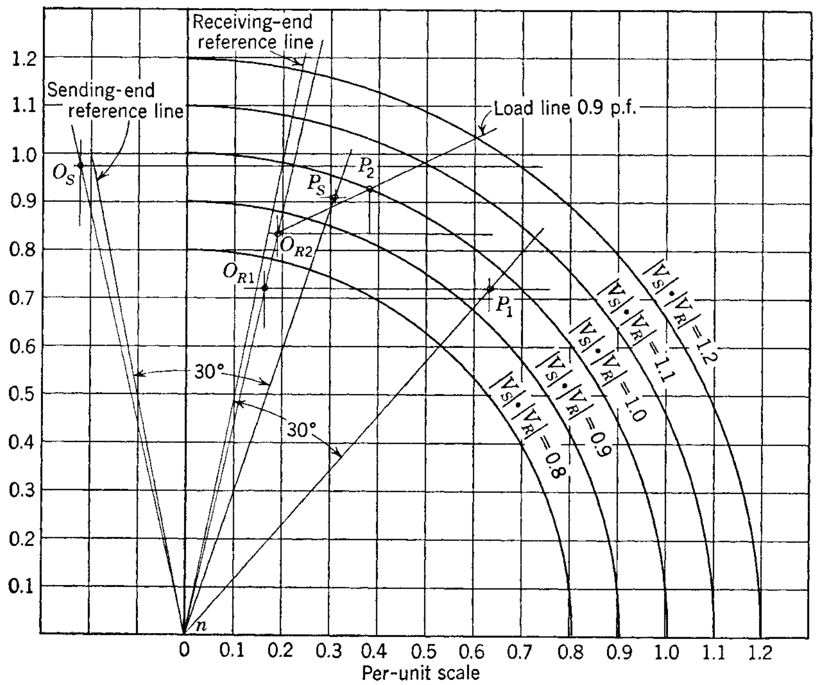


FIG. 7.12 Universal power circle diagram for Examples 7.2 and 7.3. Base voltage = 220 kv. Base volt-amperes = $(220^2/182.5) \times 10^3$ = 266,000 kva.

2. Converting 215 kv to per unit, we obtain

$$|V_R| = {}^{215}\!/_{220} = 0.977 \text{ per unit}$$

$$\text{Distance } n \text{ to } O_{R2} = 0.895(0.977)^2 = 0.855 \text{ per unit}$$

$$\text{Power} = \frac{50{,}000}{266{,}000} = 0.188 \text{ per unit}$$

The load line for 90% power factor is drawn from point ... the load point P_2 is at the intersection of the load line and the v...cal line 0.188 per unit to the right of O_{R2}. At P_2 the radius $|V_S|$ ... 1.0 per unit, and

$$|V_S| = \frac{1.0}{0.977} = 1.022 \text{ per u...}$$

or

$$|V_S| = 1.022 \times 220 = 225 \text{ kv}$$

3. To find the regulation, $|V_R|$ at no load must be found. At no load $|V_S|/|V_R| = |A|$, and

$$|V_R| = \frac{|V_S|}{|A|} = \frac{1.055}{0.895} = 1.180 \text{ per unit}$$

The regulation is

$$\frac{1.180 - 0.977}{0.977} = 20.8\%$$

The same first-quadrant circular arcs may be used for the sending-end circle diagram, and thus we have a universal circle diagram. It is necessary to plot positive reactive power downward. Although plotting positive vars downward may be confusing at first, it is advantageous because it enables us to use the same set of circles for both receiving and sending ends. We will continue to use the positive sign for the vars of loads drawing lagging current, but in plotting and reading values for the sending end on the universal diagram we will take positive vars downward.

The reason for plotting positive reactive power downward on the sending-end power circle diagram may be seen by comparing Figs. 7.8 and 7.9. By taking the center of our circles at the point n of Fig. 7.8 instead of at n in Fig. 7.9, the useful portion of the circle lies in the first quadrant with respect to n. The difference between the two figures is the sign of the reactive power. The origin O_S from which the power measurements are made lies in the second quadrant. The circles are drawn with radii equal to the same per-unit values of $|V_S| \cdot |V_R|$ as the radii of the receiving-end circles and, therefore, coincide with them when we use the same center. The reference line for the sending end is drawn at an angle of $180° - \beta$ with the horizontal axis. The line of origins makes an angle of $180° - (\beta - \Delta)$ with the horizontal axis. The origin O_S is located on the line of origins so that

$$\text{Distance } n \text{ to } O_S = |D| \cdot |V_S|^2 \qquad \text{per unit} \tag{7.14}$$

The sending-end real power is read to the right from O_S, and positive reactive power is read downward from O_S.

Example 7.3

Use the universal power circle diagram to check the sending [illegible] found in Example 5.2 for a load of 125,000 kw at 200 kv [illegible]% power factor at the receiving end of the line.

Solution

The receiving-end power for the specified load [illegible]rked P_1 on Fig. 7.12 as part of the solution of Example 7.2. The [illegible]rque angle for P_1 is

$\delta = 30°$. The sending-end reference line and line of origins is drawn in Fig. 7.12. The lines extend into the second quadrant since they make angles of $180° - 78.6° = 101.4°$ and $180° - (78.6° - 1.4°) = 102.8°$, respectively, with the horizontal. Since $|V_S| = 1.055$,

$$\text{Distance } n \text{ to } O_S = 0.895(1.055)^2 = 0.995 \text{ per unit}$$

The sending-end load point P_S must be at a radius of 0.96 per unit on the torque line making a 30° angle with the sending-end reference line. Point P_S is shown in Fig. 7.12 and found to be located 0.525 per unit to the right of O_S and 0.064 per unit below O_S. Therefore,

$$\text{Sending-end real power} = 0.525 \times 266{,}000 = 140{,}000 \text{ kw}$$
$$\text{Sending-end reactive power} = 0.064 \times 266{,}000 = 17{,}000 \text{ kvars}$$
$$\tan\theta_S = \frac{17{,}000}{140{,}000}$$
$$\theta_S = 6.93°$$
$$\text{Power factor} = \cos 6.93° = 0.9927$$

7.6 Loss Diagrams. If the receiving-end power diagram for a four-terminal network has been constructed, the power lost between the sending end and the receiving end may be calculated, without constructing the sending-end circle, by additional construction on the receiving-end diagram. An expression for the power lost is obtained by subtracting Eq. (7.6) from Eq. (7.8). Thus

$$P_L = P_S - P_R = -\frac{|V_S| \cdot |V_R|}{|B|}[\cos(\beta + \delta) + \cos(\beta - \delta)] + \frac{|D| \cdot |V_S|^2}{|B|}\cos(\beta - \Delta) + \frac{|A| \cdot |V_R|^2}{|B|}\cos(\beta - \alpha) \quad (7.15)$$

$$P_L = \frac{|A| \cdot |V_R|^2}{|B|}\cos(\beta - \alpha) + \frac{|D| \cdot |V_S|^2}{|B|}\cos(\beta - \Delta) - 2\frac{|V_S| \cdot |V_R|}{|B|}\cos\beta\cos\delta \quad (7.16)$$

If $|V_S|$ and $|V_R|$ are constant, the only variable in Eq. (7.16) is the torque angle δ. The first and second terms are the magnitudes of the horizontal coordinates of the centers of the receiving-end and sending-end circles, respectively. Equation (7.16) is not difficult [illegible] provides an easy way to obtain sending-end power [illegible] the power received is determined from the receiving-end diagram for a specified condition.

A simple loss diagram, which is the graphical solution of Eq. (7.16), may be constructed as a supplement to the receiving-end power diagram, either on the ordinary circle diagram or on the universal diagram. To measure loss on the universal circle diagram, we divide both sides of Eq. (7.16) by 2 cos β and by the base volt-amperes, $|V|^2/|B|$, to obtain

$$\frac{P_L}{2\cos\beta} = \frac{|A|\cdot|V_R|^2\cos(\beta-\alpha) + |D|\cdot|V_S|^2\cos(\beta-\Delta)}{2\cos\beta} - |V_S|\cdot|V_R|\cos\delta \qquad (7.17)$$

where $|V_S|$ and $|V_R|$ are in per unit. A distance equal to the fractional term on the right-hand side of Eq. (7.17) is measured along the reference line from n to the point marked k. At k a line called the loss line is erected perpendicular to the reference line. The construction is shown in Fig. 7.13. From any point on the circle of radius $|V_S|\cdot|V_R|$ cor-

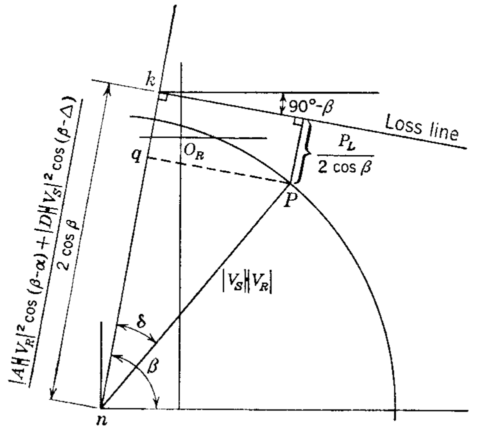


FIG. 7.13 Construction of a loss line on a universal circle diagram.

responding to the voltages for which the loss line is constructed, the distance perpendicular to the loss line is $P_L/(2\cos\beta)$, as may be seen from Eq. (7.17) and the geometry of Fig. 7.13 since the distance nq is $|V_S|\cdot|V_R|\cos\delta$.

To determine the power loss for any load, the load point P is located, and the loss line is constructed corresponding to the receiving- and sending-end voltages. The perpendicular distance from P to the loss line is measured in per unit and multiplied by 2 cos β to obtain the loss in per unit. When the loss line is constructed, sending-end power may be determined without constructing the sending-end reference line and of origins.

PROBLEMS

7.1 Find the power supplied at 100% power factor and at 90% power factor lagging by the transmission line for which Fig. 7.10 is the power-circle diagram if the sending- and receiving-end voltages are 220 kv and 200 kv, respectively.

7.2 If the load on the transmission line for which Fig. 7.10 is the power circle diagram is 100,000 kw at 90% power factor lagging and 200 kv, find the sending-end voltage when synchronous condensers are supplying 25,000 kvar at the load.

7.3 Assume that generators at the sending and receiving ends of the line for which Fig. 7.12 is the universal power circle diagram supply local loads. Voltages at both ends are 220 kv. The local load at the sending end is 10,000 kw at 80% power factor lagging. The line delivers 50,000 kw at the receiving end, where the load is 100,000 kw at 90% power factor lagging. Find the kilowatts and kvars supplied by each generator. Neglect the impedances of transformers connecting the generators to the line.

7.4 Construct the receiving-end power diagram for the line of Prob. 5.11 for $V_R = 220$ kv. Draw circles for every 10-kv increment of sending-end voltage between 200 kv and 250 kv. Determine the sending-end voltage for loads of 20,000 kw, 40,000 kw, 60,000 kw and 80,000 kw at 90% power factor lagging.

7.5 Read from the circle diagram of Prob. 7.4 the ratio of $|V_S|$ to $|V_R|$ at no load, and compute the voltage regulation of the line for each of the loads specified in Prob. 7.4.

7.6 On the circle diagram of Prob. 7.4 draw the sending-end circle and determine the sending-end power and power factor for a load of 40,000 kw at 90% power factor lagging when $V_R = 220$ kv. If a synchronous motor load of 60,000 kw at 80% power factor leading is added to the 40,000-kw lagging load, what is the value of V_S for $V_R = 220$ kv?

7.7 Plot sending- and receiving-end power circles for the line of Prob. 5.11 for sending- and receiving-end voltages of 220 kv and 210 kv, respectively. From the diagram obtain data for and draw curves of sending-end real and reactive power and receiving-end reactive power versus real power at the receiving end.

7.8 Construct the universal power circle diagram for Prob. 5.11, and check the sending-end voltage found in Prob. 7.4 for the load of 40,000 kw at 220 kv and 90% power factor lagging. Choose a base voltage of 220 kv and a scale of 5 in. = 1.0 per unit. If the receiving-end voltage drops to 210 kv and the load becomes 60,000 kw at 90% power factor lagging, find the new values of sending-end voltage and sending-end power.

7.9 Determine the loss in the line of Prob. 5.11 for the 40,000-kw load at 220 kv and 90% power factor lagging by constructing a loss line on the universal power circle diagram of Prob. 7.8.

7.10 Construct a universal power circle diagram for the line described in Prob. 5.14, and from the diagram determine the voltage, power, and power factor at the sending end for a receiving-end load of 75,000 kw at 220 kv and unity power factor. Check the results with the answers to Prob. 5.15.

CHAPTER 8

REPRESENTATION OF POWER SYSTEMS

8.1 The One-line Diagram. Since a balanced three-phase system is always solved as a single-phase circuit composed of one of the three lines and a neutral return, it is seldom necessary to show more than one phase and the neutral return when drawing a diagram of the circuit. Often the diagram is simplified further by omitting the completed circuit through the neutral and by indicating the component parts by standard symbols rather than by their equivalent circuits. Such a simplified diagram of an electric system is called a one-line diagram. It indicates by a single line and standard symbols the transmission lines and associated apparatus of an electric system.

The purpose of the one-line diagram is to supply in concise form the significant information about the system. The importance of different features of a system varies with the problem under consideration, and the amount of information included on the diagram depends on the purpose for which the diagram is intended. For instance, the location of circuit breakers and relays is unimportant in making a load study. Breakers and relays are not shown if the primary function of the diagram is to provide information for such a study. On the other hand, determination of the stability of a system under transient conditions resulting from a fault depends on the speed with which relays and circuit breakers operate to isolate the faulted part of the system. Therefore, information about the circuit breakers may be of extreme importance. Sometimes one-line diagrams include information about the current and potential transformers which connect the relays to the system or which are installed for metering. The information found on a one-line diagram must be expected to vary according to the problem at hand and according to the practice of the particular company preparing the diagram.

The American Standards Association has assigned a set of standard symbols and device numbers to all the components for an electric system.[1] The basic symbol for a machine or rotating armature is a

[1] For a complete list of standard symbols, see "American Standard Graphical Symbols for Electrical Power and Control," ASA Z32.3-1946, and "Basic Graphical Symbols for Electric Apparatus," ASA Z32.12-1947, American Standards Association, New York.

circle, but so many adaptations of the basic symbol are listed that every piece of rotating electric machinery in common use may be indicated. For anyone who is not working constantly with one-line diagrams, it is clearer to indicate a particular machine by the basic symbol followed by information on its type and rating. A few of the most common symbols are shown in Fig. 8.1. Some of these symbols will be used in one-line diagrams in this book.

It is important to know the location of points where a system is connected to ground in order to calculate the amount of current flowing when an unsymmetrical fault involving ground occurs. The standard symbol to designate a three-phase Y with the neutral solidly grounded

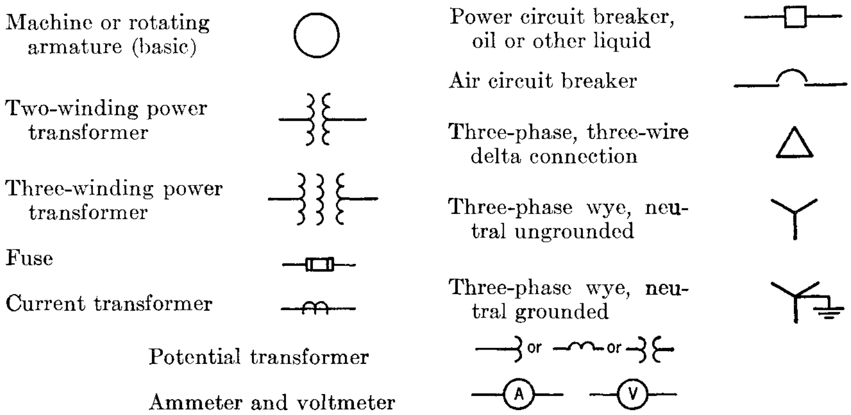


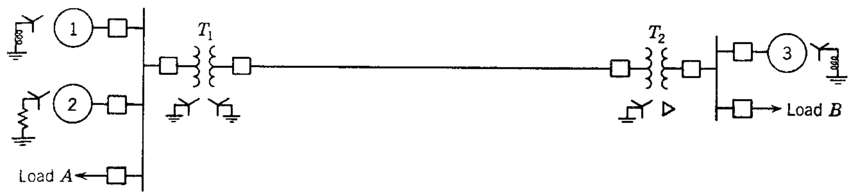
FIG. 8.1 Apparatus symbols approved by the American Standards Association.

is shown in Fig. 8.1. If a resistor or reactor is inserted between the neutral of the Y and ground to limit the flow of current to ground during a fault, the appropriate symbol for resistance or inductance may be added to the standard symbol for the grounded Y. Generator neutrals are usually grounded through resistors or inductance coils. Most transformer neutrals in transmission systems above 70 kv are solidly grounded. Below 70 kv, transformer neutrals may be solidly grounded or grounded through resistance, inductive reactance, or a coil which is tuned to provide a parallel resonant circuit in the path of the current flowing [illegible] a single line-to-ground fault. The resulting high impedance t[illegible] current permits operation of the system during this type o[illegible]. The coil is called a ground-fault neutralizer, or Petersen [illegible] becoming increasingly popular for grounding transformer neu[illegible]s below 70 kv.[2]

[2] See E. T. B. Gross, "Trends in Transmission Syste[illegible]unding Practice," *Proc. Midwest Power Conference* (now called *Proc. Am. Power* [illegible]*erence*), [illegible]

Figure 8.2 is the one-line diagram of a very simple power system. Two generators, one grounded through a reactor and one through a resistor, are connected to a bus and through a step-up transformer to a transmission line. Another generator, grounded through a reactor, is connected to a bus and through a transformer to the opposite end of the transmission line. A load is connected to each bus. On the diagram is included information about the loads, the ratings of the generators and transformers, and reactances of the different components of the circuit. Resistance is often neglected in making fault calculations and is omitted in the information accompanying Fig. 8.2. If a load study is to be made, resistance should be included.

The reactances specified for the generators of Fig. 8.2 are known as subtransient reactances. The study of a-c machinery shows that the



#1 Generator—20,000 kva, 6.6 kv, $X'' = 0.655$ ohms
#2 Generator—10,000 kva, 6.6 kv, $X'' = 1.31$ ohms
#3 Generator—30,000 kva, 3.81 kv, $X'' = 0.1452$ ohms
T_1 and T_2—each transformer in each 3-phase bank—10,000 kva, 3.81–38.1 kv, $X = 14.52$ ohms referred to the high-tension side
Reactance of the transmission line = 17.4 ohms
Load A = 15,000 kw, 6.6 kv, p.f. = 0.9 lag
Load B = 30,000 kw, 3.81 kv, p.f. = 0.9 lag

FIG. 8.2 One-line diagram of an electric system.

current flowing immediately after the occurrence of a fault depends on a different value of reactance in a generator or motor than the value which determines the current under steady-state conditions. For present purposes, it is only necessary to know that the reactance in the equivalent circuit of a rotating machine is in series with an internal generated emf of the machine. Until machine reactances and internal emfs are discussed in Chap. 9, the particular reactance and internal emf for the equivalent circuit will be specified, and the names by which they are called need cause no confusion.

395, Illinois Institute of Technology, 1951; AIEE Committee Report, "Guide for Grounding Synchronous Generator Systems," *Trans. AIEE*, vol. 72, Part III, pp. 517–530, 1953.

8.2 The Impedance and Reactance Diagrams. In order to calculate the performance of a system under load conditions, or upon the occurrence of a short circuit, the one-line diagram must be converted into an impedance diagram showing the equivalent circuit of each component of the system referred to the same side of one of the transformers. Figure 8.3 is the detailed impedance diagram of the system of Fig. 8.2. The equivalent circuit of the transmission line is represented with sufficient accuracy by the nominal π having the total resistance and inductive reactance of the line in its series arm and the total capacitance to neutral divided between its shunt arms. Resistance, leakage reactance, and a path for magnetizing current are shown for each transformer. A generated voltage in series with appropriate values of resistance and reactance represents each generator. If a load study is to be made, the lagging loads A and B are represented by resistance and inductive reactance in

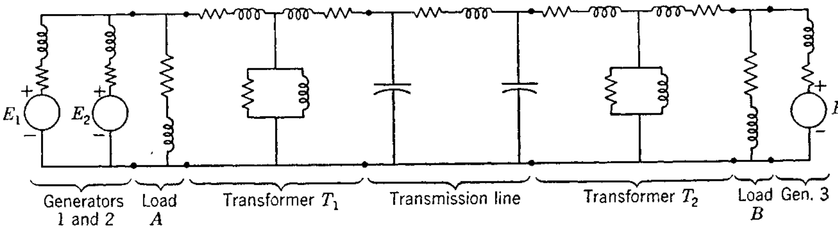


FIG. 8.3 Impedance diagram corresponding to the one-line diagram of Fig. 8.2.

series. The impedance diagram does not include the current-limiting impedances shown in the one-line diagram between the neutrals of the generators and ground because no current flows in the ground under balanced conditions and the neutrals of the generators are at the potential of the neutral of the system. Since the magnetizing current of a transformer is usually insignificant compared to the full-load current, the shunt admittance is usually omitted in the equivalent circuit of the transformer. The impedance diagram is followed in setting up a system on an a-c calculating board for making a load study.

As previously mentioned, resistance is sometimes omitted when making fault calculations. Of course, omission of resistance introduces some error, but the results may be satisfactory since the inductive reactance of a system is much larger than its resistance. Resistance and inductive reactance do not add directly, and impedance is not far d[illegible] the inductive reactance if the resistance is small. L[illegible]ich do not involve rotating machinery have little effect on the [illegible]al line current during a fault and are often omitted. Synchron[illegible] motor loads, however, are always included in making fault cal[illegible] since their generated emfs contribute to the short-circuit curr[illegible] The d[illegible] take induction motors into account by a genera[illegible] emf in series with an inductive reactance if the diagram is to be used to determine the current immediately after the occurrence of a fault. Induction motors are ignored in computing the current a few cycles after the fault occurs because the current contributed by an induction motor dies out very quickly after the induction motor is short-circuited.

If we decide to simplify our calculation of fault current by omitting all static loads, all resistances, the magnetizing current of each transformer, and the capacitance of the transmission line, the impedance diagram reduces to the reactance diagram of Fig. 8.4. The simplified reactance diagram is useful for making fault calculations analytically, but more precise fault calculations should be made on an a-c calculating board which takes into account the resistance of the circuit and the shunt capacitance of the transmission lines.

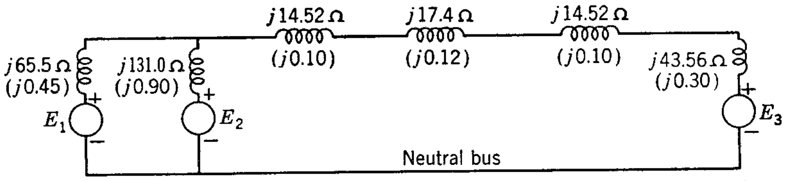


FIG. 8.4 Reactance diagram adapted from Fig. 8.3 by omitting all loads, resistances, and shunt admittances. Reactances are marked in ohms referred to the high-tension sides of the transformers. Values in parentheses are per-unit reactances on a 30,000-kva, 66-kv base.

The impedance and reactance diagrams discussed here are sometimes called the positive-sequence diagrams since they show impedances to balanced currents in a symmetrical three-phase system. The significance of this designation will become apparent when Chap. 10 is studied.

When a transformer is represented by its equivalent circuit, there is no transformation of voltage corresponding to the transformation of voltage between the high- and low-tension sides of the actual transformer. The current at both ends of the equivalent circuit is identical if magnetizing current is neglected. In an actual transformer, the current in the high- and low-tension windings would be identical only for equal turns in the primary and secondary windings with magnetizing current neglected. In a circuit where transformers are represented by their equivalent circuits, the proper impedances are those of the actual circuit referred to the side of the transformer for which the equivalent circuit is constructed.

The impedances marked on the diagram of Fig. 8.4 are in ohms referred to the high-tension circuit. Since the transmission line under consideration is in the high-tension part of the circuit, no correction is necessary in the value of the reactance placed in the equivalent circuit to represent the transmission line. The information on the one-line diagram specifies the leakage reactance of the transformers in high-tension terms, and no correction is necessary in the values of transformer leakage reactance used in the equivalent circuit.

Transformer theory shows that the impedance on the secondary side of a transformer may be transferred to the primary side by multiplying the impedance by the square of the ratio of the turns in the primary winding to the turns in the secondary winding. The generators shown in Fig. 8.2 are on the low-tension sides of the transformers, and their reactances must be referred to the high-tension circuit for which the diagram of Fig. 8.4 is drawn. Generators 1 and 2 are connected to the high-tension circuit through Y-Y transformers having a turns ratio of 10 to 1. Therefore, in high-tension terms the reactances of generators 1 and 2 are $10^2 \times 0.655 = 65.5$ ohms and $10^2 \times 1.31 = 131.0$ ohms.

The procedure in the case of generator 3, which is connected to the transmission line by a Δ-Y transformer, is not so obvious. We can arrive at the correct procedure by considering the Δ-Y transformer to be replaced by a Y-Y transformer giving the same transformation of line-to-line voltage. Since the Δ-Y transformer has a turns ratio of 10 to 1 between each high- and low-tension winding, the ratio of line voltages is 17.32 to 1. Therefore, the turns ratio between high- and low-tension windings of the Y-Y transformer having the same ratio of transformation of line voltages as the specified Δ-Y transformer must be 17.32 to 1. Looking at the equivalent circuit of generator 3 from the high-tension circuit through the Y-Y transformer, we see that the generator reactances must be referred to the high-tension circuit by multiplying them by the square of the ratio of transformation of the voltages to neutral. The ratio of line-to-line voltages is the same as the ratio of the voltages to neutral on the two sides of the transformer. Therefore, the multiplying factor is the square of the ratio of line-to-line voltages and not the square of the turns ratio of the individual windings of the Δ-Y transformer. The reactance of generator 3 in high-tension terms is $17.32^2 \times 0.1452 = 43.56$ ohms.

The internal voltages of the generators are represented in the high-tension circuit by multiplying them by the ratio of the line-to-line voltage of the high-tension circuit to the line-to-line voltage of the low-tension circuit regardless of the transformer connection.

8.3 Per-unit Quantities. Voltage, current, kva, and impedance of a circuit are often expressed as a per cent or per unit of a selected base or reference value of each of these quantities. For instance, if a base voltage of 120 kv is chosen, voltages of 108 kv, 120 kv, and 126 kv become 0.90, 1.00, and 1.05 per unit, or 90%, 100%, and 105%, respectively. The per-unit value of any quantity is defined as the ratio of the quantity to its base value expressed as a decimal. The ratio in per cent is 100 times the value in per unit. Both the per cent and per-unit methods of calculation are simpler than the use of actual amperes, ohms, and volts. The per-unit method has an advantage over the per cent method because the product of two quantities expressed in per unit is expressed in per unit itself, but the product of two quantities expressed in per cent must be divided by 100 to obtain the result in per cent.

Voltage, current, kva, and impedance are so related that selection of base values for any two of them determines the base values of the remaining two. If we specify the base values of current and voltage, base impedance and base kva can be determined. The base impedance is that impedance which will have a voltage drop across it equal to the base voltage when the current flowing in the impedance is equal to the base value of the current. The base kva in single-phase systems is the product of base voltage in kv and base current in amperes. Usually base kva and base voltage in kv are the quantities selected to specify the base. For single-phase systems, or three-phase systems where the term current refers to line current, the term voltage refers to voltage to neutral, and the term kva refers to kva per phase, the following formulas relate the various quantities:

$$\text{Base current in amperes} = \frac{\text{base kva}}{\text{base voltage in kv}} \tag{8.1}$$

$$\text{Base impedance} = \frac{\text{base voltage in volts}}{\text{base current in amperes}} \tag{8.2}$$

$$= \frac{(\text{base voltage in kv})^2 \times 1{,}000}{\text{base kva}} \tag{8.3}$$

$$\text{Base power in kw} = \text{base kva} \tag{8.4}$$

$$\begin{array}{r}\text{Per-unit impedance of} \\ \text{a circuit element}\end{array} = \frac{\text{actual impedance in ohms}}{\text{base impedance in ohms}} \tag{8.5}$$

Since three-phase circuits are solved as a single line with a neutral return, the bases for quantities in the impedance diagram are kva per phase and kv from line to neutral. Data are usually given as total three-phase kva and line-to-line kv. Because of this custom of specifying line-to-line voltage and total kva, confusion may arise regarding the relation between the per-unit value of line voltage and the per-unit value of phase voltage. Although a line voltage may be s[illegible] base, the voltage in the single-phase circuit required for solution is still the voltage to neutral. The base voltage to neutral is the base voltage from line to line divided by $\sqrt{3}$. Since this is also the ratio between line-to-line and line-to-neutral voltages of a balanced three-phase system, the per-unit value of a line-to-neutral voltage on the line-to-neutral voltage base is equal to the per-unit value of the line-to-line voltage at the same point on the line-to-line voltage base if the system is balanced. Similarly, the three-phase kva is three times the kva per phase, and the three-phase kva base is three times the base kva per phase. Therefore, the per-unit value of the three-phase kva on the three-phase kva base is identical to the per-unit value of the kva per phase on the kva-per-phase base. For instance, if the base kva is 30,000 kva and the base line-to-line voltage is 120 kv, the base values per phase are $30{,}000/3 = 10{,}000$ kva and $120/\sqrt{3} = 69.2$ kv. Then, for an actual line voltage of 108 kv, the phase voltage is $108/\sqrt{3} = 62.3$ kv, and the per-unit voltage is $108/120 = 62.3/69.2 = 0.90$. A total three-phase power of 18,000 kw is $18{,}000/3 = 6{,}000$ kw per phase, and per-unit power is $18{,}000/30{,}000 = 6{,}000/10{,}000 = 0.6$. Unless otherwise specified, a given value of base voltage in a three-phase system is a line-to-line voltage, and a given value of base kva is total three-phase kva.

Base impedance and base current can be computed directly from three-phase values of base kv and base kva. If we interpret base kva and base voltage in kv to mean base kva for the total of the three phases and base voltage from line to line, we find

$$\text{Base current in amperes} = \frac{\text{base kva}}{\sqrt{3} \times \text{base voltage in kv}} \tag{8.6}$$

and from Eq. (8.3)

$$\text{Base impedance} = \frac{(\text{base voltage in kv}/\sqrt{3})^2 \times 1{,}000}{\text{base kva}/3} \tag{8.7}$$

$$\text{Base impedance} = \frac{(\text{base voltage in kv})^2 \times 1{,}000}{\text{base kva}} \tag{8.8}$$

Since Eqs. (8.3) and (8.8) are identical, the same equation for base impedance is valid for either single-phase or three-phase circuits, provided that, in the three-phase case, line-to-line kv is used in the equation with three-phase kva or line-to-neutral kv is used with kva per phase. Equation (8.1) determines the base current for single-phase systems or for three-phase systems where the bases are specified in kva per phase and kv to neutral. Equation (8.6) determines the base current for three-phase systems where the bases are specified in total kva for the three phases and in kv from line to line.

Sometimes the per-unit impedance of a component of a system is expressed on a base other than the one selected as base for [illegible] the system in which the component is located. Since [illegible] impedances in any part of a system must be expressed on the same impedance base when making computations, it is necessary to have [illegible] means of converting per-unit impedances from one base to another. Substituting the expression for base impedance given by Eqs. (8.3) or [illegible] for base impedance in Eq. (8.5) gives

$$\text{Per-unit impedance of a circuit element} = \frac{(\text{actual impedance in ohms}) \times (\text{base kva})}{(\text{base voltage in kv})^2 \times 1{,}000} \tag{8.9}$$

Equation (8.9) shows that per-unit impedance is directly proportional to base kva and inversely proportional to the square of the base voltage. Therefore, to change from per-unit impedance on a given base to per-unit impedance on a new base, the following formula applies:

$$\text{Per-unit } Z_{\text{new}} = \text{per-unit } Z_{\text{given}} \left(\frac{\text{base kv}_{\text{given}}}{\text{base kv}_{\text{new}}}\right)^2 \times \left(\frac{\text{base kva}_{\text{new}}}{\text{base kva}_{\text{given}}}\right) \tag{8.10}$$

If we decide to convert the ohmic values of reactance shown on the diagram of Fig. 8.4 to per unit, we might select 30,000 kva and 66 kv as base. Then we would determine the base impedance from Eq. (8.8), as follows:

$$\text{Base impedance} = \frac{66^2 \times 1{,}000}{30{,}000} = 145.2 \text{ ohms}$$

Dividing each value of ohmic reactance on the diagram by the base impedance of 145.2 ohms gives the per-unit value of that reactance. The per-unit value of each reactance is shown enclosed in parentheses under each ohmic value on the diagram of Fig. 8.4.

8.4 Selection of Base for Per-unit Quantities. The selection of base values of kva and kv is made in order to reduce the work required by the calculations as much as possible. First, a base is selected for some part of the circuit. Then, the base in other parts of the circuit, separated from the original part by transformers, should be determined according to principles which will be developed in this section. The base selected should be one that yields per-unit values of rated voltage and current approximately equal to unity in order to simplify the work of computing. Time will be saved if the base is so selected that few per-unit quantities already known need be converted to a new base.

When the resistance and reactance of a device are given by the manufacturer in per cent or per unit, the base is understood to be the rated kva and kv of the apparatus. Tables are available giving approximate values of per-unit impedances of transformers, generators, synchronous motors, and induction motors.[3] Values obtained from tables are based on average values for apparatus of similar size and type. Since motors are usually rated in terms of horsepower and voltage, rated kva can be

[3] Tables A.5 and A.6 in the Appendix list some representative values. For other values, see Central Station Engineers of the Westinghouse Electric Corp., "Electrical Transmission and Distribution Reference Book," 4th ed., East Pittsburgh, Pa., 1950; "A-C Network Analyzer Manual," General Electric Company, GET-1285a, Schenectady, N.Y., 1952; A. E. Knowlton, "Standard Handbook for Electrical Engineers," McGraw-Hill Book Company, Inc., New York, 1941.

found only if the efficiency and power factor are known. If information on efficiency and power factor is lacking, the following relations, derived from average values for the particular type of motor, should be used:

Induction motors: kva = horsepower
Synchronous motors:
  Unity power factor rating: kva = 0.85 × horsepower
  0.8 power factor rating: kva = 1.10 × horsepower

The ohmic values of resistance and leakage reactance of a transformer depend on whether they are measured on the high- or low-tension side of the transformer. If they are expressed in per unit, the base kva is understood to be the kva rating of the transformer. The base voltage is understood to be the voltage rating of the low-tension winding if the ohmic values of resistance and leakage reactance are referred to the low-tension side of the transformer and to be the voltage rating of the high-tension winding if they are referred to the high-tension side of the transformer. The per-unit impedance will be the same in either case, as may be shown by the following development. Let

Z_{HT} = impedance referred to the high-tension side of the transformer
Z_{LT} = impedance referred to the low-tension side of the transformer
kv_L = rated low-tension voltage of the transformer
kv_H = rated high-tension voltage of the transformer
kva = rated kva of the transformer

Then

$$Z_{LT} = \left(\frac{\text{kv}_L}{\text{kv}_H}\right)^2 \times Z_{HT} \tag{8.11}$$

and from Eq. (8.9)

$$Z_{LT} \text{ in per unit} = \frac{(\text{kv}_L/\text{kv}_H)^2 \times Z_{HT} \times \text{kva}}{(\text{kv}_L)^2 \times 1{,}000} \tag{8.12}$$

$$= \frac{Z_{HT} \times \text{kva}}{(\text{kv}_H)^2 \times 1{,}000} \tag{8.13}$$

$$= Z_{HT} \text{ in per unit} \tag{8.14}$$

A great advantage in making per-unit computations is realized by the proper selection of different bases for circuits connected to each other through a transformer. To achieve the advantage in a single-phase system, the voltage bases for the circuits connected through the transformer must have the same ratio as the turns ratio of the transformer windings. With such a selection of voltage bases and the same kva base, the per-unit value of an impedance will be the same when it is expressed on the base selected for its own side of the transformer as when it is referred to the other side of the transformer and expressed on the base of that side.

Example 8.1

Three parts of a single-phase electric system are designated A, B, and C and are connected to each other through transformers, as shown in Fig. 8.5. The transformers are rated as follows:

A–B 10,000 kva, 13.8–138 kv, leakage reactance 10%
B–C 10,000 kva, 69–138 kv, leakage reactance 8%

If the base in circuit B is chosen as 10,000 kva, 138 kv, find the per-unit impedance of the 300-ohm resistive load in circuit C referred to circuits C, B, and A. Draw the impedance diagram neglecting magnetizing current, transformer resistances, and line impedances. Determine the voltage regulation if the voltage at the load is 66 kv with the assumption that the voltage input to circuit A remains constant.

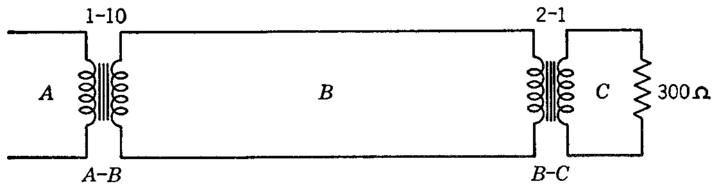


FIG. 8.5 Circuit for Example 8.1.

Solution

Base voltage for circuit A is $0.1 \times 138 = 13.8$ kv
Base voltage for circuit C is $0.5 \times 138 = 69$ kv

Base impedance of circuit C is $\dfrac{69^2 \times 1{,}000}{10{,}000} = 476$ ohms

Per-unit impedance of the load in circuit C is $300/476 = 0.63$ per unit

Because the selection of base in various parts of the system was determined by the turns ratio of the transformers, the per-unit impedance of the load referred to any part of the system will be the same. This is verified as follows:

Base impedance of circuit B is $\dfrac{138^2 \times 1{,}000}{10{,}000} = 1{,}900$ ohms

Impedance of the load referred to circuit B is $300 \times 2^2 = 1{,}200$ ohms

Per-unit impedance of the load referred to B is $\dfrac{1{,}200}{1{,}900} = 0.63$ per unit

Base impedance of circuit A is $\dfrac{13.8^2 \times 1{,}000}{10{,}000} = 19$ ohms

Impedance of the load referred to A is $300 \times 2^2 \times 0.1^2 = 12$ ohms

Per-unit impedance of the load referred to A is $12/19 = 0.63$ per unit

Figure 8.6 is the required impedance diagram with impedances marked in per unit.

The calculation of regulation proceeds as follows:

Voltage at the load is $^{66}\!/_{69} = 0.957 + j0$ per unit

Load current is $\dfrac{0.957 + j0}{0.63 + j0} = 1.52 + j0$ per unit

$$\text{Voltage input} = (1.52 + j0)(j0.10 + j0.08) + 0.957$$
$$= 0.957 + j0.274 = 0.995 \text{ per unit}$$

Voltage input = Voltage at the load with load removed

Therefore

$$\text{Regulation} = \frac{0.995 - 0.957}{0.957} \times 100 = 3.97\%$$

Because of the advantage previously pointed out, the principle followed in the above example in selecting the base for various parts of the system is always followed in making computations by per unit or per cent. The base should be the same in all parts of the system, and the selection of the base kv in one part of the system determines the base kv to be assigned, according to the turns ratios of the transformers, to the other parts of the system. Following this principle of assigning base kv allows us to combine on one impedance diagram the per-unit impedances determined in different parts of the system.

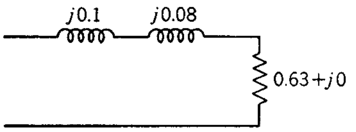


FIG. 8.6 Impedance diagram for Example 8.1. Impedances are marked in per unit.

If the above principle is applied to a three-phase circuit, the base voltages on the two sides of the transformer must have the same ratio as the rated line-to-line voltages on both sides of the transformer. Thus, the base voltages would have the same ratio as the rated line-to-neutral voltages on the two sides of the transformer and the same ratio as the turns ratio of the windings of a Y-Y transformer. For example, a 66-kv, 30,000-kva base in the line of Fig. 8.2 would require a base of 6.6 kv, for the circuit containing generators 1 and 2, and a base of 3.81 kv, 30,000 kva for the circuit containing generator 3. The per-un[illegible] rea[illegible] of generator 3 is, by Eq. (8.9),

$$\frac{0.1452 \times 30{,}000}{(3.81)^2 \times 1{,}000} = 0.30 \text{ per u[illegible]}$$

The reactance transferred to the high-tensio[illegible] is [illegible] and in per unit on the 66-kv base the reactance [illegible]

$$\frac{43.56 \times 30{,}000}{(66)^2 \times 1{,}000} = 0.30 \text{ per unit}$$

In a similar manner the reader may verify the statement that the per-unit reactances of generators 1 and 2 are 0.45 and 0.90, respectively, whether computed on the 6.6-kv base of their own circuit or referred to the high-tension side of the transformer and computed on the 66-kv base. Thus, just as in a single-phase system, the principle of selecting the base in different parts of the three-phase system allows us to combine on one impedance diagram the per-unit impedances computed in different parts of the system regardless of whether the transformers are connected Y-Y or Δ-Y. Of course, the principle is equally applicable if the transformers are connected Δ-Δ since the transformation of voltages is the same as that made by Y-Y transformers having the same line-to-line voltage ratings.

If the resistance and leakage reactance of a transformer in a three-phase circuit are specified in per unit, the per-unit value to be used in the impedance diagram is the same regardless of the three-phase connection (Y-Y, Δ-Δ, or Δ-Y). For instance, a three-phase transformer rated 10,000 kva, 138Y–13.8Δ kv may have a leakage reactance of 10%. For base in the high-tension circuit of 138 kv and 10,000 kva, the per-unit reactance is 0.1. Reactance measured on the high-tension side is $0.1\left(\frac{138^2 \times 1{,}000}{10{,}000}\right) = 190.4$ ohms. The turns ratio of the windings is $\frac{138}{13.8/\sqrt{3}} = 5.77$. Reactance measured across one low-tension winding is $190.4(1/5.77)^2 = 5.72$ ohms. The equivalent reactance to neutral—that is, the reactance of the equivalent Y—is 5.72/3. The base on this side of the transformer is 13.8 kv, and the per-unit reactance is then $\frac{5.72}{3} \times \frac{10{,}000}{13.8^2 \times 1{,}000} = 0.1$, the same as the specified value.

If the low-tension side of the transformer is connected in Y, the new rating is 10,000 kva, 138–23.9 kv. The base for the low-tension side is then 23.9 kv, and the per-unit reactance is $5.72\left(\frac{10{,}000}{23.9^2 \times 1{,}000}\right) = 0.1$, still the same as specified.

Example 8.2

A 30,000-kva 13.8-kv three-phase generator has a subtransient reactance of 15%. The generator supplies two motors over a transmission line having transformers at both ends, as shown on the one-line diagram of Fig. 8.7. The motors have rated inputs of 20,000 and 10,000 kva, both 12.5 kv with 20% subtransient reactance. The three-phase transformers are both rated 35,000 kva, 13.2Δ–115Y kv with leakage reactance of 10%. Series reactance of the transmission line is 80 ohms. Draw the reactance diagram with all reactances marked in per unit. Select the generator rating as base in the generator circuit.

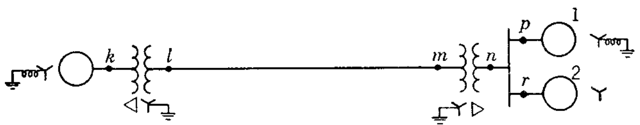


FIG. 8.7 One-line diagram for Example 8.2.

Solution

A base of 30,000 kva, 13.8 kv in the generator circuit requires a 30,000-kva base in all other circuits and the following voltage bases:

$$\text{In the transmission line: } 13.8 \times \frac{115}{13.2} = 120 \text{ kv}$$

$$\text{In the motor circuit: } \quad 120 \times \frac{13.2}{115} = 13.8 \text{ kv}$$

The reactances of the transformers must be converted from a base of 35,000 kva, 13.2 kv to a base of 30,000 kva, 13.8 kv, as follows:

$$\text{Transformer reactance} = 0.1 \times \frac{30{,}000}{35{,}000}\left(\frac{13.2}{13.8}\right)^2 = 0.0784 \text{ per unit}$$

The base impedance in the transmission line is

$$\frac{120^2 \times 1{,}000}{30{,}000} = 480 \text{ ohms}$$

and the reactance of the line is

$$80/480 = 0.167 \text{ per unit}$$

$$\text{Reactance of motor 1} = 0.2 \times \frac{30{,}000}{20{,}000}\left(\frac{12.5}{13.8}\right)^2 = 0.246 \text{ per unit}$$

$$\text{Reactance of motor 2} = 0.2 \times \frac{30{,}000}{10{,}000}\left(\frac{12.5}{13.8}\right)^2 = 0.492 \text{ per unit}$$

Figure 8.8 is the required reactance diagram.

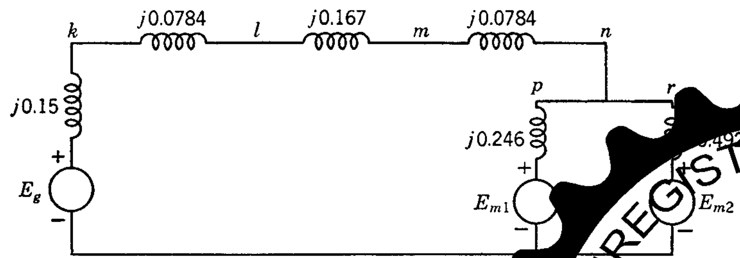


FIG. 8.8 Reactance diagram for Example 8.2. Reactances are marked in per unit on the specified base.

8.5 Per-unit Impedances of Three-winding Transformers. Both the primary and secondary windings of a two-winding transformer have the same kva rating, but all three windings of a three-winding transformer may have different kva ratings. The impedance of each winding of a three-winding transformer may be given in per cent or per unit based on the rating of its own winding, or tests may be made to determine the impedances. In any case, however, all the per-unit impedances in the impedance diagram must be expressed on the same kva base.

Three impedances may be measured by the standard short-circuit test, as follows:

Z_{ps} = leakage impedance measured in the primary with the secondary short-circuited and the tertiary open

Z_{pt} = leakage impedance measured in the primary with the tertiary short-circuited and the secondary open

Z_{st} = leakage impedance measured in the secondary with the tertiary short-circuited and the primary open

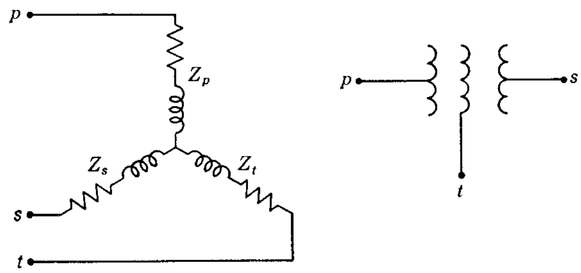
If the three impedances measured in ohms are referred to the voltage of one of the windings, transformer theory shows the impedances of each separate winding referred to that same winding to be related to the measured impedances so referred as follows:

$$\begin{aligned} Z_{ps} &= Z_p + Z_s \\ Z_{pt} &= Z_p + Z_t \\ Z_{st} &= Z_s + Z_t \end{aligned} \tag{8.15}$$

where Z_p, Z_s, and Z_t are the impedances of the primary, secondary, and tertiary windings referred to the primary circuit if Z_{ps}, Z_{pt}, and Z_{st} are the measured impedances referred to the primary circuit. Solving Eqs. (8.15) simultaneously yields

$$\begin{aligned} Z_p &= \tfrac{1}{2}(Z_{ps} + Z_{pt} - Z_{st}) \\ Z_s &= \tfrac{1}{2}(Z_{ps} + Z_{st} - Z_{pt}) \\ Z_t &= \tfrac{1}{2}(Z_{pt} + Z_{st} - Z_{ps}) \end{aligned} \tag{8.16}$$

The impedances of the three windings are connected in star to represent the single-phase equivalent circuit of the three-winding transformer [illegible] magnetizing current neglected, as shown in Fig. 8.9. The com[illegible] is fictitious and unrelated to the neutral of the system. [illegible] and t are connected to the parts of the impedance diagra[illegible] the parts of the system connected to the primary, sec[illegible] tertiary windings of the transformer. Since the ohmic value[illegible] impedances must be referred to the same voltage, it follows [illegible] unit impedance requires the same kva base for [illegible] three circuits and requires voltage bases in the three circuits that are in the same ratio as the rated line-to-line voltages of the three circuits of the transformer.



(*a*) Equivalent circuit (*b*) Symbol for one-line diagram

FIG. 8.9 The equivalent circuit of a three-winding transformer and the corresponding symbol to be used in a one-line diagram. Points *p*, *s*, and *t* link the circuit of the transformer to the appropriate equivalent circuits representing parts of the system connected to the primary, secondary, and tertiary windings.

Example 8.3

The three-phase ratings of a three-winding transformer are:

Primary: Y-connected, 66 kv, 10,000 kva
Secondary: Y-connected, 13.2 kv, 7,500 kva
Tertiary: Δ-connected, 2.3 kv, 5,000 kva

Neglecting resistance, the leakage impedances are:

$Z_{ps} = 7\%$ on 10,000-kva, 66-kv base
$Z_{pt} = 9\%$ on 10,000-kva, 66-kv base
$Z_{st} = 6\%$ on 7,500-kva, 13.2-kv base

Find the per-unit impedances of the star-connected equivalent circuit for a base of 10,000 kva, 66 kv in the primary circuit.

Solution

With a base of 10,000 kva, 66 kv in the primary circuit, the proper bases for the per-unit impedances of the equivalent circuit are 10,000 kva, 66 kv for primary-circuit quantities, 10,000 kva, 13.2 kv for secondary-circuit quantities, and 10,000 kva, 2.3 kv for tertiary-circuit quanti[illegible]s.

Z_{ps} and Z_{pt} were measured in the primary circuit and a[illegible] already expressed on the proper base for the equiv[illegible]rcuit. N[illegible] change of voltage base is required for Z_{st}. The re[illegible] change in base kva for Z_{st} is made as follows:

$$Z_{st} = 6\% \times \frac{10,000}{7,500} = 8\%$$

In per unit on the specified base

$$Z_p = \tfrac{1}{2}(j0.07 + j0.09 - j0.08) = j0.04 \text{ per unit}$$
$$Z_s = \tfrac{1}{2}(j0.07 + j0.08 - j0.09) = j0.03 \text{ per unit}$$
$$Z_t = \tfrac{1}{2}(j0.09 + j0.08 - j0.07) = j0.05 \text{ per unit}$$

Example 8.4

A constant-voltage source (infinite bus) supplies a purely resistive 5,000-kw, 2.3-kv load and a 7,500-kva, 13.2-kv synchronous motor having a subtransient reactance of $X'' = 20\%$. The source is connected to the primary of the three-winding transformer described in Example 8.3. The motor and resistive load are connected to the secondary and tertiary of the transformer. Draw the impedance diagram of the system and mark the per-unit impedances for a base of 66 kv, 10,000 kva in the primary.

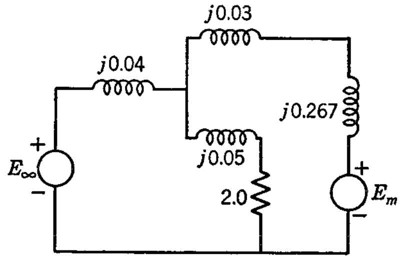


FIG. 8.10 Impedance diagram for Example 8.4.

Solution

The constant-voltage source can be represented by a generator having no internal impedance.

The resistance of the load is 1.0 per unit on a base of 5,000 kva, 2.3 kv in the tertiary. Expressed on a 10,000-kva, 2.3-kv base the load resistance is

$$R = 1.0 \times \frac{10{,}000}{5{,}000} = 2.0 \text{ per unit}$$

Changing the reactance of the motor to a base of 10,000 kva, 13.2 kv yields

$$X'' = 0.20 \times \frac{10{,}000}{7{,}500} = j0.267 \text{ per unit}$$

Figure 8.10 is the required impedance diagram.

8.6 The Advantages of Per-unit Computations. Making computations for electric systems in terms of per-unit values simplifies the w[illegible]k greatly. A real appreciation of the value of the per-unit met[illegible] through experience. Some of the advantages of the met[illegible]ized briefly below.

1. Manufacturers usually specify the impedance o[illegible] apparatus in per cent or per unit on the base of the name-plat[illegible].

2. The per-unit impedances of machines of th[illegible] different rating usually lie within a narrow rang[illegible] although the ohmic

values differ materially for machines of different ratings. For this reason, when the impedance is not known definitely, it is generally possible to select from tabulated average values a per-unit impedance which will be reasonably correct. Experience in working with per-unit values brings familiarity with the proper values of per-unit impedance for different types of apparatus.

3. When impedance in ohms is specified in an equivalent circuit, each impedance must be referred to the same circuit by multiplying it by the square of the ratio of the rated voltages of the two sides of the transformer connecting the reference circuit and the circuit containing the impedance. The per-unit impedance, once it is expressed on the proper base, is the same referred to either side of any transformer.

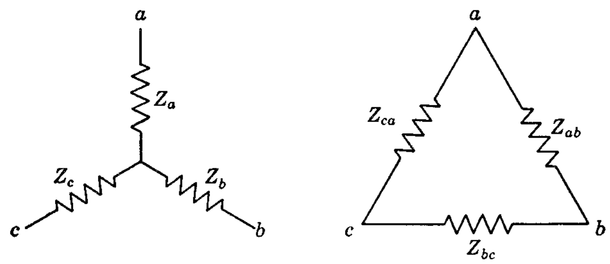


FIG. 8.11. Y-Δ equivalent circuits.

4. The way in which transformers are connected in three-phase circuits does not affect the per-unit impedances of the equivalent circuit, although the transformer connection does determine the relation between the voltage bases on the two sides of the transformer.

8.7 Network Reduction. The solution of problems involving even the simplest power system network often requires some network reduction to eliminate one or more nodes (junction points). If only three branches of the circuit terminate at a node, the node is eliminated by a Y-Δ transformation.

A Y and its equivalent Δ are shown in Fig. 8.11. The relations between the impedances are

$$Z_{ab} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_c} = Z_aZ_b \sum \frac{1}{Z_Y} \tag{8.17}$$

$$Z_{bc} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_a} = Z_bZ_c \sum \frac{1}{Z_Y} \tag{8.18}$$

$$Z_{ca} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_b} = Z_cZ_a \sum \frac{1}{Z_Y} \tag{8.19}$$

where the term $\sum \frac{1}{Z_Y}$ is the sum of the reciprocals of the three Y-connected impedances. The equations are convenient for finding the Δ-connected impedances equivalent to known Y-connected impedances.

If we desire to convert known Δ-connected impedances to an equivalent Y, convenient equations are

$$Z_a = \frac{Z_{ab}Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}} = \frac{Z_{ab}Z_{ca}}{\Sigma Z_\Delta} \tag{8.20}$$

$$Z_b = \frac{Z_{ab}Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}} = \frac{Z_{ab}Z_{bc}}{\Sigma Z_\Delta} \tag{8.21}$$

$$Z_c = \frac{Z_{bc}Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}} = \frac{Z_{bc}Z_{ca}}{\Sigma Z_\Delta} \tag{8.22}$$

where ΣZ_Δ is the sum of the three Δ-connected impedances.

If more than three impedances terminate on a node, the node may be eliminated by applying the general star-mesh conversion equations.

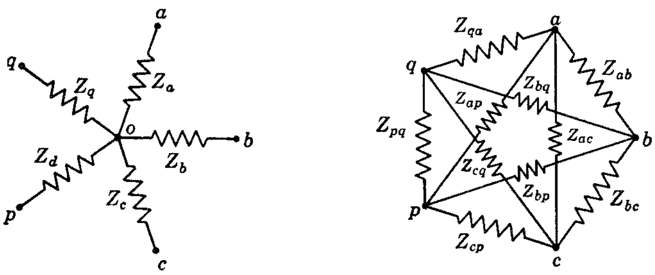


FIG. 8.12 Star-mesh equivalent circuits.

Figure 8.12 shows five star-connected impedances terminating on the node o, and the equivalent mesh-connected circuit. The equivalent mesh has an impedance connected between every possible pair of the original terminals. The impedance connected between any pair of terminals such as p and q in the mesh is given by the equation

$$Z_{pq} = Z_p Z_q \sum \frac{1}{Z_o} \tag{8.23}$$

where the term $\sum \frac{1}{Z_o}$ is the sum of the reciprocals of all the impedances connected to the node o in the original star circuit.[4]

When a number of generators are connected through a network [illegible]on- taining several nodes and the emf of each generator is known [illegible] of each can be found by eliminating all the nodes in th[illegible] except the nodes to which the emfs are connected. In the res[illegible]sh, each emf is connected directly to every other emf throu[illegible] impedance. The current flowing through each of the impedanc[illegible] difference in

[4] See Richard H. Frazier, "Elementary Electric Circu[illegible]," [illegible] Hill Book Company, Inc., New York, 1945.

potential between the two terminals of the impedance divided by the impedance.

Example 8.5

Four busses labeled a, b, c, and d are interconnected as shown by the one-line diagram of Fig. 8.13. Generators connected to busses a and b supply a synchronous motor load at bus d. For purposes of analysis all the machines at any one bus are treated as a single machine and represented by a single emf and series reactance. The reactance diagram,

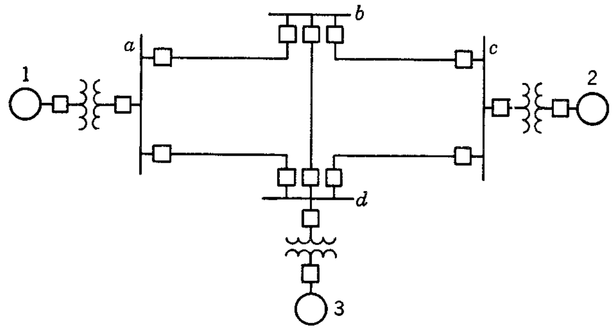


FIG. 8.13 One-line diagram for Example 8.5.

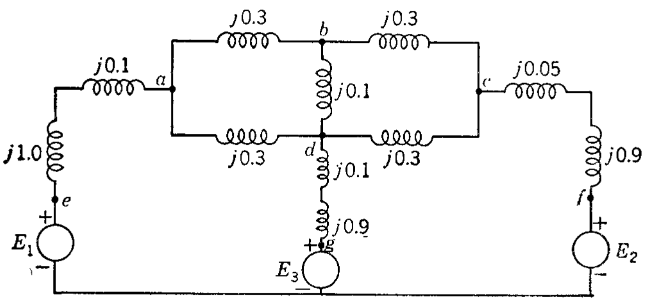


FIG. 8.14 Reactance diagram for Example 8.5.

with reactances specified in per unit, is shown in Fig. 8.14. Simplify the reactance diagram by eliminating the nodes at each bus and converting the resulting circuit to a mesh to whose terminals at e, f, and g are connected the emfs of the machines.

Solution

The successive steps in the reduction of the network are shown in Fig. 8.15. The node at b is eliminated by transforming to the equivalent delta the Y-connected reactances from a, c, and d terminating on b. Figure 8.15a is thus obtained from Fig. 8.14. The computations are

$$Z_{ac} = \frac{j0.3 \times j0.3 + j0.3 \times j0.1 + j0.3 \times j0.1}{j0.1} = \frac{-0.15}{j0.1} = j1.5 \text{ per unit}$$

$$Z_{cd} = \frac{-0.15}{j0.3} = j0.5 \text{ per unit}$$

$$Z_{da} = \frac{-0.15}{j0.3} = j0.5 \text{ per unit}$$

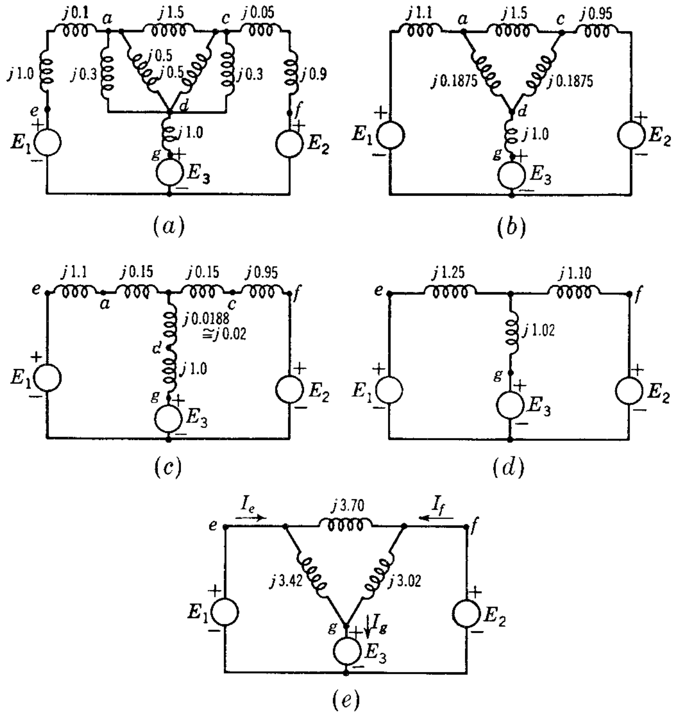


FIG. 8.15 Successive steps in the network reduction of the reactance diagram for Example 8.5.

Combining the impedances in series between e and a, between c and f, and between d and g gives

$$Z_{ea} = j1.0 + j0.1 = j1.1$$
$$Z_{cf} = j0.9 + j0.05 = j0.95$$
$$Z_{dg} = j0.9 + j0.1 = j1.0$$

and combining the parallel impedances in Fig. 8.15a between a and d and between c and d gives

$$Z_{ad} = Z_{cd} = \frac{j0.5 \times j0.3}{j0.5 + j0.3} = j0.1875$$

from which Fig. 8.15b is obtained.

Transformation of the Δ connecting points a, c, and d in Fig. 8.15b into its equivalent Y gives Fig. 8.15c. The computations are

$$Z_a = Z_c = \frac{j0.1875 \times j1.5}{j1.5 + j0.1875 + j0.1875} = j0.15$$

$$Z_d = \frac{j0.1875 \times j0.1875}{j1.5 + j0.1875 + j0.1875} = j0.0188$$

Figure 8.15d results from combining the series impedances of Fig. 8.15c. The required mesh, in this case a Δ, is obtained by a final Y-Δ transformation, and the result is shown in Fig. 8.15e. The computations are

$$Z_{ef} = \frac{j1.25 \times j1.10 + j1.10 \times j1.02 + j1.02 \times j1.25}{j1.02} = \frac{-3.77}{j1.02} = j3.70$$

$$Z_{fg} = \frac{-3.77}{j1.25} = j3.02$$

$$Z_{ge} = \frac{-3.77}{j1.10} = j3.42$$

Example 8.6

If the internal emfs in per unit at stations 1, 2, and 3 of Example 8.5 are $E_1 = 1.5\underline{/0^\circ}$, $E_2 = 1.5\underline{/15^\circ}$, and $E_3 = 1.5\underline{/-36.9^\circ}$, find the per-unit power outputs from stations 1 and 2, and find the per-unit power input to station 3.

Solution

The currents in the Δ of Fig. 8.15e are

$$I_{ef} = \frac{E_1 - E_2}{Z_{ef}} = \frac{1.5\underline{/0^\circ} - 1.5\underline{/15^\circ}}{j3.70} = -0.105 - j0.014 \text{ per unit}$$

$$I_{fg} = \frac{E_2 - E_3}{Z_{fg}} = \frac{1.5\underline{/15^\circ} - 1.5\underline{/-36.9^\circ}}{j3.02} = +0.426 - j0.083 \text{ per unit}$$

$$I_{ge} = \frac{E_3 - E_1}{Z_{ge}} = \frac{1.5\underline{/-36.9^\circ} - 1.5\underline{/0^\circ}}{j3.42} = -0.263 + j0.088 \text{ per unit}$$

The currents at the terminals of the delta are
From generator station 1:

$$I_e = I_{ef} - I_{ge} = 0.158 - j0.102 = 0.188\underline{/-32.8^\circ} \text{ per unit}$$

From generator station 2:

$$I_f = I_{fg} - I_{ef} = 0.531 - j0.069 = 0.536\underline{/-7.4^\circ} \text{ per unit}$$

Into motor station 3:

$$I_g = I_{fg} - I_{ge} = 0.689 - j0.171 = 0.710\underline{/-13.9^\circ} \text{ per unit}$$

The required power values are

Output from station 1:

$$1.5 \times 0.188 \cos 32.8° = 0.237 \text{ per unit}$$

Output from station 2:

$$1.5 \times 0.536 \cos (15° + 7.4°) = 0.743 \text{ per unit}$$

Input to load:

$$1.5 \times 0.710 \cos (36.9° - 13.9°) = 0.985 \text{ per unit}$$

The total three-phase power at each station is the product of the above per-unit values and the base three-phase kva.

The amount of work involved in reducing a complex network found in a typical power system to a minimum number of impedances is barely

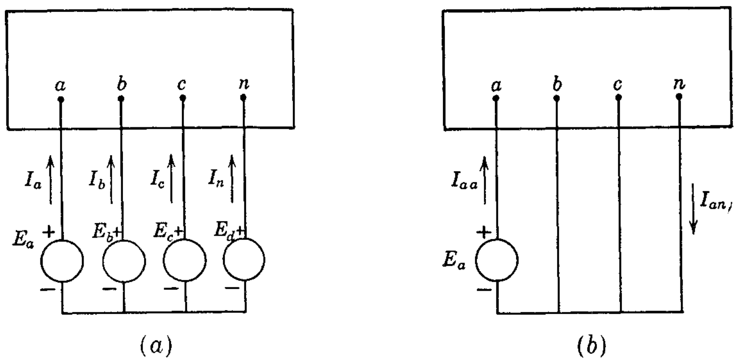


FIG. 8.16 Linear network with n terminals at which emfs may be applied.

indicated by the simple examples above. Where more nodes are present and where the general star-mesh equations are required because of the number of branches terminating on one node, the work is increased considerably. If the impedances are not pure reactances, the complication of using complex numbers adds enormously to the work. Even so, network reduction is much preferred to the solution of simultaneous equations, which would otherwise be required in any analytical solution. The time required for an analytical solution led to the development of calculating boards.

8.8 Driving-point and Transfer Admittances. The method of driving-point and transfer admittances is a convenient means of solving networks having several points where power enters and leaves the network. Figure 8.16a shows several emfs connected to a network indicated by the rectangular box. The internal impedances of the machines whose emfs are shown are incorporated in the network. The driving-point admittance at any terminal is the ratio of the current entering the terminal to the voltage applied at the terminal with all other emfs short-circuited. The transfer admittance between two points is the ratio of the current leaving the network at one terminal to the voltage applied at the other terminal with all other emfs short-circuited.

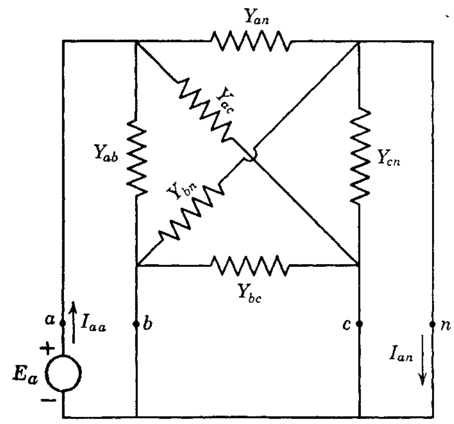


FIG. 8.17 Circuit resulting from the replacement of the n-terminal network of Fig. 8.16 by its equivalent mesh.

Figure 8.16b shows the method of determining the driving-point admittance at point a and the transfer admittance between points a and n. If connections are made as shown, the driving-point admittance at a is I_{aa}/E_a, and the transfer admittance between a and n is I_{an}/E_a. If the nodes are eliminated from the network so that the terminals are interconnected by a mesh, the resulting circuit equivalent to that of Fig. 8.16b is shown in Fig. 8.17. We see that the current entering the network at a is

$$I_{aa} = E_a(Y_{ab} + Y_{ac} + \cdots + Y_{an}) \tag{8.24}$$

from which we conclude that the driving-point admittance at any point is the sum of the admittances terminating at that point when the circuit has been reduced to the simplest mesh. The current leaving the circuit at n is

$$I_{an} = E_a Y_{an} \tag{8.25}$$

from which we conclude that the transfer admittance between two points is the admittance between the two points after the circuit has been reduced to the simplest mesh.

If the driving-point admittances are determined at all terminals, and transfer admittances are determined between all points, the currents entering the network at the terminals can be found by applying the familiar superposition principle. The superposition principle states that the current resulting from several voltage sources in a lin[ear network] is equal to the sum of the separate currents resulting f[rom each voltage] alone with all other voltages short-circuited. Adding the currents entering a point caused by each emf alone gives

$$I_a = E_aY_{aa} - E_bY_{ab} - E_cY_{ac} - \cdots - E_nY_{an} \tag{8.26}$$
$$I_b = E_bY_{bb} - E_aY_{ab} - E_cY_{bc} - \cdots - E_nY_{bn} \tag{8.27}$$

and similar equations where

$Y_{aa}, Y_{bb}, Y_{cc}, \ldots, Y_{nn}$ = driving-point admittances at points a, b, c, $\ldots$, n, respectively

$Y_{ab}, Y_{bc}, \ldots, Y_{an}, Y_{bn}$ = transfer admittances between points a and b, between points b and c, $\ldots$, between points a and n, and between points b and n, respectively

Example 8.7

Determine the current entering each terminal of the network of Example 8.5 for the emfs specified in Example 8.6 by the method of driving-point and transfer admittances. Compare the results with the values found for the same currents in Example 8.6.

Solution

The network reduction carried out in Example 8.5 enables us to calculate the required driving-point and transfer admittances, as follows:

$$Y_{ee} = \frac{1}{Z_{ef}} + \frac{1}{Z_{eg}} = \frac{1}{j3.70} + \frac{1}{j3.42} = -j0.270 - j0.292 = -j0.562 \text{ per unit}$$

$$Y_{ff} = \frac{1}{Z_{ef}} + \frac{1}{Z_{fg}} = \frac{1}{j3.70} + \frac{1}{j3.02} = -j0.270 - j0.331 = -j0.601 \text{ per unit}$$

$$Y_{gg} = \frac{1}{Z_{fg}} + \frac{1}{Z_{ge}} = \frac{1}{j3.02} + \frac{1}{j3.42} = -j0.331 - j0.292 = -j0.623 \text{ per unit}$$

$$Y_{ef} = \frac{1}{Z_{ef}} = \frac{1}{j3.70} = -j0.270 \text{ per unit}$$

$$Y_{fg} = \frac{1}{Z_{fg}} = \frac{1}{j3.02} = -j0.331 \text{ per unit}$$

$$Y_{ge} = \frac{1}{Z_{ge}} = \frac{1}{j3.42} = -j0.292 \text{ per unit}$$

The currents entering at terminals e, f, and g are

$I_e = E_1Y_{ee} - E_2Y_{ef} - E_3Y_{ge} = (1.5 + j0)(-j0.562)$
$- (1.45 + j0.388)(-j0.270) - (1.2 - j0.9)(-j0.292) = -j0.8$[illegible]
$+ j0.392 - 0.105 + j0.350 + 0.263 = 0.158 - j0.$[illegible]

$I_f = E_2Y_{ff} - E_1Y_{ef} - E_3Y_{fg} = (1.45 + j0.388)(-j0.$[illegible]
$- (1.5 + j0)(-j0.270) - (1.2 - j0.9)(-j0.331) =$ [illegible] $.871 + 0.233$
$+ j0.405 + j0.397 + 0.298 = 0.$[illegible] $j0.069$ per unit

$I_g = E_3Y_{gg} - E_1Y_{ge} - E_2Y_{fg} = (1.2 - j0.9)(-j0.$[illegible]
$- (1.5 + j0)(-j0.292) - (1.45 + j0.388)($[illegible]$.331) =$ [illegible]
$- 0.560 + j0.438 + j0.480 - 0.129 = -$[illegible]$.689 + j0.171$ per unit

The values found for the currents check exactly the values found in Example 8.6 when we recall that I_g was assumed to be flowing out of the network in the solution of Example 8.6 and into the network in the solution of Example 8.7.

8.9 D-C Calculating Boards. Calculating boards provide a method for representing transmission systems by the interconnection of the equivalent circuits of their component parts. All voltages, currents, and impedances are converted to values which are proportional to the actual values. On a board operated from a d-c supply, many variable resistance units are available to be connected to each other by flexible cords and jacks such as are found on a manually operated telephone switchboard.[5] Only the inductive reactances of the impedance diagram being studied can be represented, and each reactance is represented by a resistor on the calculating board. Generators and motors are replaced by one d-c voltage source connected to the network through resistors which represent the internal reactance of each machine. In an a-c system the amount of current flowing depends on the magnitude and phase angle of the internal voltages of the motors and generators. On a d-c board there is no adjustment of the individual motor and generator voltages and nothing to correspond to phase differences of the voltages. Therefore, the d-c board can not simulate normal load conditions.

Power companies use d-c boards extensively to study the flow of fault current caused by short circuits at various points in their systems. Some companies build d-c boards with fixed resistors permanently connected to represent their own systems and interconnections with neighboring systems. Other companies have boards composed of adjustable resistors so connected that each resistor unit terminates in two cords which connect with the cords of other units at a plugging board. The d-c board built in 1948 by Westinghouse to supplement one already operating at the Philadelphia Electric Company has resistor units adjustable from 0.2% to 110% in steps of 0.2% and from 110% to 410% in steps of 1%. A resistance of 40 ohms is 100% impedance. The voltage supply is a selenium rectifier furnishing 40 volts for 100% voltage. The voltage can be increased to 60 volts. Voltage and current at any resistor unit are read on instruments at the control panel by pressing a button corresponding to the proper unit.

Portable boards have been built in carrying cases which [illegible] of a large suitcase. They are operated by batteries [illegible]ed in the case together with an ammeter and voltmeter. A plugging arrangement is provided for interconnection of the resistors a[illegible] the insertion of meters. Although they are inexpensive, compa[illegible] and easily trans-

[5] For a description of a d-c calculating board, see W[illegible] Lewis, "A New Short-circuit Calculating Table," *Gen. Elec. Rev.*, vol. 23, no. 8, p. 669, August, 1920.

ported, the portable boards are not as accurate or as convenient to operate as the permanent installations with calibrated resistors.

To study short-circuit currents on a d-c calculating board, a reactance diagram is prepared from a one-line diagram, such as that of the local generating system of an industrial plant shown in Fig. 8.18. Of course, information must be available from which to obtain the reactances of the lines, the transformers, and the generators and motors. If the local system is connected to an external power system, as shown in Fig. 8.18, the external power system must be represented on the d-c board by a connection to the d-c supply through a resistance determined by the

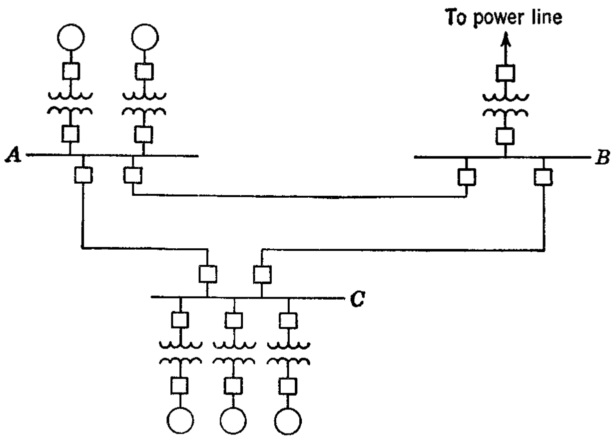


FIG. 8.18 One-line diagram of a typical industrial power system.

amount of current the power system would supply to a short circuit at the point of connection.

That the external power system is correctly represented by voltage and series impedance follows from the Helmholtz-Thévenin theorem.[6] The theorem states that a linear network terminating on two points a and b and containing any number of emfs may be replaced by a single emf and a series impedance between a and b. The emf is equal to the open-circuit voltage measured between a and b. The series impedance is the impedance of the network measured between a and b wit[illegible] the emfs short-circuited. If the emfs are constant, the imped[illegible] open-circuit voltage between a and b divided by the cu[illegible]wing in a short circuit applied between a and b. Power compa[illegible] furnish data giving the expected short-circuit current at po[illegible] throughout their systems. On a d-c board the impedances of the s[illegible]tem are considered

[6] See W. R. LePage, "Analysis of Alternating-curre[illegible] Circuits," pp. 22[illegible]–230, McGraw-Hill Book Company, Inc., New York, 1952.

to be inductive reactances or impedances having equal phase angles and, of course, must be represented by resistances.

If the external power system is large compared to that of the industrial plant, disturbances within the plant do not affect the voltage at the point of connection. In such a case the external power system is said to be an infinite bus and is represented by a constant voltage having no internal impedance. On the assumption that the external power system is an infinite bus, the reactance diagram of the industrial power system of Fig. 8.18 is shown in Fig. 8.19 with reactances of the components of the system assumed to have the values given there in per unit on whatever base is selected.

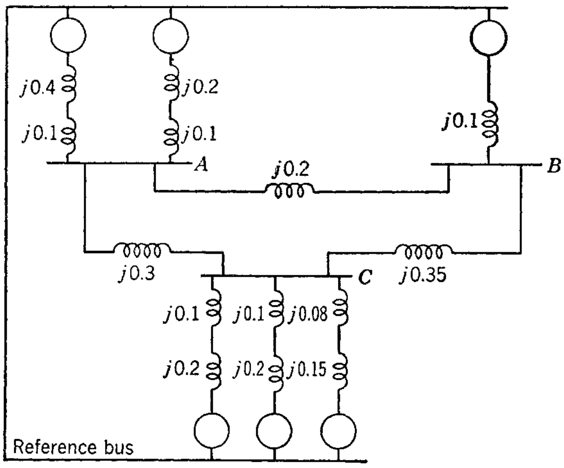


FIG. 8.19 Reactance diagram of the industrial power system of Fig. 8.18. Reactances are marked in per unit.

Figure 8.20 shows how the system is set up on a d-c calculating board. If 10 volts is selected to represent one per-unit voltage on the board and 1,000 ohms is one per-unit impedance, 10 ma is the base current. All the internal emfs of the machines of the plant system and the emf representing the external power system are assumed to have the same per-unit value, and this per-unit voltage on the base used for the calculating board is connected between the positive and the negative busses of the [illegible]rd. The interconnected resistances in the circuit of the calculating [illegible] one per-unit resistance equal to 1,000 ohms, are mark[illegible] diagram to correspond to the per-unit reactances of the react[illegible] diagram. A three-phase short circuit occurs in a system if thre[illegible]pedances are connected in Y to the three lines and if the impeda[illegible]s are then reduced to zero. Therefore, a three-phase short circ[illegible] repre[illegible] in the single-phase equivalent circuit by a short circ[illegible] from a point in the circuit to the common return. On a d-c board this is accomplished by connecting the negative bus to the junction between resistors corresponding to the point at which the short circuit occurs. In Fig. 8.20 a three-phase short circuit on bus C is simulated by closing switch S, which corresponds to connecting point C to the common return in Fig. 8.19. Until such a connection is made to the negative bus of the d-c board, no current flows in the circuit. Insertion of an ammeter in any branch of the circuit indicates the current in that branch due to the fault. The current may be recorded in per unit as read from the ammeter and converted later to amperes by multiplying by the base current, or the

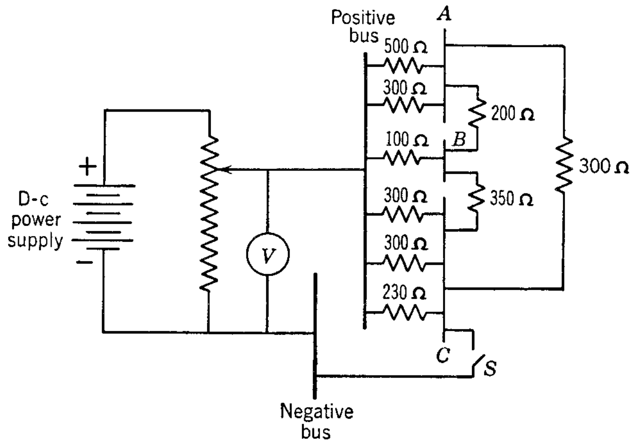


FIG. 8.20 Circuit of a d-c calculating board to represent the system of Figs. 8.18 and 8.19.

amperes flowing in the branches of the d-c board may be read and converted by a multiplying factor to the amperes which would flow in the actual system.

Although the current computed from the readings obtained from a d-c calculating board is due to the fault only and does not include the component of current due to loads on the system, the error due to the omission of load currents is not great. The total current in any part of the system during a fault is, of course, the sum of the components due to the loads and due to the fault. The load component is small, however, compared to the fault current, and the load current and fault current usually have a large difference in phase. It can easily be seen that the resultant of a small phasor and a large phasor having a large phase difference is very nearly equal to the magnitude of the large phasor. Therefore, the total current in a branch during a fault is very nearly equal in magnitude to the component due to the fault current alone.

An a-c calculating board is much more accurate and versatile than a d-c board because magnitude and phase adjustments can be made for every individual emf in the system and because all the circuit parameters can be set up on the board.

8.10 A-C Calculating Boards.[7] The first modern type of a-c calculating board was built jointly by the General Electric Company and the Massachusetts Institute of Technology in 1929. This board operates at 60 cycles. Since its installation, many boards have been built.[8] Most of the a-c boards operate at either 440 or 480 cycles, although some have been built for other frequencies, two of them for operation at 10,000 cycles. Higher frequencies allow the use of components of smaller size.

All modern a-c calculating boards operate in a similar manner and are composed of similar types of units.[9] The biggest variations between boards result from different designs of individual units and different metering arrangements. The board installed at Schenectady by the General Electric Company in 1949 is shown in Fig. 8.21 and will be described. The network units composing the various circuit elements of this board have adjusting dials calibrated in per unit. Each unit terminates in a pair of cords with plugs. Two units are connected in series by plugging a cord from each into horizontally adjacent receptacles on a large panel board.

The generator units have independent phase-angle and magnitude adjustments. They can be connected at any point in the system. A voltmeter, wattmeter, and varmeter are built into each generator unit. The series impedances of lines and associated transformers are represented by standard line units composed of resistance and inductive reactance. Capacitors are provided to represent shunt capacitance, and a nominal-π circuit is made by connecting one capacitor at each end of a standard line unit. Units to represent loads have their own

[7] Alternating-current calculating boards are called *a-c network analyzers* by the General Electric Company and *a-c network calculators* by the Westinghouse Electric Corporation.

[8] For a list of 40 a-c calculating boards, 30 of them in the United States, see S. B. Crary, I. W. Gross, and C. F. Wagner, "Progress and Future Trends in Electric Transmission," *Trans. AIEE*, vol. 71, Part III, p. 968, 1952.

[9] See for instance H. P. Kuehni and R. G. Lorraine, "A New A-C Network Analyzer," *Trans. AIEE*, vol. 57, pp. 67–73, 1938; and W. A. Morgan, F. S. Rothe, and J. J. Winsness, "An Improved A-C Network Analyzer," *Trans. AIEE*, vol. 6[illegible] pp. 891–895, 1949. For a list of the frequencies and component parts of a[illegible] boards see E. T. B. Gross, "Symposium on Network Anal[illegible] Remarks," *Proc. Am. Power Conference*, vol. 14, pp. 381–383, [illegible] Institute of Technology, Chicago, 1952. See also P. O. Bobo, "Handlin[illegible] System Problems on an A-C Network Calculator," *Elec. Eng.*, vol. 69, pp. [illegible] October, 1950; E. W. Kimbark, J. H. Starr, and J. E. Van Ness, "A Co[illegible]act, Inexpensive A-C Network Analyzer," *Trans. AIEE*, vol. 71, Part I, pp. 1[illegible] an[illegible] "A New Principle Is Employed for 60-Cycle A-C Networ[illegible] Analyzers," *Trans. AIEE*, vol. 71, Part I, pp. 18–22, 1952.

individual voltmeters and are connected to the system through autotransformers in order to keep a constant voltage on the unit regardless of the system voltage. This arrangement of the load units is helpful in adjusting the loads to precalculated values. Additional units represent synchronous condensers, autotransformers, and mutual reactance.

Push-button switches at the master-instrument and control panel connect any unit in the circuit to the master-instrument busses. Master instruments have light-beam pointers with a short time response. Readings are made on 8-in. scales at eye level in front of the operator. An

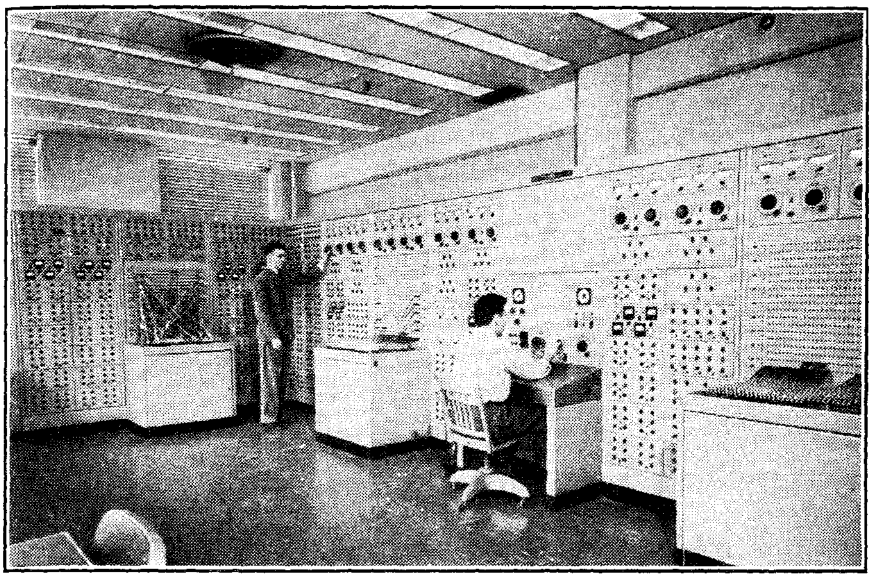


FIG. 8.21 View of the front and one side of the General Electric Network Analyzer No. 2, installed at Schenectady, N.Y. (*General Electric Company.*)

ammeter, voltmeter, and wattmeter-varmeter read magnitude and phase angle as well as real and quadrature components of current, voltage, and power. The ammeter reads directly in per unit on any of six current scales, and the wattmeter-varmeter reads either in per unit or in megawatts and megavars on a 20, 50, or 100-megavolt-amp base. The real and reactive power readings are of the proper sign looking away from the bus being metered.

An impedance diagram with impedances specified in per cent or per unit is followed in setting up the board. Operating conditions are obtained by adjustment of the magnitude and phase angle of the internal voltages of the generators. Nominal voltage of the board is 50 volts, and nominal current is 50 ma. Table 8.1 gives the number of elements of each type and their use, range, and size of step.

TABLE 8.1 ELEMENTS OF GENERAL ELECTRIC A-C NETWORK ANALYZER NO. 2
Installed at Schenectady in 1949

Number	Elements	Range*	Steps*	Rating	
				Per-unit volts	Per-unit amperes
12	Generators	V 0–2.5 ϕ $0 \pm 180°$	Continuous Continuous	2.5	10.0
12	Synchronous impedances (series $R + jX$)	R 0–0.11 X 0–1.11	0.001 0.001	1.25	5.0
146†	Line impedances (series $R + jX$)	R 0–0.51 X 0–0.81	0.001 0.001	1.25	5.0
56‡	Load units (series or parallel)	R 0–16.1 X 0–16.05	Continuous 0.05	1.25	5.0
100	Capacitors (susceptance)	0–1.1	0.01	1.25	
4	Large capacitors (susceptance)	0–50.0	1.0	1.25	
16	Autotransformers	V 0–$\pm$30.5%	0.5%	1.25	5.0
15	Mutual transformers	1:1 ratio		1.25	5.0

* In per unit unless otherwise noted.
† Of which 50 can be made into π lines by connecting a capacitor at each end.
‡ Including continuously variable autotransformers for load adjustment.

The a-c calculating boards built by the Westinghouse Electric Corporation operate at 440 cycles, and impedances are marked in ohms. Modern Westinghouse boards contain load units connected as π circuits to represent the nominal-π circuits of transmission lines. Only one setting need be made to adjust the capacitors at both ends of the π line. The board built for Commonwealth Edison Company in 1951 has, in addition to other line units and capacitors, 24 π lines, two of which are shown in Fig. 8.22.

Many modern boards are equipped with a recording [illegible] having a plastic recording surface consisting of a large number of translucent spots. The one-line diagram is placed on the recording surface. A small lamp bulb corresponding to each metered unit is placed under a spot at the point on the diagram represented by the unit. When readings are taken for a particular unit on the board, a spot of light shows through the paper at the point on the diagram at which data for that unit is to be recorded. So many spots are available for lamps that the lamp connected to any unit may be put in almost any position under the diagram.[10]

A calculating board built by the Detroit Edison Company for its own use has a fixed, permanently connected section representing the bulk

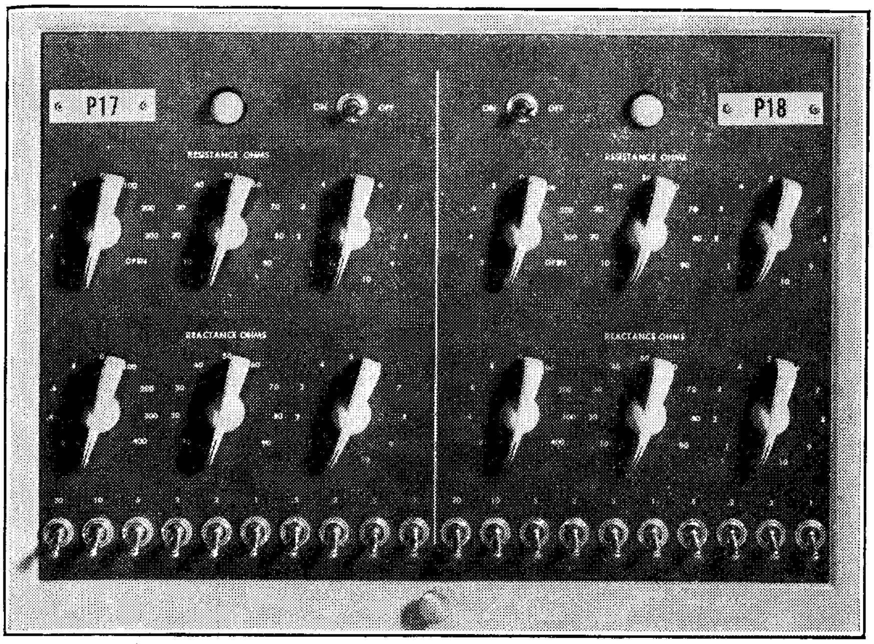


FIG. 8.22 View of two π-line units of a Westinghouse Network Calculator. Resistance and reactance values marked in ohms are equal to per-cent impedance, since 100 ohms is the base impedance. Toggle switches marked in per cent susceptance insert the indicated values in each shunt arm simultaneously. (*Westinghouse Electric Corporation.*)

power system of the company. Completely variable units are also available on this board.[11]

The calculating board is a great time saver compared with algebraic methods of solution of power networks. It is particularly advantageous in the study of the effects of changes in a system. Once a board i[illegible]set up it takes but a few moments to determine the circuit loadi[illegible]an[illegible] voltages which occur upon the temporary loss of part of t[illegible]sy[illegible]under conditions of contemplated future expansion, with the ad[illegible]of correc-

[10] Aids in the operation of calculating boards are describ[illegible] L. R. LeVesconte, "Auxiliary Equipment Facilitates Network Calculator Oper[illegible]," *Elec. Light and Power*, vol. 24, pp. 50–54, October, 1946.

[11] See E. A. Baldini and A. P. Fugill, "A Power Syst[illegible] Analogu[illegible] Computer," *Trans. AIEE*, vol. 71, Part III, pp. 291–297, [illegible]52.

tive capacitors, and for many other changes in the system. The a-c calculating board finds its most frequent use in making load studies. A load study made on an a-c calculating board is described in Chap. 1, where the data obtained on the board is shown in Figs. 1.2 and 1.3.

Short-circuit studies, which are possible on a d-c calculating board, can be performed more accurately on the a-c board. Short-circuit duty of circuit breakers, bus voltages under fault conditions, and the maximum and minimum currents for relay settings are some of the answers supplied by a board study.

As power systems grow larger and larger and as the number of interconnections increase, the subject of system stability becomes more important. The a-c board is useful in determining the steady-state and transient stability limits of a power system, the critical operating time for relays, and methods of improving stability.

PROBLEMS

8.1 Two generators are connected in parallel to the same bus and have subtransient reactances of $X'' = 10\%$. Generator 1 is rated 2,500 kva, 2.4 kv, and generator 2 is rated 5,000 kva, 2.4 kv. Find the per-unit reactance of each generator on a 15,000-kva, 2.4-kv base. What is the per-unit reactance of a single generator equivalent to the two generators in parallel on a 15,000-kva, 2.4-kv base?

8.2 Three motors rated 6.9 kv are connected to the same bus. The motors are:

No. 1: 5,000-hp, 0.8-p.f. synchronous motor, $X'' = 17\%$
No. 2: 3,000-hp, 1.0-p.f. synchronous motor, $X'' = 15\%$
No. 3: 3,500-hp, induction motor, $X'' = 20\%$

Express the subtransient reactances of these motors in per unit on a base of 10,000 kva, 6.6 kv.

8.3 A transformer bank is composed of three single-phase transformers supplying a three-phase load consisting of three identical 10-ohm resistors. Each single-phase transformer is rated 10,000 kva, 38.1–3.81 kv with a leakage reactance of 10%. Resistance may be neglected. The load is connected to the low-voltage side of the bank. The first symbol in the designation of the transformer connection in column 1 of the table included as part of the problem indicates the connection of the high-tension side of the transformer bank. Fill in the blanks in the table for a base of 30,000 kva. The impedance which would be marked on an impedance diagram is either the ohmic or the per-unit value of the impedance of one phase of the Y-connected equivalent circuit.

Column 7 refers to the impedance in ohms of the transformer plus the load viewed from the high-tension side of the transformer.

Column 8 refers to the per-unit impedance of the load computed on the base for the load circuit.

Column 9 refers to the impedance of the transformer and load viewed from the high-tension side of the transformer expressed in per unit on the base for the high-tension circuit.

8.4 A 15,000-kva, 8.5-kv three-phase generator has a subtransient reactance of 20%. It is connected through a Δ-Y transformer to a high-voltage transmission line

TABLE FOR PROB. 8.3

1	2	3	4	5	6	7	8	9
Trans-former connec-tion	Load conn.	Line-to-line base, kv		Base Z, ohms		H. T.-side total Z, ohms	Z of load, per unit	Z viewed from H. T. circuit, per unit
		L. T.	H. T.	L. T.	H. T.			
Y-Y	Y	6.6						
Y-Y	Δ	6.6						
Y-Δ	Y	3.81						
Y-Δ	Δ	3.81						
Δ-Y	Y	6.6						
Δ-Y	Δ	6.6						

having a total series reactance of 70 ohms. At the load end of the line is a Y-Y step-down transformer. Both transformer banks are composed of single-phase transformers connected for three-phase operation. Each of the three transformers composing each bank is rated 6,667 kva, 10–100 kv with a reactance of 10%. The load, represented as impedance, is drawing 10,000 kva at 12.5 kv and 80% power factor lagging. Draw the positive-sequence impedance diagram showing all impedances in per unit. Choose a base of 10,000 kva, 12.5 kv in the load circuit. Determine the voltage at the terminals of the generator.

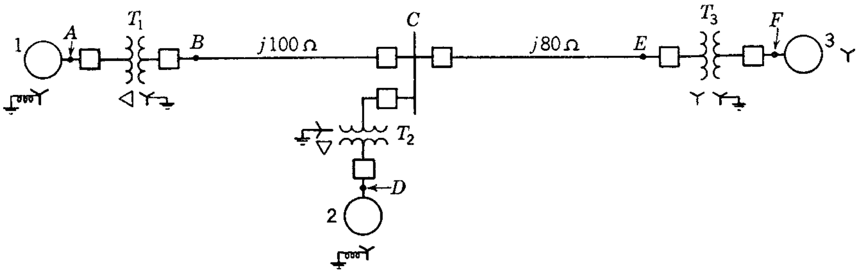


FIG. 8.23 One-line diagram for Prob. 8.5.

8.5 The one-line diagram of an unloaded power system is shown [illegible] Reactances of the two sections of transmission line are shown on the d[illegible] The generators and transformers are rated as follows:

Generator 1: 20,000 kva, 6.9 kv, $X'' = 0.15$ per unit
Generator 2: 10,000 kva, 6.9 kv, $X'' = 0.15$ per unit
Generator 3: 30,000 kva, 13.8 kv, $X'' = 0.15$ per u[illegible]
Transformer T_1: 25,000 kva, 6.9Δ–115Y kv, $X = 1$[illegible]
Transformer T_2: 12,500 kva, 6.9Δ–115Y kv, $X = 10$[illegible]

Transformer T_3: single-phase units each rated 10,000 kva, 7.5–75 kv, $X = 10\%$
Draw the impedance diagram with all reactances marked in per unit and with letters to indicate points corresponding to the one-line diagram. Choose a base of 30,000 kva, 6.9 kv in the circuit of generator 1.

8.6 Determine the driving-point and transfer admittances in per unit for the impedance diagram of Prob. 8.5 at the terminals where each of the emfs of the three machines are connected to the impedance network.

8.7 Determine by the method of driving-point and transfer admittances the power input or output of each machine in the network of Prob. 8.5 if the emfs on the base specified for that problem are $E_1 = 1.2\underline{/15^\circ}$, $E_2 = 1.2\underline{/20^\circ}$, and $E_3 = 1.2\underline{/-20^\circ}$.

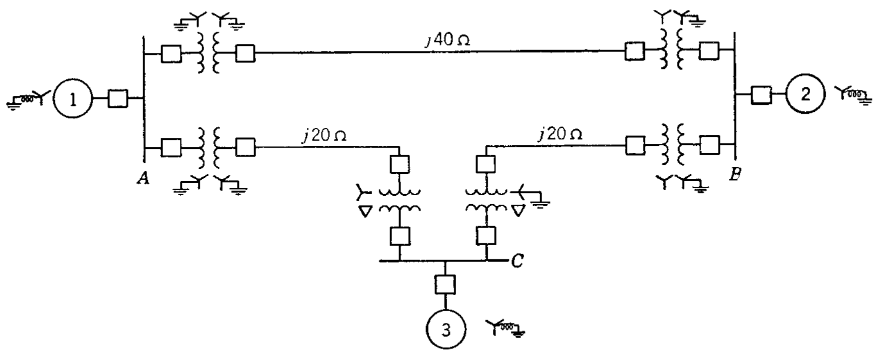


Fig. 8.24 One-line diagram for Prob. 8.8.

8.8 Draw the impedance diagram for the power system shown in Fig. 8.24. Mark impedances in per unit. Neglect resistance, and use a base of 50,000 kva, 138 kv in the 40-ohm line. The ratings of the generators, motors, and transformers are:

Generator 1: 20,000 kva, 13.2 kv, $X'' = 15\%$
Generator 2: 20,000 kva, 13.2 kv, $X'' = 15\%$
Synchronous motor 3: 30,000 kva, 6.9 kv, $X'' = 20\%$
Three-phase Y-Y transformers: 20,000 kva, 13.8Y–138Y kv, $X = 10\%$
Three-phase Y-Δ transformers: 15,000 kva, 6.9Δ–138Y kv, $X = 10\%$

All transformers are connected to step up the voltages of the generators to the transmission-line voltages.

8.9 If the voltage of bus C in Prob. 8.8 is 6.6 kv when the motor draws 24,000 kw at 0.8 power factor leading, calculate the voltages of busses A and B. Assume that the two generators divide the load equally. Give the answer in volts and in per unit on the base selected for Prob. 8.8. Find the voltages at A and B when the circuit breaker connecting generator 1 to bus A is open while the motor draws 12,000 kw at 66 kv with 0.8 power factor leading. All other circuit breakers remain closed.

8.10 Calculate the voltage regulation at bus C of Fig. 8.24 for the two conditions of Prob. 8.9. Assume that the voltage is held constant at busses A and B when the 24,000-kw load is removed while the two generators are connected and that the voltage is constant at bus B when the 12,000-kw load is removed while only generator 2 is connected. The voltages at the busses are those calculated for each case in Prob. 8.9.

8.11 The windings of a three-winding transformer are rated as follows:

Primary: Y-connected, 6.6 kv, 15,000 kva
Secondary: Y-connected, 33 kv, 10,000 kva
Tertiary: Δ-connected, 2.2kv, 7,500 kva

With resistance neglected the following leakage impedances are calculated from short-circuit tests:

Measured from the primary side: $Z_{ps} = j2.47$ ohms
$Z_{pt} = j2.90$ ohms
Measured from the secondary side: $Z_{st} = j8.70$ ohms

Find the impedances of the star-connected equivalent circuit on a base of 15,000 kva, 6.6 kv in the primary circuit.

8.12 A d-c calculating board is connected to study a three-phase short circuit on one of the busses of a system having a base of 5,000 kva, 2,300 volts. On the calculating board 100% voltage is 18 volts and 100% impedance is 3,000 ohms. If a milliammeter inserted in series with one of the resistance units of the calculating board reads 10 ma, find the expected fault current in the corresponding branch of the system.

8.13 Draw the diagram and mark on it the values of all the resistances for connecting a d-c calculating board to study the system of Prob. 8.8. The board has a 100% voltage of 50 volts and a 100% impedance of 10,000 ohms. What voltage is applied between the positive and negative busses if faults are to be studied when the transmission line is operating at a voltage of 132 kv?

CHAPTER 9

SYMMETRICAL THREE-PHASE FAULTS ON SYNCHRONOUS MACHINES

9.1 Introduction. When a fault occurs in a power network, the current flowing is determined by the internal emfs of the machines in the network, by their impedances, and by the impedances in the network between the machines and the fault. The current flowing in a synchronous machine immediately after the occurrence of a fault, that flowing a few cycles later, and the sustained or steady-state value of the fault current differ considerably because of the effect of the armature current on the flux which generates the voltage in the machine. The current changes relatively slowly from its initial value to its steady-state value. This chapter discusses the calculation of fault current at different periods, and it explains the changes in reactance and internal voltage of a synchronous machine as the current changes from its initial value upon the occurrence of a fault to its steady-state value.[1]

9.2 Transients in RL Series Circuits. As described in Sec. 1.4 the selection of a circuit breaker for a power system depends not only upon the current which the breaker is to carry under normal operating conditions but also upon the maximum current it may have to carry momentarily and the current it may have to interrupt at the voltage of the line in which it is placed. Therefore, it is always necessary to determine the initial value of current when a fault occurs on a system so as to select a breaker having a sufficient momentary rating.

In order to approach the problem of calculating the initial current when an alternator is short-circuited, consider what happens when a[illegible]-c voltage is applied to a circuit containing constant values o[illegible] and inductance. Let the applied voltage be $|V_m|$ sin [illegible], where [illegible] is zero at the time of applying the voltage. Then α [illegible]termines the magnitude of the voltage when the circuit is closed. [illegible] instantaneous

[1] The chapter is so arranged that students can omit Se[illegible] and 9.5 without loss of continuity. Such omission is recommended for students [illegible] are not familiar with the two-reaction method of analysis of synchronous mach[illegible]ery.

voltage is zero and increasing in a positive direction when it is applied by closing a switch, α is zero. If the voltage is at its positive maximum instantaneous value, α is $\pi/2$. The differential equation is

$$|V_m| \sin (\omega t + \alpha) = Ri + L\frac{di}{dt} \tag{9.1}$$

The solution[2] of this equation is

$$i = \frac{|V_m|}{|Z|}[\sin (\omega t + \alpha - \theta) - \epsilon^{-Rt/L} \sin (\alpha - \theta)] \tag{9.2}$$

where $|Z|$ is $\sqrt{R^2 + (\omega L)^2}$ and θ is $\tan^{-1} (\omega L/R)$.

The first term of Eq. (9.2) varies sinusoidally with time. The second term is nonperiodic and decays exponentially with a time constant of L/R. We recognize the sinusoidal term as the steady-state value of the current in an RL circuit for the given applied voltage. If the value of the steady-state term is not zero when $t = 0$, the nonperiodic, or d-c transient term appears in the solution in order to satisfy the physical condition of zero current at the instant of closing the switch. Note that the d-c term does not exist if the circuit is closed at a point on the voltage wave such that $\alpha - \theta = 0$ or $\alpha - \theta = \pi$. Figure 9.1 shows the variation of current with time according to Eq. (9.2) when $\alpha - \theta = 0$. If the switch is closed at a point on the voltage wave such that $\alpha - \theta = \pm\pi/2$, the d-c component has its maximum initial value, which is equal to the maximum value of the sinusoidal component. Figure 9.2 shows current versus time when $\alpha - \theta = -\pi/2$. The d-c component may have any value from 0 to $|V_m|/|Z|$, depending on the instantaneous value of the voltage when the circuit is closed and upon the power factor of the cir[illegible]t. At the instant of applying the voltage, the d-c and steady-[illegible] ponents always have the same magnitude but are opposit[illegible] in order to express the zero value of current then existing.

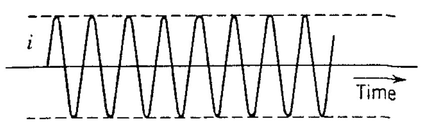


FIG. 9.1 Current as a function of time in an RL circuit for $\alpha - \theta = 0$, where $\theta = \tan^{-1} (\omega L/R)$. The voltage is $|V_m|$ $\sin (\omega t + \alpha)$ applied at $t = 0$.

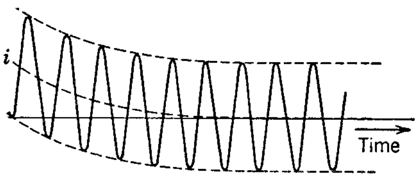


FIG. 9.2 Current as a function of time in an RL circuit for $\alpha - \theta = -90°$, where $\theta = \tan^{-1} (\omega L/R)$. The voltage is $|V_m|$ $\sin (\omega t + \alpha)$ applied at $t = 0$.

An a-c generator (alternator) consists of a rota[illegible] magnetic field which generates a voltage in an armature winding h[illegible]ing resistance and

[2] See E. B. Kurtz and G. F. Corcoran, "Introduction t[illegible]lectric T[illegible]nts," pp. 149–151, John Wiley & Sons, Inc., New York, 1935.

reactance. The current flowing when an alternator is short-circuited is similar to that flowing when an alternating voltage is suddenly applied to a resistance and an inductance in series. There are important differences, however, since the flux which crosses the air gap of an alternator and generates the voltage in the armature winding changes in value because of the effect which the current in the armature produces on the rotating field.

A good way to analyze the effect of a three-phase short circuit at the terminals of a previously unloaded alternator is to take an oscillogram

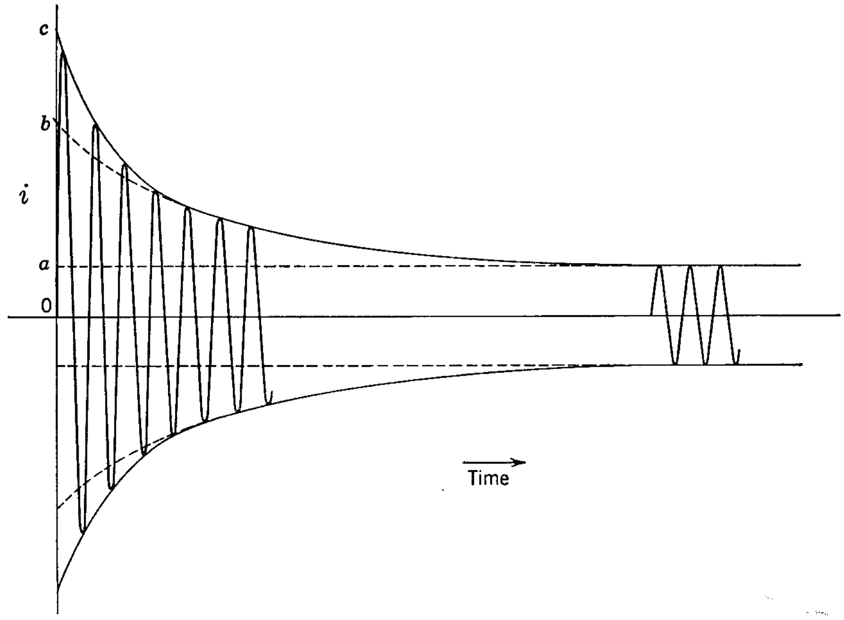


FIG. 9.3 Current as a function of time for a 208-volt 30-kw alternator short-circuited while running at no load. The unidirectional transient component of current has been eliminated in redrawing the oscillogram.

of the current in one of the phases upon the occurrence of such a fault. Since the voltages generated in the phases of a three-phase machine are displaced 120 electrical degrees from each other, the short circuit is applied at different points on the voltage wave of each phase. For this reason the unidirectional or d-c transient component of current is different in each phase. If the d-c component of current is eliminated from the current of each phase, the resulting plot of each phase current versus time is that shown in Fig. 9.3. Comparison of Figs. 9.1 and 9.3 shows the difference between applying a voltage to the ordinary RL circuit and applying a short circuit to a synchronous machine. In both of these figures there is no d-c component. In a synchronous machine the flux across the air gap of the machine is much larger at the instant the short circuit occurs than it is a few cycles later. The reduction of flux is caused by the mmf of the current in the armature. The phenomenon is called armature reaction. The resulting flux across the air gap is due to the combined mmf of the d-c winding and the armature current. Time is required for the reduction in flux to take place. As the air-gap flux diminishes, the armature current decreases because the voltage generated by the air-gap flux determines the current. This accounts for the gradual decrease in current shown in Fig. 9.3.

9.3 Short-circuit Currents and the Reactances of Synchronous Machines.[3] Certain terms which are valuable in the calculation of short-circuit current in a power system can be defined from Fig. 9.3. The reactances which we will define are called *direct-axis* reactances, a designation which is familiar to those who have studied the two-reaction theory[4] of a-c machinery and which should cause no confusion to others since it is merely applied to a value of reactance to be used for computing voltage drops caused by that component of the armature current which is in quadrature (90° out of phase) with the voltage generated at no load. Since the resistance in a faulted circuit is small compared to the inductive reactance, current during a fault is always lagging by a large angle, and the so-called direct-axis reactances are used. In the discussion to follow, it should be remembered that the current shown in the oscillogram of Fig. 9.3 is that which flows in an alternator which is operating at no load before the fault occurs.

In Fig. 9.3 the distance *oa* is the maximum value of the sustained short-circuit current. This value of current times 0.707 is the rms value I of the sustained, or steady-state, short-circuit current. The no-load voltage of the alternator E_g divided by the steady-state current I is called the *synchronous reactance* of the alternator, or the *direct-axis synchronous reactance* X_d since the power factor is low during the short circuit. The comparatively small resistance of the armature is neglected.

If the envelope of the current wave is extended back to zero time, neglecting the first few cycles where the decrement appears to be very rapid, the intercept is the distance *ob*. The rms value of the current represented by this intercept, or 0.707 times *ob* in amperes, is known as

[3] For a more complete discussion see C. F. Wagner and R. D. Evans, "S[illegible]mme[illegible] Components," Chap. V, Constants of Synchronous Machines, pp. 74–1[illegible] Hill Book Company, Inc., New York, 1933; C. F. Wagner, "Machi[illegible]cteristics," in Central Station Engineers of the Westinghouse Electric Corpo[illegible]on, "Electrical Transmission and Distribution Reference Book," 4th ed., C[illegible] pp. [illegible]5–194, East Pittsburgh, Pa., 1950. An advanced treatment is given by C[illegible]oncordia, "Synchronous Machines," John Wiley & Sons, Inc., New York, 1[illegible]

[4] See for instance A. E. Fitzgerald and C. Kingsley, "El[illegible]c Machin[illegible] [illegible]390, McGraw-Hill Book Company, Inc., New York, 1952.

the *transient current* I'. A new machine reactance may now be defined. It is called the *transient reactance*, or in this particular case the *direct-axis transient reactance* X_d', and is equal to E_g/I' for an alternator operating at no load before the fault. The point of intersection which the current envelope makes with the zero axis, if the rapid decrement of the first few cycles is neglected, can be determined more accurately by plotting on semilogarithmic paper the *excess* of the current envelope over the sustained value represented by oa, as shown in Fig. 9.4. The straight-line portion of this curve is extended to the zero-time axis, and the intercept is added to the maximum instantaneous value of the sustained current to obtain the maximum instantaneous value of transient current corresponding to ob in Fig. 9.3.

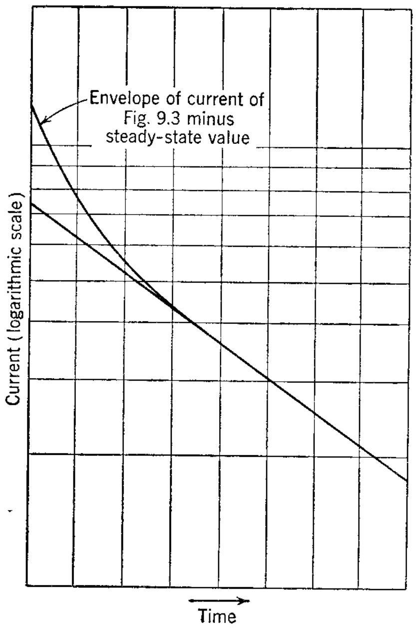


FIG. 9.4 Excess of the current envelope of Fig. 9.3 over the sustained maximum current, plotted on semilogarithmic scales.

The rms value of the current determined by the intercept of the current envelope with zero time is called the *subtransient current* I''. In Fig. 9.3 the subtransient current is 0.707 times the ordinate oc. Subtransient current is often called the *initial symmetrical rms current*, which is more descriptive because it conveys the idea of neglecting the d-c component and taking the rms value of the a-c component of current immediately after the occurrence of the fault. *Direct-axis subtransient reactance* X_d'' for an alternator operating at no load before the occurrence of a three-phase fault at its terminals is E_g/I''.

The currents and reactances discussed above are defined by the following equations, which apply to an alternator operating at no load before the occurrence of a three-phase fault at its terminals:

$$I = \frac{oa}{\sqrt{2}} = \frac{E_g}{jX_d} \qquad (9.3)$$

$$I' = \frac{ob}{\sqrt{2}} = \frac{E_g}{jX_d'} \qquad (9.4)$$

$$I'' = \frac{oc}{\sqrt{2}} = \frac{E_g}{jX_d''} \qquad (9.5)$$

where I = steady-state current, rms value

I' = transient current, rms value excluding d-c component

I'' = subtransient current, rms value excluding d-c component

X_d = direct-axis synchronous reactance

X'_d = direct-axis transient reactance

X''_d = direct-axis subtransient reactance

E_g = voltage from one terminal to neutral at no load

oa, *ob*, and *oc* are the intercepts shown in Fig. 9.3.

Equations (9.3) to (9.5) indicate the method of determining fault current in a generator when its reactances are known. If the generator is unloaded when the fault occurs, the machine is represented by the no-load voltage to neutral in series with the proper reactance. The resistance is taken into account if greater accuracy is desired. If there is impedance external to the generator between its terminals and the short circuit, the external impedance must be included in the circuit.

Example 9.1

Two generators are connected in parallel to the low-voltage side of a three-phase Δ-Y transformer as shown in Fig. 9.5. Generator 1 is rated 50,000 kva, 13.8 kv. Generator 2 is rated 25,000 kva, 13.8 kv. Each generator has a subtransient reactance of 25%. The transformer is rated 75,000 kva, 13.8Δ–69Y kv, with a reactance of 10%. Before the fault occurs, the voltage on the high-tension side of the transformer is 66 kv. The transformer is unloaded, and there is no circulating current between the generators. Find the subtransient current in each generator when a three-phase short circuit occurs on the high-tension side of the transformer.

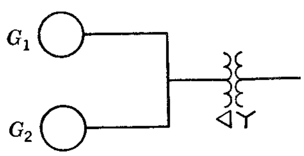


FIG. 9.5 One-line diagram for Example 9.1.

Solution

Select as base in the high-tension circuit 69 kv, 75,000 kva. Then the base voltage on the low-tension side is 13.8 kv.

$$\text{Generator 1: } X''_d = 0.25 \times \frac{75,000}{50,000} = 0.375 \text{ per unit}$$

$$E_{g1} = \frac{66}{69} = 0.957 \text{ per unit}$$

$$\text{Generator 2: } X''_d = 0.25 \times \frac{75,000}{25,000} = 0.750 \text{ per unit}$$

$$E_{g2} = \frac{66}{69} = 0.957 \text{ per unit}$$

$$\text{Transformer: } X = 0.10 \text{ per unit}$$

Figure 9.6 shows the reactance diagram before the fault. A three-phase fault at P is simulated by closing switch S. The internal voltages of the two machines may be considered to be in parallel since they must be

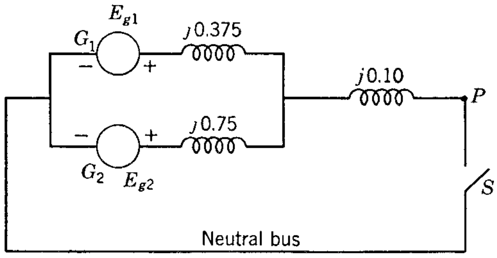


FIG. 9.6 Reactance diagram for Example 9.1.

identical in magnitude and phase if no circulating current flows between them. The equivalent parallel subtransient reactance is

$$j\,\frac{0.375 \times 0.75}{0.375 + 0.75} = j0.25 \text{ per unit}$$

Therefore,

$$I'' = \frac{0.957}{j0.25 + j0.10} = -j2.735 \text{ per unit}$$

Since this current divides between the generators inversely as the impedances of the generators,

$$\text{In generator 1: } I'' = -j2.735 \times \frac{0.75}{1.125} = -j1.823 \text{ per unit}$$

$$\text{In generator 2: } I'' = -j2.735 \times \frac{0.375}{1.125} = -j0.912 \text{ per unit}$$

To find the current in amperes, the per-unit values are multiplied by the base current of the circuit, as follows:

$$\text{In generator 1: } I'' = -j1.823 \times \frac{75{,}000}{\sqrt{3} \times 13.8} = 5{,}720 \text{ amp}$$

$$\text{In generator 2: } I'' = -j0.912 \times \frac{75{,}000}{\sqrt{3} \times 13.8} = 2{,}860 \text{ amp}$$

Finding machine reactances from an oscillogram of the current flowing when the machine is short-circuited at no load is only one of several available methods. Another method[5] of finding direct-axis subtransient reactance is discussed in Sec. 9.5.

Although machine reactances are not true constants of the machine and depend on the degree of saturation of the magnetic circuit, their

[5] For additional methods, see S. H. Wright, "Determination of Synchronous Machine Constants by Test," *Trans. AIEE*, vol. 50, pp. 1331–1351, December, 1931.

values usually lie within certain limits and can be predicted for various types of machines. Table A.5 in the Appendix gives typical values of machine reactances which are needed in making fault calculations and in stability studies. In general, subtransient reactances of generators and motors are used to determine the initial current flowing on the occurrence of a short circuit. For determining the interrupting capacity of circuit breakers, except those that open instantaneously, subtransient reactance is used for generators and transient reactance is used for synchronous motors. In stability studies where the problem is to determine whether a fault will cause a machine to lose synchronism with the rest of the system, if the fault is removed after a certain time interval, transient reactances apply.

9.4 Further Discussion of Synchronous Reactance. Although synchronous reactance has been mentioned briefly, a better understanding of its meaning and of the meaning of transient and subtransient reactance may be had through reviewing the phasor diagram of an alternator under steady-state conditions. The two-reaction theory of synchronous machine operation considers the magnetomotive force and flux acting directly in line with the poles of a salient-pole machine separately from the magnetomotive force and flux in quadrature with the poles. Information on machine reactances is generally given in terms of direct- and quadrature-axis components.

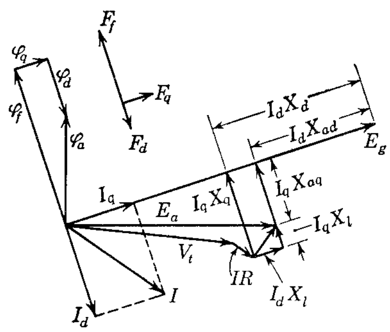
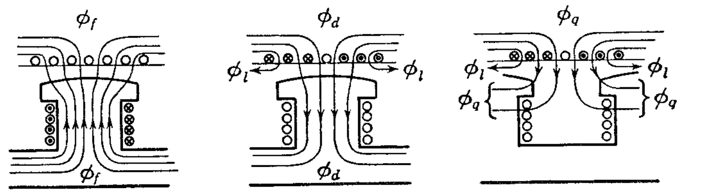


FIG. 9.7 Phasor diagram of a salient-pole alternator delivering a current I. E_g is the no-load voltage.

The two-reaction theory may be explained by referring to Fig. 9.7, which is the phasor diagram of an alternator under load. The current I is made up of two components, I_d, which is 90° out of phase with the no-load voltage E_g, and I_q in phase with E_g. The component I_d is called the direct-axis component of current because it produces the magnetomotive force F_d, which acts on the same axis as the magnetomotive force F_f, produced by the d-c field winding of the poles. F_f is the only magnetomotive force present at no load, and its flux ϕ_f induces the no-load voltage E_g. Similarly, I_q is called the quadrature-axis component of current because it produces the magnetomotive force F_q and in turn the flux ϕ_q, which is in quadrature with the main flux. Magnetomotive forces F_f and F_d add to give the resultant direct-axis mmf, which, in a salient-pole machine, acts over a low reluctance path compared to F_q, since the former acts directly on the pole while F_q acts between the poles.

The flux produced by each magnetomotive force depends on the reluctance in its path.[6] Figure 9.8*a* shows current in the field winding only, and the flux is that determined by F_f alone. Figure 9.8*b* shows the current component I_d in the armature winding and the flux which would be produced by its magnetomotive force if there were no other mmfs present. Figure 9.8*c* shows only I_q in the armature winding and flux ϕ_q set up by F_q. The voltage phasors corresponding to each component of flux produced by the component magnetomotive forces are shown in Fig. 9.7, where the no-load voltage E_g is the result of ϕ_f alone, the voltage phasor labelled I_dX_{ad} represents the voltage resulting from the decrease ϕ_d in direct-axis flux caused by F_d, and I_qX_{aq} is caused by ϕ_q. Each of these voltages lags by 90° the flux which induces it and therefore also lags the corresponding magnetomotive force and current. The voltage drop resulting from the action of F_d is considered as the product of a



(*a*) Field current and field flux (*b*) Direct-axis armature current and flux (*c*) Quadrature-axis armature current and flux

FIG. 9.8 Flux paths in a salient-pole alternator.

current and inductive reactance, namely, I_dX_{ad}, because it is proportional to I_d and lags it by 90°. Similar reasoning holds for I_qX_{aq}. Thus X_{ad} and X_{aq} are the constants of proportionality between the components of current and the voltage drops they cause by armature reaction. These values are true constants only if saturation is negligible. For a salient-pole machine X_{ad} is much larger than X_{aq} because of the lower reluctance on the direct axis.

The resultant of all the magnetomotive forces—that is, $F_d + F_q + F_f$—produces the air-gap flux ϕ_a, which in turn induces the voltage E_a in the armature. This voltage is called the *air-gap* voltage. In Fig. 9.7, phasors representing component fluxes ϕ_f, ϕ_d, and ϕ_q are shown adding up to produce ϕ_a. As stated above, ϕ_f, is flux produced by F_f alone, and ϕ_d is the decrease in direct-axis flux caused by F_d and not necess[illegible]ly the flux which F_d would produce if acting alone. In tern[illegible] voltages, E_a is the sum of the no-load voltage E_g and the [illegible]ges induced by the magnetomotive forces of armature reaction, or [illegible] is equal to E_a plus the voltage drops I_dX_{ad} and I_qX_{aq}. The termi[illegible] voltage V_t differs

[6] A discussion of the flux paths in a salient-pole machi[illegible] given by L. P. Schildneck, "Synchronous Machine Reactances, a Fundamenta[illegible]d Physi[illegible]," *Gen. Elec. Rev.*, vol. 35, pp. 560–565, November, 1937.

from the air-gap voltage at any time only by the armature resistance and leakage reactance drops $IR + jIX_l$. This is an important concept. After a three-phase short circuit has occurred at the terminals of an alternator, the terminal voltage is zero, and the steady-state current is

$$I = \frac{E_a}{R + jX_l} \tag{9.6}$$

where E_a is the steady-state value of air-gap voltage.

When the voltage drops of armature reaction are combined with the drop caused by leakage reactance, the total direct-axis voltage drop divided by I_d is called the *direct-axis synchronous reactance* X_d, and the total quadrature-axis voltage drop divided by I_q is the *quadrature-axis synchronous reactance* X_q. These relations are given by the following equations for direct- and quadrature-axis synchronous reactance:

$$X_d = X_{ad} + X_l \qquad \text{and} \qquad X_q = X_{aq} + X_l \tag{9.7}$$

It should be noted here that the diagram of Fig. 9.7 is for a salient-pole machine and that for low power factors the angle between E_g and I approaches 90°. Direct-axis synchronous reactance X_d is often the only machine impedance that need be considered for steady-state fault calculations since the armature resistance is usually small and the current usually lags E_g by a very large angle. This may result in a small error for faults remote from the alternator terminals if the resistance in the external circuit is appreciable, but the neglect of resistance and quadrature-axis reactance simplifies the calculation greatly.

For a three-phase short circuit at the generator terminals, the steady-state short-circuit current is given by

$$I = \frac{E_a}{jX_l} = \frac{E_g}{jX_d} \tag{9.8}$$

If there is external impedance Z_{ext} between the terminals and the fault, the current will be

$$I = \frac{E_a}{Z_{\text{ext}} + jX_l} = \frac{E_g}{Z_{\text{ext}} + jX_d} \tag{9.9}$$

In general E_a is not known, but E_g is known or can be determined.

9.5 The Significance of Transient and Subtransient Reactances. Section 9.4 shows that the steady-state current in a generator with armature resistance neglected is determined by the air-gap voltage E_a and the leakage reactance of the armature circuit X_l plus any external impedance present. When the current in the armature is changing rapidly, after the occurrence of a short circuit, the current goes rapidly from its subtransient value to its transient and steady-state values. The current is still determined by the air-gap voltage and the leakage reactance of the armature plus any external impedance. The magnitude of the air-gap voltage is changing during this period. For a three-phase short circuit at the generator terminals,

$$I' = \frac{E'_a}{jX_l} = \frac{E_g}{jX'_d} \tag{9.10}$$

$$I'' = \frac{E''_a}{jX_l} = \frac{E_g}{jX''_d} \tag{9.11}$$

where E'_a and E''_a are, respectively, the air-gap voltages when transient and subtransient currents are flowing. The difference between the subtransient current and the steady-state current is caused entirely by the change in the air-gap voltage. At no load, E_g is equal to the air-gap voltage. With a steady-state load, as shown in Fig. 9.7, E_a is much smaller than E_g because of armature reaction. If the machine is suddenly short-circuited, the current which starts to flow in the armature circuit produces a magnetomotive force which tends to decrease the flux linking the field circuit and the rotor iron of a turbine generator or the damper winding of a salient-pole machine.

According to Lenz's law, any change in the flux linking a circuit will induce an emf in the circuit tending to cause the flow of current in a direction to oppose the change in flux. An instantaneous change in the total flux linkages of a circuit would produce an infinitely large induced voltage still opposing the change. Therefore, the total flux linkages of a circuit tend to remain constant instantaneously. This phenomenon is called the principle of constant flux linkages.[7] The demagnetizing effect of the armature current (armature reaction) would reduce the flux linkages of the circuits in the path of the flux except for the fact that current is immediately induced in the circuits linked. The principle of constant flux linkages explains the reason for induced current in the rotor iron, the damper windings, and the field windings in order to maintain a constant value of flux linkages of these circuits for an instant after a sudden change occurs in the armature current. Some of the flux which links the current induced in the damper winding or rotor iron is leakage flux, or flux which does not link the armature circuit. Similarly, the increased current in the field winding will result in increased field leakage flux. The flux across the air gap is reduced slightly since the constant total value of flux linkages of the damper winding, the rotor iron, and the field winding consists of both air-gap flux and leakage flux. Thus, E''_a is slightly less than E_g and, since

[7] See W. C. Johnson, "Mathematical and Physical Principles of Engineering Analysis," pp. 26–30, McGraw-Hill Book Company, Inc., 1944.

Eq. (9.11) yields the relation

$$\frac{X''_d}{X_l} = \frac{E_g}{E''_a} \tag{9.12}$$

X''_d is slightly larger than X_l. Usually E_g is known or can be calculated, and the subtransient current is found by the equation

$$I'' = \frac{E_g}{X''_d} \tag{9.13}$$

As the current induced in the damper winding, or other eddy-current path, dies out, the air-gap flux decreases. The result is a reduced air-gap voltage and, in turn, a decrease in armature current. The rate of decrease of armature current is determined by the time constant L/R of the path of the induced current. This time constant is comparatively short. After a few cycles the eddy-current paths cease to be the major determining factor in the decay of air-gap flux and armature current, but the rate of decay is then determined by the considerably larger time constant of the field winding. This is the reason for the change in the rate of decay of the armature current after the first few cycles. Since I' is less than I'', X'_d is greater than X''_d, but both are much less than the synchronous reactance.

Any change of current in the armature circuit will change the magnetomotive force of armature reaction. The resulting change in flux linking the damper winding and field circuit is governed by the time constants of these circuits since it is the current flowing in the windings which opposes the change of flux. For any transient condition in the armature circuit, either subtransient or transient reactance should be used. The choice of reactance depends on whether the current to be found is the initial value for a sudden change in circuit conditions or the value after the transient in the damper winding has died out and the change in flux has penetrated to the field circuit.

If a single-phase voltage is applied to two terminals of the armature circuit, a pulsating, nonrotating field is set up by the armature current. If the field circuit is short-circuited and the rotor is turned slowly so that the pulsating field is sometimes lined up directly with the axis of the rotor and sometimes is at 90 electrical degrees with it, the current through the armature will have different values depending on the position of the rotor. The pulsating field of the armature induces currents in the field circuit and damper windings. These induced currents oppose the flux which produces them. The mmfs of the armature current and of the current induced in the field winding are the same as those which act during the flow of subtransient current in the armature circuit. The voltage induced by the flux set up by these mmfs is the product of the armature current times subtransient reactance. If the applied single-phase voltage is divided by 2 to obtain the voltage per phase and then by the current measured for various positions of the rotor as it changes position, the reactive components of the impedances obtained are direct-axis subtransient reactance when the armature magnetomotive force is lined up with the rotor, or quadrature-axis subtransient reactance when the rotor is shifted by 90 electrical degrees. The reactances so measured for a salient-pole machine with and without damper windings are plotted in Fig. 9.9. This demonstrates the effect of the damper winding. The damper winding which extends around the complete circumference of the

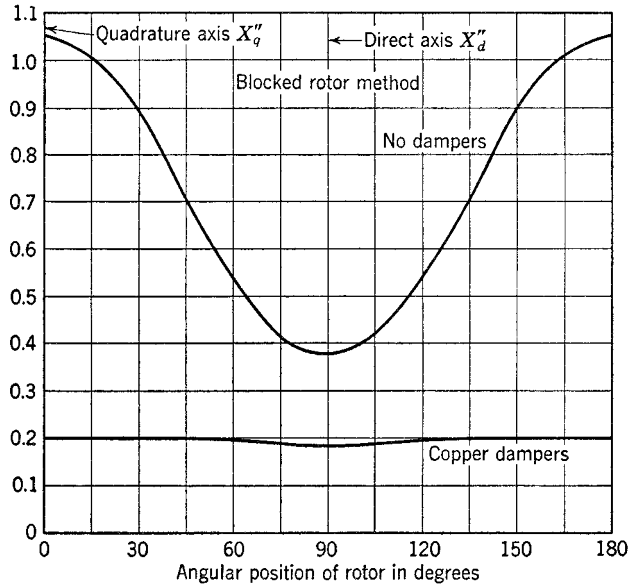
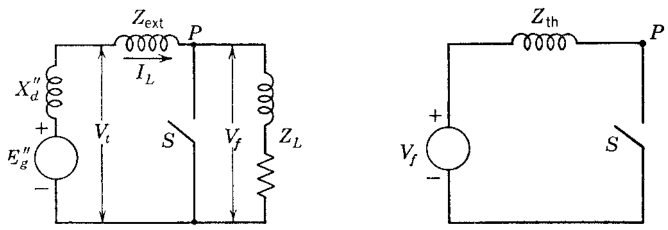


Fig. 9.9 Subtransient reactance as a function of rotor position for a salient-pole machine with and without a damper winding. (*From C. F. Wagner and R. D. Evans, "Symmetrical Components," McGraw-Hill Book Company, Inc., New York,* 1933, *by permission.*)

rotor is so effective in opposing the change of flux linking the armature circuit that the presence or absence of the rotor poles is hardly noticeable, and the effect is similar to that in a round-rotor machine. When there are no damper windings, X_q'' is much greater than X_d'' in a salient-pole machine because there is no closely coupled circuit on the quadrature axis to oppose the changing flux, and the flux change is large. On the direct axis the field circuit opposes the change of flux by acting like the short-circuited secondary of a transformer, and the flux change is small.

9.6 Internal Voltages of Loaded Machines under Transient Conditions. All the preceding discussion pertains to an alternator which carries no current at the time a three-phase fault occurs at the terminals of the machine. Now consider an alternator which is loaded when the fault occurs. Figure 9.10*a* is the equivalent circuit of an alternator which has a balanced three-phase load. External impedance is shown between the alternator terminals and the point P where the fault occurs. A three-phase fault is simulated by closing the switch S. Before the switch is closed the current flowing is I_L, the voltage at the fault is V_f, and the terminal voltage of the alternator is V_t.

The circuit may be solved by the Helmholtz-Thévenin theorem, which is applicable to linear, bilateral circuits.[8] When constant values are used for the reactances of synchronous machines, linearity is assumed. When the theorem is applied to the circuit of Fig. 9.10*a*, the equivalent circuit is a single generator and a single impedance terminating at the point of application of the fault. The new generator has an internal



(*a*) Original circuit (*b*) Thévenin equivalent circuit at fault

FIG. 9.10 Equivalent circuits of an alternator supplying a balanced three-phase load. Application of a three-phase fault is simulated by closing switch S.

voltage equal to V_f, the voltage at the fault point before the fault occurs. The impedance is that measured at the point of application of the fault looking back into the circuit with all the generated voltages short-circuited. Subtransient reactances should be used if the initial current is desired. Figure 9.10*b* is the Helmholtz-Thévenin equivalent of the original circuit at the fault. The impedance Z_{th} is equal to $(Z_{ext} + jX_d'')Z_L/(Z_L + Z_{ext} + jX_d'')$. Upon the occurrence of a three-phase short circuit at P, simulated by closing S, the subtransient current in the fault is

$$I'' = \frac{V_f}{Z_{th}} = \frac{V_f(Z_L + Z_{ext} + jX_d'')}{Z_L(Z_{ext} + jX_d'')} \tag{9.14}$$

In making steady-state calculations for synchronous machines, [illegible]e machine is regarded as having an internal voltage equal to its [illegible] voltage at no load E_g and an internal reactance called [illegible] reactance. In a similar manner the machine may be r[illegible]ing an internal voltage called the *voltage behind subtransient* [illegible] E'', which drives the subtransient current through the subtrans[illegible]ctance when there is a sudden change in load or a short circuit. [illegible]

[8] The Helmholtz-Thévenin theorem is stated in Sec. 8.9[illegible]

may be regarded as being driven through the transient reactance by the *voltage behind transient reactance* E'. Equations (9.3) to (9.5) indicate that E_g, E'', and E' are all equal when there is no load on the machine before a short circuit occurs, since the equations show E_g determining the current for each condition. Such is not the case if the machine is loaded when the fault occurs. The values of E' and E'' when a machine is loaded before a disturbance occurs are found by noting that Eq. (9.14) gives the value of I'' and that, by definition for the circuit of Fig. 9.10*a*,

$$I'' = \frac{E''}{Z_{ext} + jX''_d} \tag{9.15}$$

Eliminating I'' from Eqs. (9.14) and (9.15), and noting that $I_L = V_f/Z_L$, we obtain

$$E'' = I_L(Z_L + Z_{ext} + jX''_d) \tag{9.16}$$

and, since $I_L(Z_L + Z_{ext}) = V_t$,

$$E'' = V_t + jI_LX''_d \tag{9.17}$$

Similarly,

$$E' = V_t + jI_LX'_d \tag{9.18}$$

Equations (9.17) and (9.18) are useful in computing E'' and E' when the terminal voltage of an alternator and the load current are known before the fault occurs.

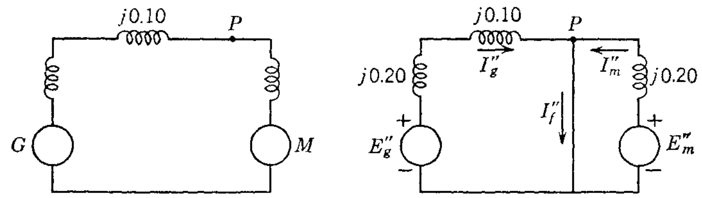
Synchronous motors have reactances of the same type as alternators. When a motor is short-circuited it no longer receives electric energy from the power line, but its field remains energized, and the inertia of its rotor and connected load keeps it rotating for an indefinite period. The internal voltage of a synchronous motor causes it to contribute current to the system, for it is then acting like an alternator. By comparison with the corresponding formulas for an alternator, the voltage behind subtransient reactance and the voltage behind transient reactance for a synchronous motor are found to be given by the following equations:

$$E'' = V_t - jI_LX''_d \tag{9.19}$$
$$E' = V_t - jI_LX'_d \tag{9.20}$$

Systems which contain alternators and motors under load may be solved either by the Helmholtz-Thévenin theorem or by the use of voltages behind transient or subtransient reactance, as is illustrated in the following examples.

Example 9.2

An alternator and a synchronous motor are rated [illegible] kva, 13.2 kv, and both have subtransient reactances of 20%. The line connecting them has a reactance of 10% on the base of the machine ratings. The motor is drawing 20,000 kw at 0.8 power factor leading and a terminal voltage of 12.8 kv when a symmetrical three-phase fault occurs at the motor terminals. Find the subtransient current in the alternator, motor, and fault by using the internal voltages of the machines.



(a) Before the fault (b) During the fault

FIG. 9.11 Equivalent circuits for Example 9.2.

Solution

Choose as base 30,000 kva, 13.2 kv.

Figure 9.11*a* shows the equivalent circuit of the system described.

Use the voltage at the fault V_f as the reference phasor.

$$V_f = \frac{12.8}{13.2} = 0.97\underline{/0^\circ} \text{ per unit}$$

$$\text{Base current} = \frac{30{,}000}{\sqrt{3} \times 13.2} = 1{,}310 \text{ amp}$$

$$I_L = \frac{20{,}000}{0.8 \times \sqrt{3} \times 12.8} = 1{,}128\underline{/36.9^\circ} \text{ amp}$$

$$= \frac{1{,}128}{1{,}310} = 0.86\underline{/36.9^\circ} \text{ per unit}$$

$$= 0.86(0.8 + j0.6) = 0.69 + j0.52 \text{ per unit}$$

For the generator,

$$V_t = 0.970 + j0.1(0.69 + j0.52)$$

$$= 0.970 + j0.069 - 0.052 = 0.918 + j0.069 \text{ per unit}$$

$$E''_g = 0.918 + j0.069 + j0.2(0.69 + j0.52)$$

$$= 0.918 + j0.069 + j0.138 - 0.104 = 0.814 + j0.207 \text{ per unit}$$

$$I''_g = \frac{0.814 + j0.207}{j0.3} = 0.69 - j2.71 \text{ per unit}$$

$$= 1{,}310(0.69 - j2.71) = 904 - j3{,}550 \text{ amp}$$

For the motor,

$$V_t = V_f = 0.97\underline{/0^\circ} \text{ per unit}$$

$$E''_m = 0.97 + j0 - j0.2(0.69 + j0.52) = 0.97 - j0. \quad 0.104$$

$$= 1.074 - j0.138 \text{ per unit}$$

$$I''_m = \frac{1.074 - j0.138}{j0.2} = -0.69 - j5.37 \text{ per uni}$$

$$= 1{,}310(-0.69 - j5.37) = -904 - j7{,}0 \quad \text{amp}$$

In the fault,

$$I_f'' = I_g'' + I_m'' = 0.69 - j2.71 - 0.69 - j5.37 = -j8.08 \text{ per unit}$$
$$= -j8.08 \times 1{,}310 = -j10{,}590 \text{ amp}$$

Figure 9.11*b* shows the paths of I_g'', I_m'', and I_f''.

Example 9.3

Solve Example 9.2 by the use of the Helmholtz-Thévenin theorem.

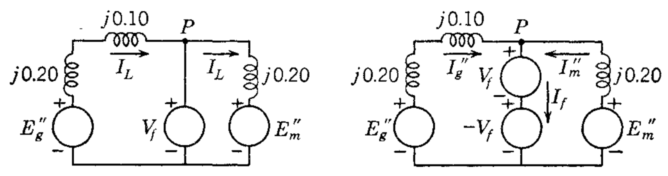
Solution

$$Z_{th} = \frac{j0.3 \times j0.2}{j0.3 + j0.2} = j0.12 \text{ per unit}$$
$$V_f = 0.97\underline{/0^\circ} \text{ per unit}$$

In the fault,

$$I_f'' = \frac{0.97 + j0}{j0.12} = -j8.08 \text{ per unit}$$

The above current, found by applying the Helmholtz-Thévenin theorem, is that which flows out of the circuit at the fault because of the



(*a*) Before the fault (*b*) During the fault

FIG. 9.12 Circuits illustrating the application of the superposition theorem to determine the proportion of the fault current in each branch of the system.

reduction of the voltage to zero at that point. If this current caused by the fault is divided between the parallel circuits of the generators inversely as their impedances, the resulting values are the currents from each machine due only to the *change* in voltage at the fault point. To the fault currents thus attributed to the two machines must be added the current flowing in each before the fault occurred to find the total current in the machines after the fault. The superposition theorem supplies the reason for adding the current flowing before the fault to the current computed by the Helmholtz-Thévenin theorem. [illegible] shows a generator having a voltage V_f connected at th[illegible] equal to the voltage at the fault before the fault occurs. T[illegible] generator has no effect on the current flowing before the fault [illegible]s, and the circuit corresponds to that of Fig. 9.11*a*. Adding in se[illegible]s with V_f another generator having an emf of equal magnitude b[illegible]0° out of phase with V_f gives the circuit of Fig. 9.12*b*, which correspo[illegible]s to that of Fig. 9.11*b*.

The principle of superposition, applied by first shorting E_g'', E_m'', and V_f, gives the currents found by distributing the fault current between the two generators inversely as the impedances of their circuits. Then shorting the remaining generator $-V_f$ with E_g'', E_m'', and V_f in the circuit gives the current flowing before the fault. Adding the two values of current in each branch gives the current in the branch after the fault. A further discussion of the method is given in Sec. 13.5. Applying the above principle to the present example gives

$$\text{Fault current from the generator} = -j8.08 \times \frac{j0.2}{j0.5} = -j3.23 \text{ per unit}$$

$$\text{Fault current from the motor} = -j8.08 \times \frac{j0.3}{j0.5} = -j4.85 \text{ per unit}$$

To these currents must be added the prefault current I_L to obtain the total subtransient currents in the machines.

$$I_g'' = 0.69 + j0.52 - j3.23 = 0.69 - j2.71 \text{ per unit}$$
$$I_m'' = -0.69 - j0.52 - j4.85 = -0.69 - j5.37 \text{ per unit}$$

Note that I_L is in the same direction as I_g'' but opposite to I_m''.

As in Example 9.2,

$$I_g'' = 904 - j3{,}550 \text{ amp}$$
$$I_m'' = -904 - j7{,}040 \text{ amp}$$
$$I_f'' = -j10{,}590 \text{ amp}$$

9.7 The Selection of Circuit Breakers. The subtransient current is the initial *symmetrical* rms current and does not include the d-c component of the transient fault current. Exact calculation of the rms value of the fault current in a power system is exceedingly complicated. Approximate methods are more practical and usually sufficiently accurate. The method recommended by the AIEE Switchgear Committee[9] takes the d-c component into account by applying a multiplying factor to a symmetrical rms current calculated, according to certain rules, for the type and location of fault which provides the heaviest duty on the breaker. In determining current which a breaker must carry immediately after a fault occurs and which is called the momentary duty of a breaker, the initial symmetrical rms current is computed by network reduction [illegible] on an a-c or d-c calculating board with subtransient reactances [illegible] the generators, synchronous motors, and induction [illegible]. Current flowing before the fault occurs is neglected. The multiplying factor recommended is 1.6, except that at 5 kv and [illegible] the multiplying factor is 1.5, unless the circuit is fed predominantly [illegible] directly connected

[9] AIEE Switchgear Committee, "Simplified Calculation [illegible] Fault Currents," *Trans. AIEE*, vol. 67, Part II, pp. 1433–1435, 1948.

machines or through reactors. A factor of 1.25 is recommended for air circuit breakers rated 600 volts or less. Such breakers are rated on an average for all three phases.

In computing the current to which to apply a multiplying factor to determine current which a breaker must be able to interrupt at the time its contacts part and which is called the interrupting rating, it is recommended that subtransient reactance be used for generators and transient reactance for synchronous motors and that induction motors be neglected. The suggested multiplying factors to obtain the interrupting rating depend on the speed of the breaker. For the general case, the multiplying factors are:

8-cycle or slower breakers	1.0
5-cycle breaker	1.1
3-cycle breaker	1.2
2-cycle breaker	1.4

If the circuit breakers are on a generator bus and the three-phase kva exceeds 500,000 before the application of any multiplying factor, the factors given above should be increased by adding 0.1 to each. Air circuit breakers below 600 volts are considered to open instantaneously, and their momentary and interrupting currents are the same.

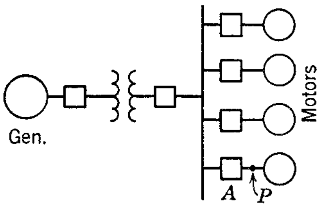


Fig. 9.13 One-line diagram for Example 9.4.

Example 9.4

A 25,000-kva, 13.8-kv generator with $X_d'' = 15\%$ is connected through a transformer to a bus which supplies four identical motors as shown in Fig. 9.13. Each motor has $X_d'' = 20\%$ and $X_d' = 30\%$ on a base of 5,000 kva, 6.9 kv. The three-phase rating of the transformer is 25,000 kva, 13.8–6.9 kv with a leakage reactance of 10%. The bus voltage at the motors is 6.9 kv when a three-phase fault occurs at the point P. For the fault specified, determine:

1. The subtransient current in the fault.
2. The subtransient current in breaker A.
3. The momentary current in breaker A.
4. The current to be interrupted by breaker A in 5 cycles.

Solution

1. For a base of 25,000 kva, 13.8 kv in the generator circuit, the base for the motors is 25,000 kva, 6.9 kv. The reactances for each motor are

$$X''_d = j0.20 \times \frac{25{,}000}{5{,}000} = j1.0 \text{ per unit}$$

$$X'_d = j0.30 \times \frac{25{,}000}{5{,}000} = j1.5 \text{ per unit}$$

Figure 9.14 is the reactance diagram with subtransient values of reactance marked.

For a fault at P,

$$V_f = 1.0 \text{ per unit}$$

$$Z_{th} = j0.125 \text{ per unit}$$

$$I''_f = \frac{1.0}{j0.125} = -j8.0 \text{ per unit}$$

The base current in the 6.9-kv circuit is $\dfrac{25{,}000}{\sqrt{3} \times 6.9} = 2{,}090$ amp.

$$I''_f = 8 \times 2{,}090 = 16{,}720 \text{ amp}$$

2. Through breaker A comes the contribution from the generator and three of the four motors.

The generator contributes a current of $-j8.0 \times \dfrac{0.25}{0.50} = -j4.0$ per unit.

Each motor contributes 25% of the remaining fault current or $-j1.0$ per-unit amp.

Through breaker A,

$$I'' = -j4.0 + 3(-j1.0) = -j7.0 \text{ per unit} = 7 \times 2{,}090 = 14{,}630 \text{ amp}$$

3. The momentary current in breaker A is $1.6 \times 14{,}630 = 23{,}450$ amp

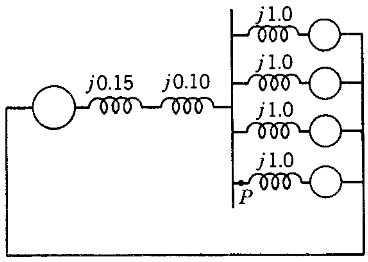


FIG. 9.14 Reactance diagram for Example 9.4.

4. To compute the current to be interrupted, replace the subtransient reactance of $j1.0$ by the transient reactance of $j1.5$ in the motor circuits of Fig. 9.14. Then,

$$Z_{th} = j\frac{0.375 \times 0.25}{0.375 + 0.25} = j0.15 \text{ per unit}$$

The generator contributes a current of

$$\frac{1}{j0.15} \times \frac{0.375}{0.625} = -j4.0 \text{ per unit}$$

Each motor contributes a current of

$$\frac{1}{4} \times \frac{1}{j0.15} \times \frac{0.25}{0.625} = -j0.67 \text{ per unit}$$

The current to be interrupted is

$$1.1(4.0 + 3 \times 0.67) \times 2{,}090 = 13{,}800 \text{ amp}$$

The interrupting kva is

$$\sqrt{3} \times 13{,}800 \times 6.9 = 165{,}000 \text{ kva}$$

9.8 Summary. The machines in a power system may be represented by an internal voltage and series impedance. The values of the internal voltages and the impedances depend on the conditions being studied. Each emf contributes to the fault current, which may be found by computing the contribution caused by each individual emf. An alternative method is to find the current in the fault by the Helmholtz-Thévenin theorem and to divide this current between the parts of the system which supply it. To obtain results by the Thévenin method equivalent to considering each individual emf, the prefault current in the various parts of the system must be added to the component of current in each part due to the fault. Because of the complexity of fault calculations, simplified methods, disregarding the load current and omitting resistance, are justified in many cases.

PROBLEMS

9.1 A 60-cycle alternating voltage having a rms value of 100 volts is applied to a series RL circuit by closing a switch. The resistance is 10 ohms and the inductance is 0.1 henry.

(*a*) Find the value of the d-c component of current upon closing the switch if the instantaneous value of the voltage is 50 volts at that time.

(*b*) What is the instantaneous value of the voltage which will produce the maximum d-c component of current upon closing the switch?

(*c*) What is the instantaneous value of the voltage which will result in the absence of any d-c component of current upon closing the switch?

(*d*) If the switch is closed when the instantaneous voltage is zero, find the instantaneous current 0.5, 1.5, and 5.5 cycles later.

9.2 A generator connected through a 5-cycle circuit breaker to a transformer is rated 7,500 kva, 6.9 kv with reactances of $X_d'' = 9\%$, $X_d' = 15\%$, and $X_d = 100\%$. It is operating at no load and rated voltage when a three-phase short circuit occurs between the breaker and the transformer. Find (*a*) the sustained short-circuit current in the breaker, (*b*) the initial symmetrical rms current in the breaker, (*c*) the maximum possible d-c component of the short-circuit current in the breaker, (*d*) the momentary current rating required for the breaker, (*e*) the current to be interrupted by the breaker, and the interrupting kva.

9.3 The three-phase transformer connected to the generator described in Prob. 9.2 is rated 7,500 kva, 6.9Δ–115Y kv, $X = 10\%$. If a three-phase short circuit occurs on the high-tension side of the transformer at rated voltage and no load, find (*a*) the initial symmetrical rms current in the transformer windings on the high-tension side, (*b*) the initial symmetrical rms current in the line on the low-tension side.

9.4 A 60-cycle generator is rated 625 kv, 480 volts with $X_d'' = 0.08$ per unit. It supplies a purely resistive load of 500 kw at 480 volts. The load is connected directly across the terminals of the generator. If all three phases of the load are short-circuited simultaneously, find the initial symmetrical rms current in the generator in per unit on a base of 625 kva, 480 volts.

9.5 A generator is connected through a transformer to a synchronous motor. Reduced to the same base the per-unit subtransient reactances of the generator and motor are 0.15 and 0.35, respectively, and the leakage reactance of the transformer is 0.10 per unit. A three-phase fault occurs at the terminals of the motor when the terminal voltage of the generator is 0.9 per unit and the output current of the generator is 1.0 per unit at 0.8 power factor leading. Find the subtransient current in per unit in the fault, in the generator, and in the motor. Use the terminal voltage of the generator as the reference phasor, and obtain the solution (*a*) by computing the voltages behind subtransient reactance in the generator and motor and (*b*) by using the Helmholtz-Thévenin theorem.

9.6 A 625-kva, 2.4-kv generator with $X_d'' = 0.08$ per unit is connected to a bus through a circuit breaker as shown in Fig. 9.15. Connected through circuit breakers to the same bus are three synchronous motors rated 250 hp, 2.4 kv, 1.0 power factor, 90% efficiency with $X_d'' = 0.20$ per unit. The motors are operating at full load, unity power factor, and rated voltage with the load equally divided between the machines.

(*a*) Draw the impedance diagram with the impedances marked in per unit on a base of 625 kva, 2.4 kv.

(*b*) Find the initial symmetrical rms current in per unit in the fault and in breakers *A* and *B* for a three-phase fault at point *P*. Simplify the calculations by neglecting the prefault current.

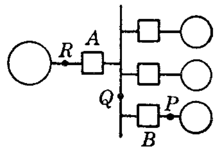


FIG. 9.15 One-line diagram for Example 9.6.

(*c*) Repeat (*b*) for a three-phase fault at point *Q*.

(*d*) Repeat (*b*) for a three-phase fault at point *R*.

(*e*) Find the highest value of momentary current expected for any three-phase fault for breakers *A* and *B*.

9.7 The system shown in Fig. 9.16 is delivering 60,000 kva at 12.5 kv, 0.8 power factor lag to a large metropolitan system which may be represented by an infinite bus. The generator is rated 60,000 kva, 12.7 kv, $X_d'' = 0.20$ per unit. Each three-phase

FIG. 9.16 One-line diagram for Example 9.7.

transformer is rated 75,000 kva, 13.8Δ–69Y kv, $X = 8\%$. The reactance of the transmission line is 10 ohms. A three-phase fault occurs at point *P*. Determine the momentary current in breakers *A* and *B* for the specified fault. Find the initial symmetrical rms current in the fault. Use the generator rating as base [illegible] of the generator.

CHAPTER 10

SYMMETRICAL COMPONENTS

10.1 Analysis by Symmetrical Components. In 1918 one of the most powerful tools for dealing with unbalanced polyphase circuits was discussed by Dr. C. L. Fortescue at a meeting of the American Institute of Electrical Engineers, where he presented a paper entitled "Method of Symmetrical Coordinates Applied to the Solution of Polyphase Networks."[1] Since that time the method of symmetrical components has become of great importance and has been the subject of many articles and experimental investigations. Unsymmetrical faults on transmission systems, which may consist of short circuits, impedance between lines, impedance from one or two lines to ground, or open conductors, are studied by the method of symmetrical components. The method is applicable to analytical solutions or to calculating boards.

Fortescue's work proves that an unbalanced system of n related phasors can be resolved into n systems of balanced phasors called the symmetrical components of the original phasors. The n phasors of each set of components are equal in length, and the angles between adjacent phasors of the set are equal. Although the method is applicable to any unbalanced polyphase system, we will confine our discussion to three-phase systems.

According to Fortescue's theorem, three unbalanced phasors of a three-phase system can be resolved into three balanced systems of phasors. The balanced sets of components are:

1. Positive-sequence components consisting of three phasors equal in magnitude, displaced from each other by 120° in phase, and having the same phase sequence as the original phasors.

2. Negative-sequence components consisting of three phasors equal in magnitude, displaced from each other by 120° in phase, and having the phase sequence opposite to that of the original phasors.

3. Zero-sequence components consisting of three phasors equal in magnitude and with zero phase displacement from each other.

[1] C. L. Fortescue, "Method of Symmetrical Coordinates Applied to the Solution of Polyphase Networks," *Trans. AIEE*, vol. 37, pp. 1027–1140, 1918.

It is customary, when solving a problem by symmetrical components, to designate the three phases of the system as a, b, and c in such a manner that the phase sequence of the voltages and currents in the system is abc. Thus the phase sequence of the positive-sequence components of the unbalanced phasors is abc, and the phase sequence of the negative-sequence components is acb. If the original phasors are voltages, they

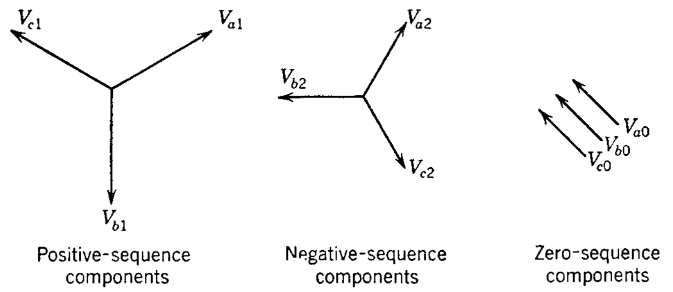


FIG. 10.1 Three sets of balanced phasors which are the symmetrical components of three unbalanced phasors.

may be designated V_a, V_b, and V_c. The three sets of symmetrical components are designated by the additional subscript 1 for the positive-sequence components, 2 for the negative-sequence components, and 0 for the zero-sequence components. The positive-sequence components of V_a, V_b, and V_c are V_{a1}, V_{b1}, and V_{c1}. Similarly, the negative-sequence components are V_{a2}, V_{b2}, and V_{c2}, and the zero-sequence components are V_{a0}, V_{b0}, and V_{c0}. Phasors representing currents will be designated by I with subscripts as for voltages.

Since each of the original unbalanced phasors is the sum of its components, the original phasors expressed in terms of their components are:

$$V_a = V_{a1} + V_{a2} + V_{a0} \quad (10.1)$$
$$V_b = V_{b1} + V_{b2} + V_{b0} \quad (10.2)$$
$$V_c = V_{c1} + V_{c2} + V_{c0} \quad (10.3)$$

The synthesis of a set of three unbalanced phasors in accord with Eqs. (10.1) to (10.3) is shown graphically in Figs. 10.1 and 10.2. The three sets of balanced phasors which are the symmetrical components of three unbalanced phasors are shown in Fig. 10.1. The graphical addition of the components and the resulting unbalanced phasors are shown in Fig. 10.2.

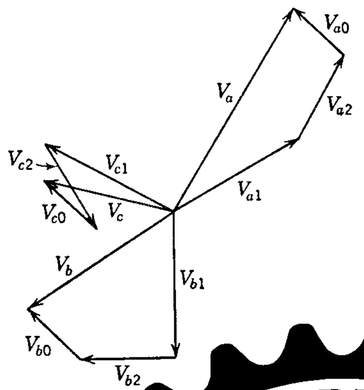


FIG. 10.2 Graphical addition of the components shown in Fig. 10.1 to obtain three unbalanced phasors.

The many advantages of analysis of power systems by the method of symmetrical components will become apparent gradually as we apply the method to the study of unsymmetrical faults on otherwise symmetrical systems and to the study of unbalanced systems and unbalanced loads. It is sufficient to say here that the method consists of finding the symmetrical components of current at the fault or the point of unbalance. Then the values of current and voltage at various points in the system can be found. The method is simple and leads to accurate predictions of system behavior under conditions of unbalance.

10.2 Operators. Because of the phase displacement of the symmetrical components of the voltages and currents in a three-phase system by 120°, it is convenient to have a shorthand method of indicating the rotation of a phasor through 120°. The result of the multiplication of two complex numbers is the product of their magnitudes and the sum of their angles. If the complex number expressing a phasor is multiplied by a complex number of unit magnitude and angle θ, the resulting complex number represents a phasor equal to the original phasor displaced by the angle θ.

The complex number of unit magnitude and associated angle θ is an operator which rotates the phasor on which it operates through the angle θ.

We are already familiar with the operator j, which causes rotation through 90°, and the operator -1, which causes rotation through 180°. Two successive applications of the operator j cause rotation through $90° + 90°$, which leads us to the conclusion that $j \times j$ causes rotation through 180°, and thus we recognize that j^2 is equal to -1. Other powers of the operator j are found by similar analysis. Some of the many combinations of the operator j are given in Table 10.1.

TABLE 10.1 FUNCTIONS OF THE OPERATOR j

$$
\begin{aligned}
j &= 1\angle 90^\circ = 1\angle -270^\circ = 0 + j1 \\
j^2 &= 1\angle 180^\circ = 1\angle -180^\circ = -1 + j0 = -1 \\
j^3 &= 1\angle 270^\circ = 1\angle -90^\circ = 0 - j1 = -j \\
j^4 &= 1\angle 360^\circ = 1\angle 0^\circ = 1 + j0 = 1 \\
j^5 &= 1\angle 450^\circ = 1\angle 90^\circ = 0 + j1 = j \\
j + j^2 &= \sqrt{2}\angle 135^\circ = \sqrt{2}\angle -225^\circ = -1 + j1 \\
j - j^2 &= \sqrt{2}\angle 45^\circ = \sqrt{2}\angle -315^\circ = 1 + j1 \\
j + j^3 &= 0 = 0 + j0 \\
j - j^3 &= 2\angle 90^\circ = 2\angle -270^\circ = 0 + j2
\end{aligned}
$$

The letter a is commonly used to designate the operator which causes a rotation of 120° in the counterclockwise direction. Such an operator is a complex number of unit magnitude with an angle of 120° and is defined by the following expressions:

$$a = 1\angle 120^\circ = 1\epsilon^{j2\pi/3} = -0.5 + j0.866$$

If the operator a is applied to a phasor twice in succession, the phasor is rotated through 240°. Three successive applications of a rotate the phasor through 360°. Thus,

$$a^2 = 1\underline{/240^\circ} = -0.5 - j0.866$$

and

$$a^3 = 1\underline{/360^\circ} = 1\underline{/0^\circ} = 1$$

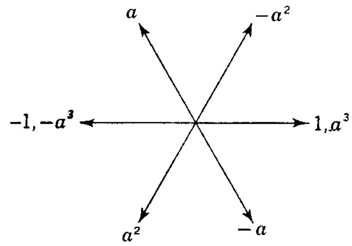


Fig. 10.3 Phasor diagram of the various powers of the operator a.

Figure 10.3 shows phasors representing various powers of a. Various combinations of the operator a are given in Table 10.2.

Table 10.2 Functions of the Operator a

$$\begin{aligned}
a &= 1\underline{/120^\circ} = -0.5 + j0.866 \\
a^2 &= 1\underline{/240^\circ} = -0.5 - j0.866 \\
a^3 &= 1\underline{/360^\circ} = 1 + j0 \\
a^4 &= 1\underline{/120^\circ} = -0.5 + j0.866 = a \\
1 + a &= 1\underline{/60^\circ} = 0.5 + j0.866 = -a^2 \\
1 - a &= \sqrt{3}\underline{/-30^\circ} = 1.5 - j0.866 \\
1 + a^2 &= 1\underline{/-60^\circ} = 0.5 - j0.866 = -a \\
1 - a^2 &= \sqrt{3}\underline{/30^\circ} = 1.5 + j0.866 \\
a + a^2 &= 1\underline{/180^\circ} = -1 - j0 \\
a - a^2 &= \sqrt{3}\underline{/90^\circ} = 0 + j1.732 \\
1 + a + a^2 &= 0 = 0 + j0
\end{aligned}$$

An important difference must be noted between the use of the operators j and a. The operator j is unit magnitude at $+90^\circ$, and $-j$ means that the complex number j is changed by an angle of 180° to give unit magnitude at 270°. Thus,

$$j = 1\underline{/90^\circ} \qquad \text{and} \qquad -j = 1\underline{/270^\circ} = 1\underline{/-90^\circ}$$

Hence, it is sometimes said that $+j$ indicates rotation through $+90^\circ$ and $-j$ indicates rotation through -90°. The statement is correct, but a similar statement does not apply to the operator a, since

$$a = 1\underline{/120^\circ}$$

but

$$-a = 1\underline{/120^\circ} \times 1\underline{/180^\circ} = 1\underline{/300^\circ} = 1\underline{/-60^\circ}$$

To clarify the situation assume that the complex number $p + jq$ is equal to $1\underline{/\theta}$, an operator causing rotation through a positive angle θ, where $\theta = \tan^{-1} q/p$. Then $p - jq$ equals $1\underline{/-\theta}$, an operator causing rotation through a negative angle θ, where $\theta = \tan^{-1} q/p$. Thus, we may form the general statement that two complex numbers of unit magnitude are operators that cause rotation through equal angles in opposite directions if the complex numbers are the conjugates of each other.

10.3 The Symmetrical Components of Unsymmetrical Phasors. We have seen (Fig. 10.2) the synthesis of three unsymmetrical phasors from three sets of symmetrical phasors. The synthesis was made in accordance with Eqs. (10.1) to (10.3). Now let us examine these same equations to determine how to resolve three unsymmetrical phasors into their symmetrical components.

First, we note that the number of unknown quantities can be reduced by expressing each component of V_a and V_b as the product of some function of the operator a and a component of V_a. Reference to Fig. 10.1 verifies the following relations:

$$\begin{aligned} V_{b1} &= a^2V_{a1} & V_{c1} &= aV_{a1} \\ V_{b2} &= aV_{a2} & V_{c2} &= a^2V_{a2} \\ V_{b0} &= V_{a0} & V_{c0} &= V_{a0} \end{aligned} \tag{10.4}$$

Upon substitution of Eqs. (10.4) in Eqs. (10.1) to (10.3), we obtain

$$V_a = V_{a1} + V_{a2} + V_{a0} \tag{10.5}$$
$$V_b = a^2V_{a1} + aV_{a2} + V_{a0} \tag{10.6}$$
$$V_c = aV_{a1} + a^2V_{a2} + V_{a0} \tag{10.7}$$

Adding Eqs. (10.5), (10.6), and (10.7) gives

$$V_a + V_b + V_c = (1 + a + a^2)V_{a1} + (1 + a + a^2)V_{a2} + 3V_{a0} \tag{10.8}$$

and, since $1 + a + a^2 = 0$,

$$V_{a0} = \tfrac{1}{3}(V_a + V_b + V_c) \tag{10.9}$$

Equation (10.9) enables us to find the zero-sequence components of three unsymmetrical phasors. We see that no zero-sequence components exist if the sum of the phasors is zero. Since the sum of the line-to-line voltage phasors in a three-phase system is always zero, zero-sequence components are never present in the line voltages, regardless of the amount of unbalance. The sum of the three line-to-neutral voltage phasors is not necessarily zero, and voltages to neutral may contain zero-sequence components.

The positive sequence components of three unsymmetrical phasors may be found by manipulation of Eqs. (10.6) and (10.7). Multiplying Eq. (10.6) by a and Eq. (10.7) by a^2 gives, after substituting 1 for a^3 and a for a^4,

$$aV_b = V_{a1} + a^2V_{a2} + aV_{a0} \tag{10.10}$$
$$a^2V_c = V_{a1} + aV_{a2} + a^2V_{a0} \tag{10.11}$$

Adding Eqs. (10.10) and (10.11) to Eq. (10.5) gives

$$V_a + aV_b + a^2V_c = 3V_{a1} + (1 + a + a^2)V_{a2} + (1 + a + a^2)V_{a0} \tag{10.12}$$

and, since $1 + a + a^2 = 0$,

$$V_{a1} = \tfrac{1}{3}(V_a + aV_b + a^2V_c) \tag{10.13}$$

Negative-sequence components may be found in a similar manner. First, we multiply Eq. (10.6) by a^2 and Eq. (10.7) by a. Adding the resulting equations to Eq. (10.5) gives

$$V_a + a^2V_b + aV_c = (1 + a + a^2)V_{a1} + 3V_{a2} + (1 + a + a^2)V_{a0} \tag{10.14}$$

and, since $1 + a + a^2 = 0$,

$$V_{a2} = \tfrac{1}{3}(V_a + a^2V_b + aV_c) \tag{10.15}$$

Equations (10.9), (10.13), and (10.15) enable us to find the complete sets of symmetrical components of a given set of three phasors since we can find the components of V_b and V_c from the components of V_a by the relations given in Eqs. (10.4). The equations could have been written for any set of related phasors, and we might have written them for currents instead of for voltages. They may be solved either analytically or graphically. Because some of the preceding equations are so fundamental, they are summarized below for currents.

$$I_a = I_{a1} + I_{a2} + I_{a0} \tag{10.16}$$
$$I_b = a^2I_{a1} + aI_{a2} + I_{a0} \tag{10.17}$$
$$I_c = aI_{a1} + a^2I_{a2} + I_{a0} \tag{10.18}$$
$$I_{a1} = \tfrac{1}{3}(I_a + aI_b + a^2I_c) \tag{10.19}$$
$$I_{a2} = \tfrac{1}{3}(I_a + a^2I_b + aI_c) \tag{10.20}$$
$$I_{a0} = \tfrac{1}{3}(I_a + I_b + I_c) \tag{10.21}$$

In a three-phase system the sum of the line currents is equal to the current I_n in the return path through the neutral. Thus,

$$I_a + I_b + I_c = I_n \tag{10.22}$$

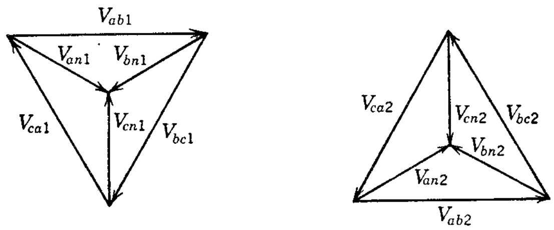
Comparing Eqs. (10.21) and (10.22) gives

$$I_n = 3I_{a0} \tag{10.23}$$

In the absence of a path through the neutral of a three-phase system, I_n is zero, and the line currents contain no zero-sequence components. A Δ-connected load provides no path to neutral, and the line currents flowing to a Δ-connected load can contain no negative-sequence components.

10.4 Determination of Phase Voltages from Unsymmetrical Line Voltages. When unsymmetrical three-phase line voltages are applied to the terminals of a balanced Y-connected load, the method of symmetrical components provides one means of determining the voltage to neutral of each phase if no zero-sequence currents flow in the impedances. The voltages to neutral caused by the positive-sequence currents will be of positive sequence only, because each component of each voltage to neutral will be displaced equally in phase from the component of current causing it. Similarly, negative-sequence currents will cause only negative-sequence voltage drops in a balanced system. If no zero-sequence currents are present, there can be no zero-sequence voltage drops in the balanced Y-connected impedances.

We proceed to resolve the line voltages into their symmetrical components and to determine the voltages to neutral for each sequence from the known relations between line and phase voltages when balanced



Positive-sequence components   Negative-sequence components

FIG. 10.4 Positive- and negative-sequence components of line-to-line and line-to-neutral voltages of a three-phase system.

voltages are applied to balanced loads. The positive- and negative-sequence components of line-to-line and line-to-neutral voltages are shown in Fig. 10.4, from which we see that

$$V_{an1} = \frac{1}{\sqrt{3}} V_{ab1}\underline{/-30^\circ} \tag{10.24}$$

and

$$V_{an2} = \frac{1}{\sqrt{3}} V_{ab2}\underline{/30^\circ} \tag{10.25}$$

We find V_{an} as the sum of its components. So,

$$V_{an} = V_{an1} + V_{an2} \tag{10.26}$$

The other voltages to neutral are found by obtaining their components from V_{an1} and V_{an2} by Eqs. (10.4). If the voltages to neutral are in per unit referred to the base voltage to neutral and the line voltages are in per unit referred to the base voltage from line to line, the $1/\sqrt{3}$ factor must be omitted in Eqs. (10.24) and (10.25). If both voltages are referred to the same base or are in actual volts, the equations are correct as given.

Example 10.1

Three identical resistors are Y-connected and rated 2,300 volts, 500 kva as a three-phase unit. The neutral point is not available. The resistor unit is connected to an unsymmetrical three-phase system whose line voltages are measured and found to be

$$|V_{ab}| = 1{,}840 \text{ volts} \qquad |V_{bc}| = 2{,}760 \text{ volts} \qquad |V_{ca}| = 2{,}300 \text{ volts}$$

Find the current in each line by the method of symmetrical components.

Solution

Select as base for the circuit 2,300 volts, 500 kva so that each resistor in the three-phase load has a resistance of 1.0 per unit. The base current is

$$\frac{500{,}000}{\sqrt{3} \times 2{,}300} = 125.5 \text{ amp}$$

and

$$|V_{ab}| = \frac{1{,}840}{2{,}300} = 0.8 \text{ per unit}$$

$$|V_{bc}| = \frac{2{,}760}{2{,}300} = 1.2 \text{ per unit}$$

$$|V_{ca}| = \frac{2{,}300}{2{,}300} = 1.0 \text{ per unit}$$

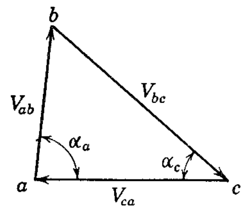


FIG. 10.5 Phasor diagram of the line voltages of Example 10.1.

Since the sum of the line voltages in a three-phase circuit is zero, the triangle formed by the phasors of the line voltages can be solved to find the angle of each phasor. The triangle of line-voltage phasors is shown in Fig. 10.5, where V_{ca} is taken as reference and the phase sequence is *abc*. Any phase angle could have been assumed for any one of the voltages. The angles of the phasors can be found by the law of cosines, as follows:

$$1.44 = 1.0 + 0.64 - 1.6 \cos \alpha_a$$

$$\cos \alpha_a = \frac{0.20}{1.6} = 0.125$$

$$\alpha_a = 82.8°$$

$$0.64 = 1.44 + 1.0 - 2.4 \cos \alpha_c$$

$$\cos \alpha_c = \frac{1.80}{2.40} = 0.75$$

$$\alpha_c = 41.4°$$

Then

$$V_{ab} = 0.8\underline{/82.8°} \text{ per unit}$$

$$V_{bc} = 1.2\underline{/-41.4°} \text{ per unit}$$

$$V_{ca} = 1.0\underline{/180°} \text{ per unit}$$

The symmetrical components of the line voltages are

$$\begin{aligned}
V_{ab1} &= \tfrac{1}{3}(0.8\angle 82.8^\circ + 1.2\angle 120^\circ - 41.4^\circ + 1.0\angle 240^\circ + 180^\circ) \\
&= \tfrac{1}{3}(0.1 + j0.794 + 0.237 + j1.177 + 0.5 + j0.866) \\
&= 0.279 + j0.946 = 0.985\angle 73.6^\circ \text{ per unit} \\
V_{ab2} &= \tfrac{1}{3}(0.8\angle 82.8^\circ + 1.2\angle 240^\circ - 41.4^\circ + 1.0\angle 120^\circ + 180^\circ) \\
&= \tfrac{1}{3}(1.0 + j0.794 - 1.138 - j0.383 + 0.5 - j0.866) \\
&= -0.179 - j0.152 = 0.235\angle 220.3^\circ \text{ per unit}
\end{aligned}$$

Check:

$$\begin{aligned}
&V_{ab} = V_{ab1} + V_{ab2} \\
&0.1 + j0.794 = 0.279 + j0.946 - 0.179 - j0.152 \\
&0.1 + j0.794 = 0.1 + j0.794
\end{aligned}$$

The absence of a neutral connection means that zero-sequence currents are not present. Therefore, the phase voltages at the load contain positive- and negative-sequence components only. The phase voltages are found from Eqs. (10.24) and (10.25) with the factor $1/\sqrt{3}$ omitted, since the line voltages are expressed in terms of the base voltage from line to line and the phase voltage is desired in per unit of the base voltage to neutral. Thus

$$\begin{aligned}
V_{an1} &= 0.985\angle 73.6^\circ - 30^\circ \\
&= 0.985\angle 43.6^\circ &&= 0.713 + j0.680 \\
V_{an2} &= 0.235\angle 220.3^\circ + 30^\circ \\
&= 0.235\angle 250.3^\circ &&= -0.079 - j0.221 \\
&\quad V_{an} = V_{an1} + V_{an2} &&= 0.634 + j0.459 = 0.783\angle 35.9^\circ \text{ per unit} \\
V_{bn1} &= 0.985\angle 43.6^\circ + 240^\circ &&= 0.232 - j0.958 \\
V_{bn2} &= 0.235\angle 250.3^\circ + 120^\circ &&= 0.231 + j0.042 \\
&\quad V_{bn} = V_{bn1} + V_{bn2} &&= 0.463 - j0.916 = 1.027\angle -63.2^\circ \text{ per unit} \\
V_{cn1} &= 0.985\angle 43.6^\circ + 120^\circ &&= -0.945 + j0.278 \\
V_{cn2} &= 0.235\angle 250.3^\circ + 240^\circ &&= -0.152 + j0.179 \\
&\quad V_{cn} = V_{cn1} + V_{cn2} &&= -1.097 + j0.457 = 1.19\angle 157.3^\circ \text{ per unit}
\end{aligned}$$

The currents in each line are equal to the per-unit voltages to neutral divided by the per-unit resistance in each phase. Thus

$$I_{an} = \frac{0.783\angle 35.9^\circ}{1\angle 0^\circ} \times 125.5 = 98.4\angle 35.9^\circ \text{ amp}$$

$$I_{bn} = \frac{1.027\angle -63.2^\circ}{1\angle 0^\circ} \times 125.5 = 129.0\angle -63.2^\circ \text{ amp}$$

$$I_{cn} = \frac{1.19\angle 157.3^\circ}{1\angle 0^\circ} \times 125.5 = 148.8\angle 157.3^\circ \text{ amp}$$

Example 10.1 could have been solved by transforming the Y-connected load to its equivalent Δ, but the example serves to illustrate the symmetrical-component method. Equations (10.24) and (10.25) show that the phase voltages of positive sequence are displaced from the reference positive-sequence line voltage in the direction opposite to the displacement of the phase voltages of negative sequence from the reference negative-sequence line voltage. This important principle will be encountered later in our discussion of phase shift in Y-Δ transformer banks.

10.5 Relations between Line and Phase Currents in Δ-connected Circuits. A relation similar to that between line and phase voltages exists between Δ phase currents and their related line currents. A Δ load of identical impedances connected to unsymmetrical line voltages results in unbalanced line currents. The line currents, as noted in Sec. 10.4, contain no zero-sequence components since the sum of the line

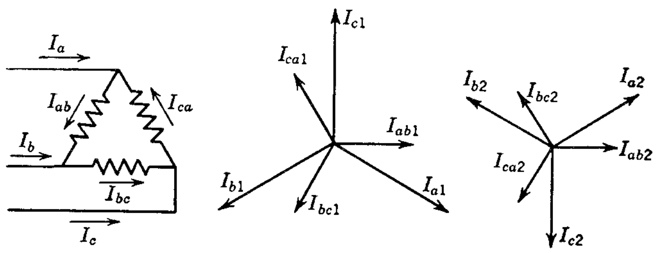


FIG. 10.6 Wiring diagram of a Δ-connected load and phasor diagrams of the positive- and negative-sequence components of line and phase currents.

currents must be zero due to the absence of a return path. Figure 10.6 shows a Δ-connected load and the phasor diagrams of positive- and negative-sequence line and phase currents. Inspection of the diagrams shows the following relations between the components of I_a and I_{ab}:

$$I_{ab1} = \frac{1}{\sqrt{3}} I_{a1}\underline{/30^\circ} \tag{10.27}$$

$$I_{ab2} = \frac{1}{\sqrt{3}} I_{a2}\underline{/-30^\circ} \tag{10.28}$$

Example 10.2

The line currents entering a balanced Δ load are $I_a = 5$ amp, $I_b = 4$ amp, and $I_c = 3$ amp. Find the current in phase ab of the load by the method of symmetrical components.

Solution

If the current in phase c is taken as reference, the line currents are:

$$\begin{aligned} I_a &= -3 - j4 \text{ amp} \\ I_b &= 0 + j4 \text{ amp} \\ I_c &= 3 + j0 \text{ amp} \end{aligned}$$

The symmetrical components of the line currents are

$$
\begin{aligned}
I_{a1} &= \tfrac{1}{3}(-3 - j4 + 4\angle 90^\circ + 120^\circ + 3\angle 240^\circ) \\
&= \tfrac{1}{3}(-3 - j4 - 3.465 - j2 - 1.5 - j2.6) \\
&= -2.655 - j2.867 = 3.9\angle 227.25^\circ \text{ amp} \\
I_{a2} &= \tfrac{1}{3}(-3 - j4 + 4\angle 90^\circ + 240^\circ + 3\angle 120^\circ) \\
&= \tfrac{1}{3}(-3 - j4 + 3.465 - j2 - 1.5 + j2.6) \\
&= -0.345 - j1.133 = 1.19\angle 253.1^\circ \text{ amp}
\end{aligned}
$$

From Eqs. (10.27) and (10.28)

$$I_{ab1} = \frac{3.9}{\sqrt{3}}\angle 257.25^\circ = -0.496 - j2.20 \text{ amp}$$

$$I_{ab2} = \frac{1.19}{\sqrt{3}}\angle 223.1^\circ = -0.501 - j0.470 \text{ amp}$$

If the load contains no emf to induce zero-sequence current, I_{ab} is composed of positive- and negative-sequence current only, and

$$I_{ab} = I_{ab1} + I_{ab2} = -0.997 - j2.670 = 2.85\angle 249.5^\circ \text{ amp}$$

10.6 Phase Shift in Y-Δ Transformer Banks. Section 10.4 has shown the phase shift between the symmetrical components of line and phase voltages for balanced Y-connected loads. Section 10.5 has shown the phase shift between the symmetrical components of line and phase currents for balanced Δ-connected loads. A phase shift occurs in the current and voltage between one side of a Y-Δ transformer and the other side. The single-phase equivalent circuit of such a transformer does not take into account the phase shift except in so far as the phase shift of voltage is due to the impedance of the transformer. If the resistance, leakage reactance, and magnetizing current of the transformer are neglected, the solution of the equivalent circuit shows the same per-unit voltages and the same per-unit currents on both sides of the transformer. The usual procedure is to calculate the currents and voltages without regard to phase shift caused by the Y-Δ connection of transformers. If phase shift is of importance, it can be taken into account in a manner about to be discussed. We shall learn how to account for phase shift in a Y-Δ transformer by determining the phase shift in the symmetrical components of voltage and current on opposite sides of the transformer with impedance neglected.

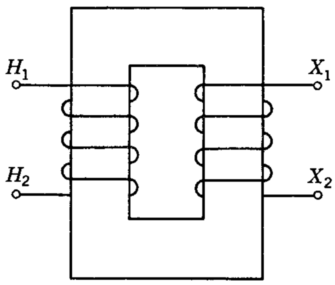


FIG. 10.7 Standard markings of single-phase transformer windings.

Before proceeding with the discussion of three-phase transformers, let us examine the standard method of marking transformer terminals.

Consider the primary and secondary windings shown on a common core in Fig. 10.7. The high-tension winding is marked H_1 and H_2, and the low-tension winding is marked X_1 and X_2. Current flowing from H_1 to H_2 tends to produce flux in the common core in the same direction as current flowing from X_1 to X_2. Transformer theory shows that current must flow out at terminal X_1 when it flows into terminal H_1, with magnetizing current neglected. Without such a standard convention for marking windings, a schematic diagram like Fig. 10.8 would not indicate whether the currents I_s and I_p were in phase with each other or 180° out of phase. With the standard markings shown on Fig. 10.8 we know that I_s and I_p are in phase. Terminals H_1 and X_1 are positive at the same time with respect to H_2 and X_2. If the direction of the arrow marked I_s were reversed while the direction of the arrow marked I_p remained the same, I_s and I_p would be 180° out of phase. Therefore, the primary and secondary currents are either in phase or 180° out of phase depending on the direction assumed to be positive for the flow of current. Similarly, primary and secondary voltages may be in phase or 180° out of phase depending on which terminal is assumed to be positive for specifying voltage drop.

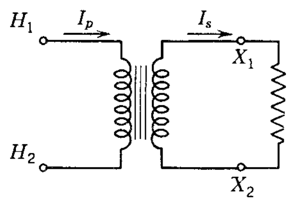


FIG. 10.8 Schematic diagram of single-phase transformer windings showing standard markings and the directions assumed positive for primary and secondary current.

The high-tension terminals of three-phase transformers are marked H_1, H_2, and H_3, and the low-tension terminals are marked X_1, X_2, and X_3. In Y-Y or Δ-Δ transformers the markings are such that voltages to neutral from terminals H_1, H_2, and H_3 are in phase with the voltages to neutral from terminals X_1, X_2, and X_3, respectively.

Figure 10.9*a* is the wiring diagram of a Y-Δ transformer. The high-tension terminals H_1, H_2, and H_3 are connected to phases a, b, and c. The arrangement and notation of the diagram conform to a convention which we will follow in all our computations. Windings which are drawn in parallel directions are those linked magnetically by being wound on the same core. The winding an is the phase on the Y-connected side which is linked magnetically with the phase winding BC on the Δ-connected side. When capital letters are assigned to phases on one side of the transformer, lower-case letters will be assigned to [illegible] on the other side. Once phase designations are arbitrarily assigned to any two terminals on one side of the transformer, the phases to be assigned to the other terminals are definitely determined. To follow our adopted convention the phase sequence is to be abc on one side and ABC on the other, and phase a on the Y side must be linked magnetically with phase BC on the Δ side. Likewise, phase b is linked magnetically with phase CA, and phase c with phase AB. Of course, capital letters could be used for either the Y or the Δ windings. If H_1 is the terminal to which line a is connected, it is customary to connect phase b to H_2 and c to H_3.

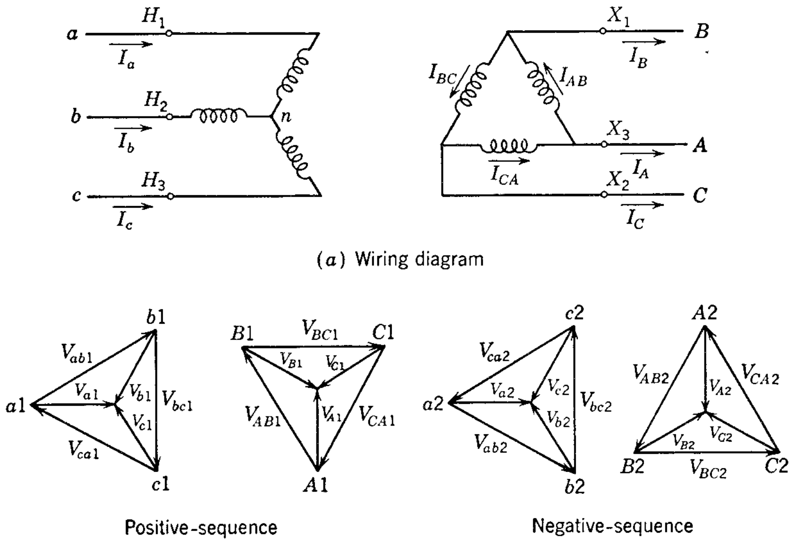
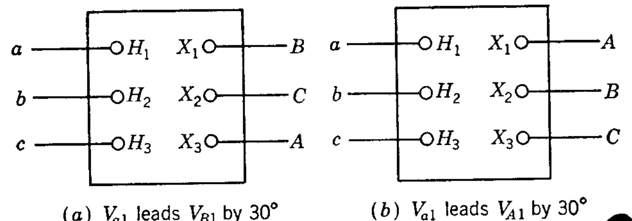


Fig. 10.9 Wiring diagram and voltage phasors for a three-phase transformer connected Y-Δ.

The phasor diagrams for the sequence components of voltage are shown in Fig. 10.9b. We see that V_{a1} leads V_{B1} by 30°, which enables us to determine that the terminal to which phase B is connected should be labeled X_1. The labeling of X_1 conforms to the American standard for three-phase transformers, which requires that the voltage drop from



(a) V_{a1} leads V_{B1} by 30° (b) V_{a1} leads V_{A1} by 30°

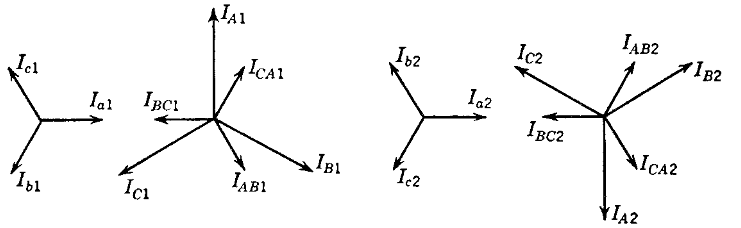
Fig. 10.10 Labeling of lines connected to a three-phase Y-Δ transformer.

H_1 to neutral lead the voltage drop from X_1 to neutral in Y-Δ transformers, regardless of whether the Y or the Δ windings is on the high-tension side. Similarly, the voltage at H_2 leads that at X_2 by 30°, and the voltage at H_3 leads that at X_3 by 30°.

Figure 10.10a shows the connections of the phases to the transformer so that the positive-sequence voltage to neutral V_{a1} leads the positive-sequence voltage to neutral B_{B1} by 30°. The American standard for naming the phases connected to a Y-Δ transformer with the high-tension side Y-connected is shown in Fig. 10.10*b* and results in V_{a1} leading V_{A1} by 30°. We will follow the scheme of Fig. 10.10*a*, which conforms to the wiring and phasor diagrams of Fig. 10.9, since such nomenclature is the most convenient for computations.

Inspection of the phasor diagrams of Fig. 10.9 shows that V_{A1} leads V_{a1} by 90° and that V_{A2} lags V_{a2} by 90°. The diagrams show V_{a1} and V_{a2} in phase, which is not necessarily true, but phase shift between V_{a1} and V_{a2} does not alter the 90° relation between V_{A1} and V_{a1} or between V_{A2} and V_{a2}.

Transformer theory shows that I_a and I_{BC} are 180° out of phase if V_a and V_{BC} are in phase. Therefore, the phase relation between the Y and Δ currents are as shown in Fig. 10.11. We note that I_{A1} leads I_{a1} by 90°



Positive-sequence components   Negative-sequence components

FIG. 10.11 Current phasors of a three-phase transformer connected Y-Δ.

and I_{A2} lags I_{a2} by 90°. Summarizing the relations between the symmetrical components of the line-to-neutral voltages and between the symmetrical components of the line currents on the two sides of the transformer gives

$$\begin{aligned} V_{A1} &= +jV_{a1} \qquad & I_{A1} &= +jI_{a1} \\ V_{A2} &= -jV_{a2} \qquad & I_{A2} &= -jI_{a2} \end{aligned} \tag{10.29}$$

where each voltage and current is expressed in per unit. Transformer impedance is neglected.

The phases connected to the windings could have been so named that V_{BC1} and V_{a1} would be 180° out of phase by interchanging phases a and c on the Y side at the same time A and C are interchanged on the Δ side. The following equations specify the resulting phase shift of [illegible] current:

$$\begin{aligned} V_{A1} &= -jV_{a1} \qquad & I_{A1} &= -jI_{a1} \\ V_{A2} &= +jV_{a2} \qquad & I_{A2} &= +jI_{a2} \end{aligned} \tag{10.30}$$

When we find it necessary to account for the phase shift in Y-Δ transformers, we will name the phases so that Eqs. (10.29) or (10.30) apply.

Y-Y and Δ-Δ transformers are connected so that the phase shift for both currents and voltages is either 0° or 180°.

Example 10.3

If the resistors of Example 10.1 are connected to the Y side of a Δ-Y transformer whose output voltages are those specified in Example 10.1, find the line voltages and currents on the Δ side in per unit. Use the same base on the Y side as in Example 10.1, and neglect the impedance of the transformer.

Solution

From Example 10.1,

$$V_{an1} = V_{a1} = 0.985\underline{/43.6^\circ} \text{ per unit}$$
$$V_{an2} = V_{a2} = 0.235\underline{/250.3^\circ} \text{ per unit}$$

Therefore,

$$I_{a1} = \frac{V_{a1}}{1.0\underline{/0^\circ}} = 0.985\underline{/43.6^\circ} \text{ per unit}$$

$$I_{a2} = \frac{V_{a2}}{1.0\underline{/0^\circ}} = 0.235\underline{/250.3^\circ} \text{ per unit}$$

The direction assumed to be positive for the currents is from the supply toward the Δ primary of the transformer and away from the Y side toward the load. The phasor diagrams are similar to those of Fig. 10.9*a* and Fig. 10.11 except that the directions of all current phasors are reversed.

By Eqs. (10.29)

$$V_{A1} = jV_{a1} = 0.985\underline{/133.6^\circ} = -0.680 + j0.713$$
$$V_{A2} = -jV_{a2} = 0.235\underline{/160.3^\circ} = -0.221 + j0.079$$
$$V_A = V_{A1} + V_{A2} = -0.901 + j0.792 = 1.20\underline{/138.6^\circ} \text{ per unit}$$
$$V_{B1} = a^2V_{A1} = 0.985\underline{/373.6^\circ} = 0.958 + j0.232$$
$$V_{B2} = aV_{A2} = 0.235\underline{/280.3^\circ} = 0.042 - j0.232$$
$$V_B = V_{B1} + V_{B2} = 1.00 + j0 = 1.0\underline{/0^\circ} \text{ per unit}$$
$$V_{C1} = aV_{A1} = 0.985\underline{/253.6^\circ} = -0.278 - j0.944$$
$$V_{C2} = a^2V_{A2} = 0.235\underline{/400.3^\circ} = 0.179 + j0.152$$
$$V_C = V_{C1} + V_{C2} = -0.099 - j0.792 = 0.8\underline{/262}\text{[illegible] per unit}$$

$$V_{AB} = V_A - V_B = -0.901 + j0.792 - 1.0 = -1.901\text{[illegible]}$$
$$= 2.06\underline{/157.3^\circ} \text{ per unit (phase-to-neutral voltage [illegible])}$$
$$= \frac{2.06}{\sqrt{3}}\underline{/157.3^\circ} = 1.19\underline{/157.3^\circ} \text{ per unit (line vo[illegible]ge base)}$$
$$V_{BC} = V_B - V_C = 1.0 + 0.099 + j0.792 = 1.\text{[illegible]} + j0.79\text{[illegible]}$$
$$= 1.355\underline{/35.8^\circ} \text{ per unit (phase-to-neutral vo[illegible]age base)}$$

$$= \frac{1.355}{\sqrt{3}} \underline{/35.8^\circ} = 0.783\underline{/35.8^\circ} \text{ per unit (line voltage base)}$$

$$V_{CA} = V_C - V_A = -0.099 - j0.792 + 0.901 - j0.792 = 0.802 - j1.584$$
$$= 1.78\underline{/296.8^\circ} \text{ per unit (phase-to-neutral voltage base)}$$
$$= \frac{1.78}{\sqrt{3}} \underline{/243.2^\circ} = 1.027\underline{/243.2^\circ} \text{ per unit (line voltage base)}$$

Since the load impedance in each phase is resistance of $1.0\underline{/0^\circ}$ per unit, I_{a1} and V_{a1} are found to have identical per-unit values in this problem. Likewise, I_{a2} and V_{a2} are identical in per unit. Therefore, I_A must be identical to V_A expressed in per unit. Thus

$$I_A = 1.20\underline{/138.6^\circ} \text{ per unit}$$
$$I_B = 1.0\underline{/0} \text{ per unit}$$
$$I_C = 0.80\underline{/262.8} \text{ per unit}$$

When problems involving unsymmetrical faults are solved on a calculating board, positive- and negative-sequence components are read separately, and phase shift is taken into account, if necessary, by applying Eqs. (10.29) or (10.30). Further examples of phase shift through Y-Δ transformers will be found in Chap. 13. In that chapter the method of accounting for transformer impedance in relation to phase shift of voltages will be discussed.

PROBLEMS

10.1 Evaluate the following expressions in polar form:

(a) $a^2 - 1$ (c) $2a^2 + 3 + 2a$
(b) $1 - a - a^2$ (d) ja

10.2 Determine analytically the voltages to neutral V_{an}, V_{bn}, and V_{cn} in a circuit where $V_{an1} = 50\underline{/0^\circ}$, $V_{an2} = 10\underline{/90^\circ}$, and $V_{an0} = 10\underline{/180^\circ}$ volts.

10.3 Solve Prob. 10.2 graphically.

10.4 Determine the symmetrical components of the three currents $I_a = 10\underline{/0^\circ}$, $I_b = 10\underline{/250^\circ}$, and $I_c = 10\underline{/110^\circ}$ amp.

10.5 One conductor of a three-phase line is open. The current flowing to the Δ-connected load in line a is 10 amp. With the current in line a as reference and assuming line c to be open, find (a) the symmetrical components of the line currents, (b) the positive-sequence current in line c, (c) the negative-sequence current i[illegible]e c, (d) the zero-sequence current in line c, (e) the total of parts (b), (c), and [illegible]

10.6 The currents flowing in the line feeding a balanced loa[illegible]ed in Δ are $I_a = 100\underline{/0^\circ}$, $I_b = 141.4\underline{/225^\circ}$, and $I_c = 100\underline{/90^\circ}$ amp. Find th[illegible]rent in phase ab of the load by symmetrical components.

10.7 The voltages at the terminals of a balanced load c[illegible]sting of three 10-ohm resistors connected in Y are $V_{ab} = 100\underline{/0^\circ}$, $V_{bc} = 90\underline{/240}$[illegible] and $V_{ca} = 95.5\underline{/125.2^\circ}$ volts. Find the current in line a by symmetrical com[illegible]ts.

10.8 Assume that the currents specified in Prob. 10[illegible] are flowing toward a load from lines connected to the Y side of a Δ-Y transfor[illegible] rated 10,000 kva, 13.2Δ-

66Y kv. Determine the currents flowing in the lines on the Δ side by converting the symmetrical components of the currents to per unit on the base of the transformer rating and by shifting the components according to Eqs. (10.29) or (10.30). Check the results by computing the currents in each phase of the Δ windings in amperes directly from the currents on the Y side by multiplying by the turns ratio of the windings. Complete the check by computing the line currents from the phase currents on the Δ side.

CHAPTER 11

UNSYMMETRICAL SHORT CIRCUITS ON AN UNLOADED GENERATOR

11.1 Introduction. Most of the faults that occur on power systems are unsymmetrical faults, which may consist of unsymmetrical short circuits, unsymmetrical faults through impedances, or open conductors. Unsymmetrical faults occur as single line-to-ground faults, line-to-line faults, or double line-to-ground faults. The path of the fault current from line to line or line to ground may or may not contain impedance. One or two open conductors result in unsymmetrical faults either through the breaking of one or two conductors or through the action of fuses and other devices which may not open the three phases simultaneously.

Since any unsymmetrical fault causes unbalanced currents to flow in the system, the method of symmetrical components is very useful in an analysis to determine the currents and voltages in all parts of the system after the occurrence of the fault. In this chapter we shall discuss line-to-line faults and faults between one or two lines and ground at the terminals of an unloaded generator at no load. We shall postpone until Chap. 13 the discussion of faults on loaded generators and on systems containing more than one emf, whether loaded or not. Unsymmetrical faults resulting from open conductors will also be discussed in Chap. 13.

11.2 Fundamental Relations. The study of a fault at the terminals of a generator which is disconnected from a power system provides an elementary approach to the somewhat more involved study of a fault in a complex network fed by a number of power sources. We can derive some fundamental equations which apply to such an isolated generator regardless of the type of fault at its terminals.

An unloaded generator, grounded through a reactor, is shown in Fig. 11.1. When a fault (not indicated in the figure) occurs at the terminals of the generator, currents I_a, I_b, and I_c flow in the lines. If the fault involves ground, the current flowing into the neutral of the generator is designated I_n. One or two of the line currents may be zero, but the currents can be resolved into their symmetrical components regardless of how unbalanced they may be. As we saw in Chap. 10, currents of a given phase sequence in a balanced system produce voltage drops of the same sequence only. In any part of a circuit, the voltage drop caused by current of a certain sequence depends on the impedance of that part of the circuit to current of that sequence. The impedance of any section of a balanced network to current of one sequence may be different from the impedance to current of another sequence. In Chap. 12 we shall discuss the reasons for different impedances to the flow of currents of different phase sequence. For our present discussion we need know only that impedances may differ according to the phase sequence of the currents flowing.

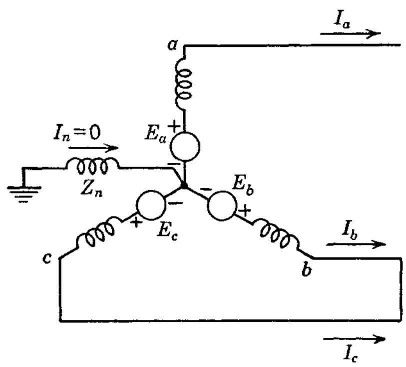
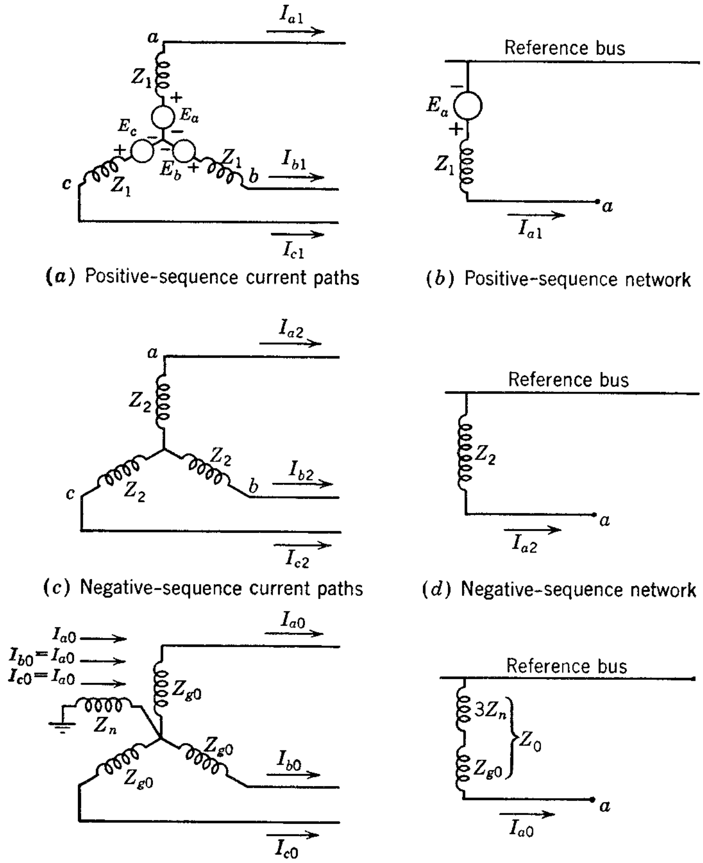


FIG. 11.1 Circuit diagram of an unloaded generator grounded through a reactance. The emfs of each phase are E_a, E_b, and E_c.

The impedance of a circuit when positive-sequence currents alone are flowing is called the *impedance to positive-sequence current.* Similarly, when only negative-sequence currents are present, the impedance is called the *impedance to negative-sequence current.* When only zero-sequence currents are present, the impedance is called the *impedance to zero-sequence current.* These names of the impedances of a circuit to currents of the different sequences are usually shortened to the less descriptive terms, *positive-sequence impedance, negative-sequence impedance,* and *zero-sequence impedance.*

The analysis of an unsymmetrical fault on a symmetrical system consists of finding the symmetrical components of the unbalanced currents which are flowing. Since the component currents of one phase sequence cause voltage drops of like sequence only and are independent of currents of other sequences, in a balanced system, currents of any one sequence may be considered to flow in an independent network composed of the impedances to the current of that sequence only. The single-phase equivalent circuit composed of the impedances to current of any one sequence only is called the *sequence network* for that particular sequence. The sequence network includes any gene[illegible]ted [illegible]fs of like sequence. Sequence networks carrying the curr[illegible]ts [illegible]$_{a2}$, and I_{a0} are interconnected to represent various unbalanced [illegible] conditions. Therefore, to calculate the effect of a fault by the m[illegible] of symmetrical components, it is essential to determine the sequen[illegible]mpedances and to combine them to form the sequence networks.

In this chapter our task is simple because one [illegible]enerator [illegible] an impedance in the neutral comprise the entire circuit. The generated voltages are of positive sequence only, since the generator is designed to supply balanced three-phase voltages. Therefore the positive-sequence network is composed of an emf in series with the positive-sequence impedance of the generator. The negative- and zero-sequence networks



(*a*) Positive-sequence current paths (*b*) Positive-sequence network

(*c*) Negative-sequence current paths (*d*) Negative-sequence network

(*e*) Zero-sequence current paths (*f*) Zero-sequence network

FIG. 11.2 Paths for current of each sequence in a generator, and the corresponding sequence networks.

contain no emfs but include the impedances of the generator to negative- and zero-sequence currents, respectively. The sequence components of current are shown in Fig. 11.2. They are flowing through impedances of their own sequence only, as indicated by the appropriate subscripts on the impedances shown in the figure. The sequence networks shown in Fig. 11.2 are the single-phase equivalent circuits of the balanced three-phase circuits through which the symmetrical components of the unbalanced currents are considered to flow. The generated emf in the positive-sequence network is the no-load terminal voltage to neutral, which is also equal to the voltages behind transient and subtransient reactances and to the voltage behind synchronous reactance since the generator is not loaded. The reactance in the positive-sequence network is the subtransient, transient, or synchronous reactance, depending on whether subtransient, transient, or steady-state conditions are being studied.

The reference bus for the positive- and negative-sequence networks is the neutral of the generator. So far as positive- and negative-sequence components are concerned the neutral of the generator is at ground potential since only zero-sequence current flows in the impedance between neutral and ground. The reference bus for the zero-sequence network is the ground at the generator.

The current flowing in the impedance Z_n between neutral and ground is $3I_{a0}$. By referring to Fig. 11.2*e*, we see that the voltage *drop* of zero sequence *from point a to ground* is $-3I_{a0}Z_n - I_{a0}Z_{g0}$, where Z_{g0} is the zero-sequence impedance per phase of the generator. The zero-sequence network, which is a single-phase circuit assumed to carry only the zero-sequence current of one phase, must, therefore, have an impedance of $3Z_n + Z_{g0}$, as shown in Fig. 11.2*f*. The total zero-sequence impedance through which I_{a0} flows is

$$Z_0 = 3Z_n + Z_{g0} \tag{11.1}$$

Usually the components of current and voltage for phase a are found from equations determined by the sequence networks. The equations for the components of voltage drop from point a of phase a to the reference bus (or ground) are, as may be deduced from Fig. 11.2,

$$V_{a1} = E_a - I_{a1}Z_1 \tag{11.2}$$
$$V_{a2} = -I_{a2}Z_2 \tag{11.3}$$
$$V_{a0} = -I_{a0}Z_0 \tag{11.4}$$

where E_a is the positive-sequence no-load voltage to neutral, Z_1 and Z_2 are the positive- and negative-sequence impedances of the generator, and Z_0 is defined by Eq. (11.1). The above equations, which apply to any generator carrying unbalanced currents, are the starting points for the derivation of equations for the components of current for different types of faults. They apply to the case of a loaded generator if E_a is given the value computed for the voltage behind subtransient, transient, or synchronous reactance for the load existing before the fault.

11.3 Single Line-to-ground Fault on an Unloaded Generator. The circuit diagram for a single line-to-ground fault on an unloaded Y-connected generator with its neutral grounded through a reactance is shown in Fig. 11.3, where phase a is the one on which the fault occurs. The relations to be developed for this type of fault will apply only when the fault is on phase a, but this should cause no difficulty since the phases are labeled arbitrarily and any phase may be designated as phase a. The conditions at the fault are expressed by the following equations:

$$I_b = 0 \qquad I_c = 0 \qquad V_a = 0$$

When $I_b = 0$ and $I_c = 0$ are substituted in Eqs. (10.19) to (10.21), we obtain

$$I_{a1} = \tfrac{1}{3}(I_a + aI_b + a^2I_c) = \frac{I_a}{3}$$

$$I_{a2} = \tfrac{1}{3}(I_a + a^2I_b + aI_c) = \frac{I_a}{3}$$

and

$$I_{a0} = \tfrac{1}{3}(I_a + I_b + I_c) = \frac{I_a}{3}$$

Therefore,

$$I_{a1} = I_{a2} = I_{a0} \qquad (11.5)$$

By Eq. (10.5), since $V_a = 0$,

$$V_a = V_{a1} + V_{a2} + V_{a0} = 0$$

and

$$V_{a1} = -V_{a2} - V_{a0}$$

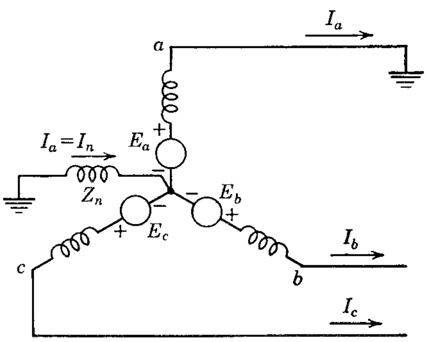


FIG. 11.3 Circuit diagram for a single line-to-ground fault on phase a at the terminals of an unloaded generator whose neutral is grounded through a reactance.

Then, by Eq. (11.2),

$$V_{a1} = -V_{a2} - V_{a0} = E_a - I_{a1}Z_1$$

and from Eqs. (11.3) and (11.4)

$$I_{a2}Z_2 + I_{a0}Z_0 = E_a - I_{a1}Z_1$$

but, since $I_{a1} = I_{a2} = I_{a0}$,

$$I_{a1}Z_2 + I_{a1}Z_0 = E_a - I_{a1}Z_1$$

and, solving for I_{a1}, we obtain

$$I_{a1} = \frac{E_a}{Z_1 + Z_2 + Z_0} \qquad (11.6)$$

Equations (11.5) and (11.6) are the special equations for a single line-to-ground fault. They are used with Eqs. (11.2) to (11.4) together with the symmetrical-component relations to determine all the voltages and currents at the fault. If the three sequence networks of Fig. 11.2 are connected in series as shown in Fig. 11.4, we see that the currents and voltages resulting therefrom satisfy the equations above, for the three sequence impedances are then in series with the voltage E_a. With the sequence networks so connected, the voltage across each sequence network is the symmetrical component of V_a of that sequence. The connection of the sequence networks as shown in Fig. 11.4 is a convenient means of remembering the equations for the solution of the single line-to-ground fault, for all the necessary equations can be determined from the sequence network connection.

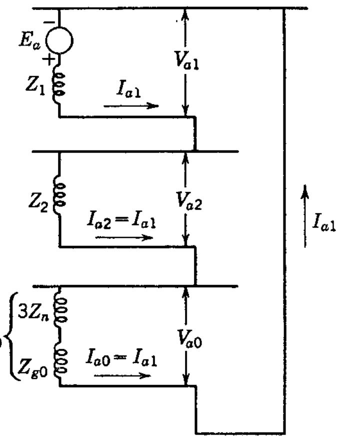


FIG. 11.4 Connection of the sequence networks of an unloaded generator for a single line-to-ground fault on phase a at the terminals of the generator.

If the neutral of the generator is not grounded, the zero-sequence network is open-circuited, and Z_0 is infinite. Since Eq. 11.6 shows that I_{a1} is zero when Z_0 is infinite, I_{a2} and I_{a0} must be zero. Thus no current flows in line a since I_a is the sum of its components, all of which are zero. The same result can be seen without the use of symmetrical components since inspection of the circuit shows that no path exists for the flow of current in the fault unless there is a ground at the generator neutral.

Example 11.1

A 20,000-kva, 13.8-kv generator has a direct-axis subtransient reactance of 0.25 per unit. The negative- and zero-sequence reactances are, respectively, 0.35 and 0.10 per unit. The neutral of the generator is solidly grounded. Determine the subtransient current in the generator and the line-to-line voltages for subtransient conditions when a single line-to-ground fault occurs at the generator terminals with the generator operating unloaded at rated voltage. Neglect resistance.

Solution

On a base of 20,000 kva, 13.8 kv, $E_a = 1.0$ per unit, since the internal voltage is equal to the terminal voltage at no load.

Then, in per unit,

$$I_{a1} = \frac{E_a}{Z_1 + Z_2 + Z_0} = \frac{1.0 + j0}{j0.25 + j0.35 + j0.10} = -\text{[illegible]} \text{ per unit}$$

$$I_a = 3I_{a1} = -j4.29 \text{ per unit}$$

$$\text{Base current} = \frac{20,000}{\sqrt{3} \times 13.8} = 83\text{[illegible]} \text{ amp}$$

Subtransient current in line a is $I_a = -j4.29 \times$ [illegible] $= -j3,585$ amp

The symmetrical components of the voltage from point a to ground are:

$$\begin{aligned}
V_{a1} &= E_a - I_{a1}Z_1 = 1.0 - (-j1.43)(j0.25) \\
&= 1.0 - 0.357 = 0.643 \text{ per unit} \\
V_{a2} &= -I_{a2}Z_2 = -(-j1.43)(j0.35) = -0.50 \text{ per unit} \\
V_{a0} &= -I_{a0}Z_0 = -(-j1.43)(j0.10) = -0.143 \text{ per unit}
\end{aligned}$$

Line-to-ground voltages are:

$$\begin{aligned}
V_a &= V_{a1} + V_{a2} + V_{a0} = 0.643 - 0.50 - 0.143 = 0 \\
V_b &= a^2V_{a1} + aV_{a2} + V_{a0} \\
&= 0.643(-0.5 - j0.866) - 0.50(-0.5 + j0.866) - 0.143 \\
&= -0.322 - j0.556 + 0.25 - j0.433 - 0.143 \\
&= -0.215 - j0.989 \text{ per unit} \\
V_c &= aV_{a1} + a^2V_{a2} + V_{c0} \\
&= 0.643(-0.5 + j0.866) - 0.50(-0.5 - j0.866) - 0.143 \\
&= -0.322 + j0.556 + 0.25 + j0.433 - 0.143 \\
&= -0.215 + j0.989 \text{ per unit}
\end{aligned}$$

Line-to-line voltages are:

$$\begin{aligned}
V_{ab} &= V_a - V_b = 0.215 + j0.989 = 1.01\underline{/77.7^\circ} \text{ per unit} \\
V_{bc} &= V_b - V_c = 0 - j1.978 = 1.978\underline{/270^\circ} \text{ per unit} \\
V_{ca} &= V_c - V_a = -0.215 + j0.989 = 1.01\underline{/102.3^\circ} \text{ per unit}
\end{aligned}$$

Since the generated voltage-to-neutral E_a was taken as 1.0 per unit, the above line-to-line voltages are expressed in per unit of the base voltage-to-neutral. When expressed in volts the postfault line voltages are:

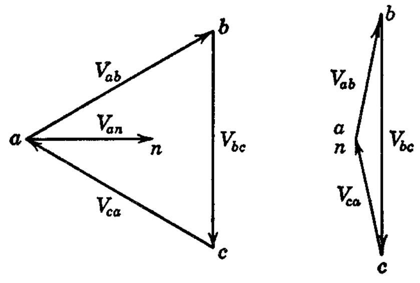
$$\begin{aligned}
V_{ab} &= 1.01 \times \frac{13.8}{\sqrt{3}}\underline{/77.7^\circ} = 8.05\underline{/77.7^\circ} \text{ kv} \\
V_{bc} &= 1.978 \times \frac{13.8}{\sqrt{3}}\underline{/270^\circ} = 15.73\underline{/270^\circ} \text{ kv} \\
V_{ca} &= 1.01 \times \frac{13.8}{\sqrt{3}}\underline{/102.3^\circ} = 8.05\underline{/102.3^\circ} \text{ kv}
\end{aligned}$$

Before the fault the line voltages were balanced and equal to 13.8 kv. For comparison with the line voltages after the fault occurs, the prefault voltages are given below with $V_{an} = E_a$ as reference. Prefault line voltages are

$$V_{ab} = 13.8\underline{/30^\circ} \text{ kv} \qquad V_{bc} = 13.8\underline{/270^\circ} \text{ kv} \qquad V_{ca} = 13.8\underline{/150^\circ} \text{ kv}$$

The phasor diagrams of prefault and postfault voltages are shown in Fig. 11.5.

11.4 Line-to-line Fault on an Unloaded Generator. The circuit diagram for a line-to-line fault on an unloaded, Y-connected generator is



(*a*) Pre-fault (*b*) Post-fault

FIG. 11.5 Phasor diagrams of the line voltages of Example 11.1 before and after the fault.

shown in Fig. 11.6 with the fault on phases b and c. The conditions at the fault are expressed by the following equations:

$$V_b = V_c \qquad I_a = 0 \qquad I_b = -I_c$$

Substituting the relations given by Eqs. (10.6) and (10.7) for V_b and V_c in the equation $V_b = V_c$ gives

$$a^2V_{a1} + aV_{a2} + V_{a0} = aV_{a1} + a^2V_{a2} + V_{a0}$$
$$(a^2 - a)V_{a1} = (a^2 - a)V_{a2}$$
$$V_{a1} = V_{a2} \tag{11.7}$$

Substituting $I_a = 0$ and $I_b = -I_c$ in Eqs. (10.19) to (10.21) gives

$$I_{a1} = \tfrac{1}{3}(0 + aI_b - a^2I_b) = \frac{jI_b}{\sqrt{3}}$$
$$I_{a2} = \tfrac{1}{3}(0 + a^2I_b - aI_b) = -\frac{jI_b}{\sqrt{3}}$$
$$I_{a0} = \tfrac{1}{3}(0 + I_b - I_b) = 0$$

Therefore,

$$I_{a1} = -I_{a2} \tag{11.8}$$

Then, from Eqs. (11.2) and (11.7),

$$V_{a2} = V_{a1} = E_a - I_{a1}Z_1$$

which becomes, by Eqs. (11.3) and (11.8),

$$-I_{a2}Z_2 = I_{a1}Z_2 = E_a - I_{a1}Z_1$$

Solving for I_{a1} gives

$$I_{a1} = \frac{E_a}{Z_1 + Z_2} \tag{11.9}$$

Equations (11.7) to (11.9) are the special equations for a line-to-line fault. They are used with Eqs. (11.2) to (11.4) and the symmetrical-component relations to determine all the voltages and currents at the fault. The special equations indicate the way in which the sequence networks are connected to represent the fault. Since Z_0 does not enter into the equations, the zero-sequence network is not used. The positive- and negative-sequence networks must be in parallel since $V_{a1} = V_{a2}$. The parallel connection of the positive- and negative-sequence networks without the zero-sequence network makes $I_{a1} = -I_{a2}$, as specified by Eq. (11.8). The connection of the sequence networks for a line-to-line fault is shown in Fig. 11.7. The currents and voltages in the sequence

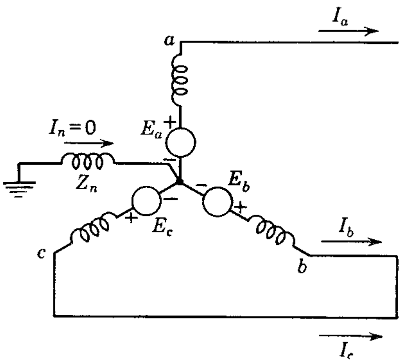


FIG. 11.6 Circuit diagram for a line-to-line fault between phases b and c at the terminals of an unloaded generator whose neutral is grounded through a reactor.

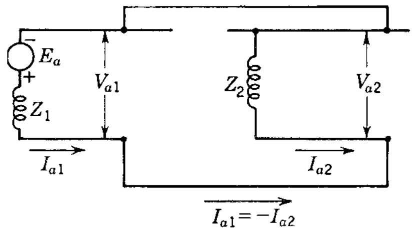


FIG. 11.7 Connection of the sequence networks of an unloaded generator for a line-to-line fault between phases b and c at the terminals of the generator.

networks, when so connected, satisfy all the equations derived for the line-to-line fault.

Since there is no ground at the fault, there is only one ground in the circuit at the generator neutral, and no current can flow in the ground. In the derivation of the relations for the line-to-line fault we found that $I_{a0} = 0$. This is consistent with the fact that no ground current can flow, since the ground current I_n is equal to $3I_{a0}$. The presence or absence of a grounded neutral at the generator does not affect the fault current. If the generator neutral is not grounded, Z_0 is infinite and I_{a0} is indeterminate, but line-to-line voltages may still be found since they contain no zero-sequence components.

Example 11.2

Find the subtransient currents and the line-to-line voltages at the fault under subtransient conditions when a line-to-line fault occurs at the terminals of the generator described in Example 11.1. Assume that the generator is unloaded and operating at rated terminal voltage when the fault occurs. Neglect resistance.

Solution

$$
\begin{aligned}
I_{a1} &= \frac{1.0 + j0}{j0.25 + j0.35} = -j1.667 \text{ per unit} \\
I_{a2} &= -I_{a1} = j1.667 \text{ per unit} \\
I_{a0} &= 0 \\
I_a &= I_{a1} + I_{a2} + I_{a0} = -j1.667 + j1.667 = 0 \\
I_b &= a^2 I_{a1} + a I_{a2} + I_{a0} \\
&= -j1.667(-0.5 - j0.866) + j1.667(-0.5 + j0.866) \\
&= j0.834 - 1.446 - j0.834 - 1.446 = -2.892 + j0 \text{ per unit} \\
I_c &= -I_b = 2.892 + j0 \text{ per unit}
\end{aligned}
$$

As in Example 11.1, base current is 836 amp. So

$$
\begin{aligned}
I_a &= 0 \\
I_b &= -2.892 \times 836 = 2{,}420/\underline{180^\circ} \text{ amp} \\
I_c &= 2.892 \times 836 = 2{,}420/\underline{0^\circ} \text{ amp}
\end{aligned}
$$

The symmetrical components of the voltage from a to ground are:

$$
\begin{aligned}
V_{a1} &= V_{a2} = 1 - (-j1.667)(j0.25) = 1 - 0.416 = 0.584 \text{ per unit} \\
V_{a0} &= 0 \text{ (neutral of the generator grounded)}
\end{aligned}
$$

Line-to-ground voltages are

$$
\begin{aligned}
V_a &= V_{a1} + V_{a2} + V_{a0} = 0.584 + 0.584 = 1.168/\underline{0^\circ} \text{ per unit} \\
V_b &= a^2 V_{a1} + a V_{a2} + V_{a0} \\
V_c &= V_b = 0.584(-0.5 - j0.866) + 0.584(-0.5 + j0.866) \\
&= -0.584 \text{ per unit}
\end{aligned}
$$

Line-to-line voltages are

$$
\begin{aligned}
V_{ab} &= V_a - V_b = 1.168 + 0.584 = 1.752/\underline{0^\circ} \text{ per unit} \\
V_{bc} &= V_b - V_c = -0.584 + 0.584 = 0 \text{ per unit} \\
V_{ca} &= V_c - V_a = -0.584 - 1.168 = 1.752/\underline{180^\circ} \text{ per unit}
\end{aligned}
$$

Expressed in volts the line-to-line voltages are

$$
\begin{aligned}
V_{ab} &= 1.752 \times \frac{13.8}{\sqrt{3}} = 13.95/\underline{0^\circ} \text{ kv} \\
V_{bc} &= 0 \text{ kv} \\
V_{ca} &= -1.752 \times \frac{13.8}{\sqrt{3}} = 13.95/\underline{18\ldots}
\end{aligned}
$$

11.5 Double Line-to-ground Fault on an Unloaded Generator. The circuit diagram for a double line-to-ground fault on an unloaded, Y-connected generator having a grounded neutral is shown in Fig. 11.8. The faulted phases are b and c. The conditions at the fault are expressed by the following equations:

$$V_b = V_c = 0$$
$$I_a = 0$$

Substituting $V_b = 0$ and $V_c = 0$ in Eqs. (10.9), (10.13), and (10.15) gives

$$V_{a1} = \tfrac{1}{3}(V_a + 0 + 0) = \frac{V_a}{3}$$
$$V_{a2} = \tfrac{1}{3}(V_a + 0 + 0) = \frac{V_a}{3}$$
$$V_{a0} = \tfrac{1}{3}(V_a + 0 + 0) = \frac{V_a}{3}$$

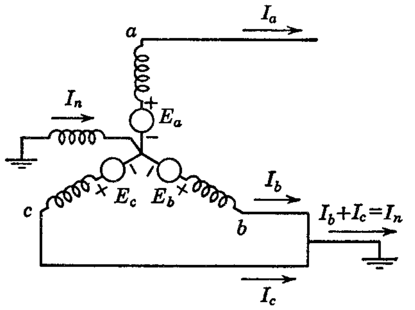


FIG. 11.8 Circuit for a double line-to-ground fault on phases b and c at the terminals of an unloaded generator whose neutral is grounded through a reactance.

Therefore,

$$V_{a1} = V_{a2} = V_{a0} \tag{11.10}$$

Solving Eqs. (11.3) and (11.4) for I_{a2} and I_{a0} and substituting V_{a1} for V_{a2} and V_{a0}, we obtain

$$I_{a2} = -\frac{V_{a2}}{Z_2} = -\frac{V_{a1}}{Z_2}$$
$$I_{a0} = -\frac{V_{a0}}{Z_0} = -\frac{V_{a1}}{Z_0}$$

Replacing V_{a1} by $E_a - I_{a1}Z_1$ gives

$$I_{a2} = -\frac{E_a - I_{a1}Z_1}{Z_2}$$

and

$$I_{a0} = -\frac{E_a - I_{a1}Z_1}{Z_0}$$

Since $I_a = 0$,

$$I_{a1} + I_{a2} + I_{a0} = 0$$

and

$$I_{a1} - \frac{E_a - I_{a1}Z_1}{Z_2} - \frac{E_a - I_{a1}Z_1}{Z_0} = 0$$

$$I_{a1}Z_2Z_0 - E_aZ_0 + I_{a1}Z_1Z_0 - E_aZ_2 + I_{a1}Z_1Z_2 = 0$$

$$I_{a1} = \frac{E_a(Z_2 + Z_0)}{Z_1Z_2 + Z_1Z_0 + Z_2Z_0} = \frac{E_a}{Z_1 + Z_2Z_0/(Z_2 + Z_0)} \tag{11.11}$$

Equations (11.10) and (11.11) are the special equations for a double line-to-ground fault. They are used with Eqs. (11.2) to (11.4) and the symmetrical component relations to determine all the voltages and currents at the fault. Equation (11.10) indicates that the sequence networks should be connected in parallel, as shown in Fig. 11.9, since the positive-, negative-, and zero-sequence voltages are equal at the fault. Examination of Fig. 11.9 shows that all the conditions derived above for the double line-to-ground fault are satisfied by this connection.

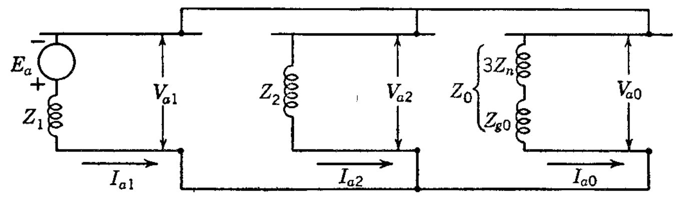


FIG. 11.9 Connection of the sequence networks of an unloaded generator for a double line-to-ground fault on phases b and c at the terminals of the generator.

The diagram of network connections shows that the positive-sequence current I_{a1} is determined by the voltage E_a impressed on Z_1 in series with the parallel combination of Z_2 and Z_0. The same relation is given by Eq. (11.11).

In the absence of a ground connection at the generator no current can flow into the ground at the fault. In this case Z_0 would be infinite and I_{a0} would be zero. In so far as current is concerned the result would be the same as in a line-to-line fault. Equation (11.11) for a double line-to-ground fault approaches Eq. (11.9) for a line-to-line fault as Z_0 approaches infinity, as may be seen by dividing the numerator and denominator of the second term in the denominator of Eq. (11.11) by Z_0 and letting Z_0 be infinitely large.

Example 11.3

Find the subtransient currents and the line-to-line voltages at the fault under subtransient conditions when a double line-to-ground fault occurs at the terminals of the generator described in [illegible]

Assume that the generator is unloaded and operating at rated voltage when the fault occurs. Neglect resistance.

Solution

$$I_{a1} = \frac{E_a}{Z_1 + Z_2Z_0/(Z_2 + Z_0)} = \frac{1.0 + j0}{j0.25 + \dfrac{j0.35 \times j0.10}{j0.35 + j0.10}}$$

$$= \frac{1.0}{j0.25 + j0.0778} = \frac{1.0}{j0.3277} = -j3.05 \text{ per unit}$$

$$V_{a1} = V_{a2} = V_{a0} = E_a - I_{a1}Z_1 = 1 - (-j3.05)(j0.25)$$

$$= 1.0 - 0.763 = 0.237 \text{ per unit}$$

$$I_{a2} = -\frac{V_{a2}}{Z_2} = -\frac{0.237}{j0.35} = j0.68 \text{ per unit}$$

$$I_{a0} = -\frac{V_{a0}}{Z_0} = -\frac{0.237}{j0.10} = j2.37 \text{ per unit}$$

$$I_a = I_{a1} + I_{a2} + I_{a0} = -j3.05 + j0.68 + j2.37 = 0$$

$$I_b = a^2I_{a1} + aI_{a2} + I_{a0}$$

$$= (-0.5 - j0.866)(-j3.05) + (-0.5 + j0.866)(j0.68) + j2.37$$

$$= j1.525 - 2.64 - j0.34 - 0.589 + j2.37$$

$$= -3.229 + j3.555 = 4.81\angle 132.5^\circ \text{ per unit}$$

$$I_c = aI_{a1} + a^2I_{a2} + I_{a0}$$

$$= (-0.5 + j0.866)(-j3.05) + (-0.5 - j0.866)(j0.68) + j2.37$$

$$= j1.525 + 2.64 - j0.34 + 0.589 + j2.37 = 3.229 + j3.555$$

$$= 4.81\angle 47.5^\circ \text{ per unit}$$

$$I_n = 3I_{a0} = 3 \times j2.37 = j7.11 \text{ per unit}$$

$$I_n = I_b + I_c = -3.229 + j3.555 + 3.229 + j3.555 = j7.11 \text{ per unit}$$

$$V_a = V_{a1} + V_{a2} + V_{a0} = 3V_{a1} = 3 \times 0.237 = 0.711 \text{ per unit}$$

$$V_b = V_c = 0$$

$$V_{ab} = V_a - V_b = 0.711 \text{ per unit}$$

$$V_{bc} = 0$$

$$V_{ca} = V_c - V_a = -0.711 \text{ per unit}$$

Expressed in amperes and volts

$$I_a = 0$$

$$I_b = 836 \times 4.81\angle 132.5^\circ = 4{,}025\angle 132.5^\circ \text{ amp}$$

$$I_c = 836 \times 4.81\angle 47.5^\circ = 4{,}025\angle 47.5^\circ \text{ amp}$$

$$I_n = 836 \times 7.11\angle 90^\circ = 5{,}950\angle 90^\circ \text{ amp}$$

$$V_{ab} = 0.711 \times \frac{13.8}{\sqrt{3}} = 5.66\angle 0^\circ \text{ kv}$$

$$V_{bc} = 0$$

$$V_{ca} = -0.711 \times \frac{13.8}{\sqrt{3}} = 5.66\angle 180^\circ \text{ k}$$

PROBLEMS

11.1 A 60-cycle turbogenerator is rated 10,000 kva, 13.8 kv. It is Y-connected, solidly grounded, and is operating at rated voltage at no load. It is disconnected from the rest of the system. Its reactances are $X'' = X_2 = 0.15$ and $X_0 = 0.05$ per unit. Find the ratio of the subtransient line current for a single line-to-ground fault to the subtransient line current for a symmetrical three-phase fault.

11.2 Find the ratio of the subtransient line current for a line-to-line fault to the subtransient current for a symmetrical three-phase fault on the generator of Prob. 11.1.

11.3 Determine the ohms of inductive reactance to be inserted in the neutral connection of the generator of Prob. 11.1 to limit the subtransient line current for a single line-to-ground fault to that for a three-phase fault.

11.4 With the inductive reactance found in Prob. 11.3 inserted in the neutral of the generator of Prob. 11.1, find the ratios of the subtransient line currents for the following faults to the subtransient line current for a three-phase fault: (*a*) single line-to-ground fault, (*b*) line-to-line fault, (*c*) double line-to-ground fault.

11.5 How many ohms of resistance in the neutral connection of the generator of Prob. 11.1 would limit the subtransient line current for a single line-to-ground fault to that for a three-phase fault?

11.6 A generator rated 10,000 kva, 6.9 kv has $X'' = X_2 = 15\%$ and $X_0 = 5\%$. Its neutral is grounded through a reactor of 0.381 ohms. The generator is operating at 6.9 kv without load and is disconnected from the system when a single line-to-ground fault occurs at its terminals. Find the subtransient current in the faulted phase.

11.7 A 10,000-kva, 13.8-kv turbogenerator having $X'' = X_2 = 15\%$ and $X_0 = 5\%$ is about to be connected to a power system. The generator has a current-limiting reactor of 0.7 ohms in the neutral. Before the generator is connected to the system its voltage is adjusted to 13.2 kv when a double line-to-ground fault develops at terminals b and c. Find the initial symmetrical rms current in the ground and in line b.

11.8 A 15,000-kva, 6.9-kv generator, Y-connected, has positive-, negative-, and zero-sequence reactances of 25%, 25%, and 8%, respectively. In order to reduce the short-circuit current in case of a fault to ground, an inductive reactor with 6% reactance based on the rating of the generator is placed in the line from neutral to ground. A line-to-line fault occurs at the terminals of the generator when it is operating at rated voltage and disconnected from the system. Find the initial symmetrical rms line and neutral currents and the line-to-line and line-to-neutral voltages (*a*) if the fault does not involve ground; (*b*) if the fault is solidly grounded at the instant of its occurrence.

CHAPTER 12

SEQUENCE IMPEDANCES AND SEQUENCE NETWORKS

12.1 Introduction. In Chap. 11 we defined a particular sequence impedance as the impedance to the flow of current of that sequence. We accepted the fact that negative-sequence impedance and zero-sequence impedance of a generator were not generally equal to any of the positive-sequence impedances applicable to subtransient, transient, or steady-state conditions. This chapter will discuss negative- and zero-sequence impedances of rotating machines and compare them with positive-sequence impedance. We shall also discuss the sequence impedances of transformers and transmission lines.

For an unloaded generator we found that the symmetrical components of current flowing during short circuits could be determined by certain interconnections of sequence networks. The sequence networks were simple because we were dealing only with an isolated generator. In a system consisting of several generators and motors and of transformers and transmission lines between various points, each sequence network is composed of sequence impedances connected to provide the proper paths for currents of one phase of the particular sequence for which each network is synthesized. The equivalent circuits of Chaps. 8 and 9 consist of positive-sequence emfs and positive-sequence impedances only and may be called positive-sequence networks. Similar networks providing paths for negative- and zero-sequence currents contain negative- and zero-sequence impedances, respectively. Since currents of one sequence produce only voltage drops of like sequence in a symmetrical system, the current in any sequence network of a balanced system carries only current of the same sequence as the impedances in the network. Thus we say there is no coupling between the sequence networks. This chapter discusses the synthesis of such networks.

12.2 Sequence Impedances. The positive-sequence and negative-sequence impedances of linear, symmetrical, static circuits are identical because the impedance of such circuits is independent of phase order provided the applied voltages are balanced. The impedance of such circuits to zero-sequence currents may differ from the impedance to positive- and negative-sequence currents. The impedances of rotating machines to currents of the three sequences will generally be different for each sequence.

In deriving the equations for inductance and capacitance of transposed transmission lines, we assumed balanced three-phase currents and did not specify phase order. Therefore, the resulting equations are valid for both positive- and negative-sequence impedances. The inductance and capacitance of transmission lines for zero-sequence currents will be discussed later.

For symmetrical three-phase static loads consisting of lumped constants or loads which can be analyzed as having lumped constants, the impedances to current of positive, negative, and zero sequences are the same because each phase is isolated from, and independent of, the other phases. If the load is Δ-connected or Y-connected with an ungrounded neutral, no path exists for the flow of zero-sequence current. The absence of a path for zero-sequence current means that an infinite impedance is in series with the zero-sequence impedance of the load in the zero-sequence network. If there is an impedance Z_n in the connection between neutral and ground, an impedance of $3Z_n$ is placed in series with the zero-sequence impedance of the load, since $3I_{a0}$ flows in Z_n if there is a completed path for neutral current while only I_{a0} flows in the zero-sequence network. In general, Z_n may have any value between zero and infinity depending on whether the neutral is solidly grounded, is grounded through impedance, or is isolated from ground.

The study of impedance to negative- and zero-sequence current becomes very complex when a complete analysis is attempted for many different kinds of circuits and apparatus. Only the more elementary cases and those which will be encountered most frequently in our discussions will be considered. Positive-sequence impedances need no extended treatment because they have been discussed previously in connection with transmission lines or are of a type with which we are already familiar.

12.3 Transformer Impedances. The equivalent circuit of a transformer as developed in a-c machinery theory and as shown in part of Fig. 8.3 consists of series impedance composed of effective resistance and leakage reactance and shunt admittance to provide a path for the magnetizing current. The series impedance is that impedance measured by the short-circuit test, and the shunt admittance is that admittance measured by the open-circuit test. The shunt admittance for positive- and negative-sequence current and sometimes for zero-sequence current is so low that it may be neglected. In some cases the series resistance is neglected when it is very small compared to the leakage reactance, as is the case in power transformers of over 500 kva. Typical values of transformer series impedances are given in the Appendix in Table A.6.

A three-phase transformer bank may be made up of three individual single-phase units, or it may be a three-phase core or shell transformer. Originally, single-phase units were more popular than three-phase units, and they still have the advantages of providing partial service on open Δ in the event of damage to one unit, of easier portability, and of readily available spare capacity in the form of an additional single-phase unit to replace a damaged unit. Most of the modern installations are three-phase units, because of their lower initial cost, smaller space requirements, and higher efficiency. Improved methods of system protection and increased reliability of transformers have reduced the importance of immediately available spare capacity. The types of transformers for three-phase service are shown in Figs. 12.1 to 12.3, where only the primary windings are indicated.

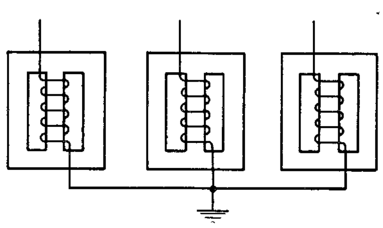


FIG. 12.1 Y-connected primary windings of a three-phase bank consisting of three single-phase transformers.

The paths of leakage flux are partially through air, regardless of phase sequence or type of three-phase bank. There is no difference

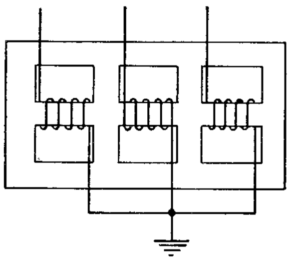


FIG. 12.2 Y-connected primary windings of a three-phase shell-type transformer.

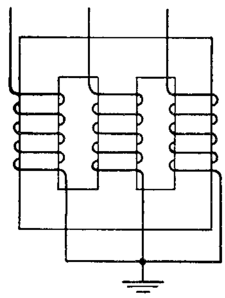


FIG. 12.3 Y-connected primary windings of a three-phase core-type transformer.

between positive- and negative-sequence series impedances as measured by the short-circuit test regardless of the type of bank. Theoretically the zero-sequence short-circuit impedance differs somewhat from the positive-sequence impedance in a shell- or core-type transformer, especially in the core type where it may be appreciably lower than the positive-sequence impedance. The difference is small enough, however, so that the best approximation for the series impedances is to assume equality for impedances of all sequences regardless of type of bank.[1]

In all three types of transformers for three-phase service, the same path is followed by excitation flux resulting from negative-sequence applied voltage as is followed by the excitation flux resulting from positive-sequence applied voltage. Therefore, for the same applied voltage, the shunt admittance measured on open circuit will be the same for both positive- and negative-sequence exciting currents and will be so low that it will be neglected.

Zero-sequence exciting current produces flux in the same path in three single-phase units as do the exciting currents of positive and negative sequence. The admittance to zero-sequence exciting current may be neglected. In three-phase shell-type transformers the paths of excitation flux are completely in iron, as in the three-phase bank of single-phase units, but comparison of Figs. 12.1 and 12.2 shows that saturation will occur at a lower voltage in the three-phase shell-type transformer than in the single-phase units when the exciting currents are in phase. The shunt admittance to zero-sequence current in three-phase shell-type transformers varies widely with saturation but will be low enough to neglect even in very accurate work unless the zero-sequence voltage approaches the value of the positive-sequence voltage.

Examination of Fig. 12.3 shows that zero-sequence excitation flux paths in the three-phase core-type transformer are partly through air, because the zero-sequence mmfs are in phase and the magnetic circuit cannot be completed through the iron. On the other hand, the paths of positive- and negative-sequence fluxes are completely in the iron because of the phase displacement of their mmfs. Therefore, the admittance to zero-sequence exciting current is considerably higher for the three-phase core-type transformer than for the shell type. For simplicity in our analytical calculations we shall omit shunt admittance of all sequences for all types of three-phase transformers.

12.4 Negative-sequence Impedance of Synchronous Machines. When the three phase currents of a synchronous machine have the same phase sequence as the generated voltages of the machine, the mmf produced by the armature current rotates in the same direction as the mmf produced by the d-c field winding. The resulting flux across the air gap is determined by the sum of these two mmfs. Positive-sequence impedance is the impedance met by the armature current having the same phase sequence as the generated voltages. The reactive component of the positive-sequence impedance includes leakage reactance and an additional component to account for the change in flux caused by the mmf

[1] See A. N. Garin, "Zero-phase-sequence Characteristics of Transformers," Parts I and II, *Gen. Elec. Rev.*, vol. 43, pp. 131–136 and 174–179, March and April, 1940.

of the armature current. The amount of this additional component depends upon whether the reactance is that during the subtransient, transient, or steady-state period.

Negative-sequence impedance of a synchronous machine is the impedance met by armature current of phase sequence opposite to that of the generated phase voltages. The mmf produced by the negative-sequence armature current rotates in the direction opposite to that of the rotor and its d-c field winding. Unlike the flux produced by positive-sequence current, which is stationary with respect to the rotor, the flux produced by the negative-sequence current is sweeping rapidly over the face of the rotor. The currents induced in the field and damper windings by the rotating armature flux keep the flux from penetrating the rotor. The flux path is the same as in the case of subtransient reactance. We saw in Chap. 9 that flux resulting from the changing positive-sequence armature current is changing rapidly and is kept from penetrating the rotor, but the subtransient reactance depends on whether the armature mmf acts on the direct axis, the quadrature axis, or somewhere between the two axes. In sweeping over the entire circumference of the rotor face, the mmf of negative-sequence current is constantly varying its position with respect to the direct and quadrature axes of the rotor. Therefore, negative-sequence reactance would be expected to be the average of the direct and quadrature subtransient reactances. Indeed, one definition of negative-sequence reactance of a synchronous machine is

$$X_2 = \frac{X''_d + X''_q}{2} \tag{12.1}$$

We discussed in Chap. 9 the fact that the subtransient reactance of machines having damper windings is practically constant regardless of the position of the axis of armature mmf with respect to the direct and quadrature axes of the rotor. Therefore, in turbine generators and salient-pole generators with copper dampers continuous between the poles, X''_d and X''_q are nearly equal in value, and X_2 is equal to X''_d, as may be verified by referring to Table A.5 in the Appendix.

Negative-sequence resistance is the resistance by which the square of the negative-sequence current is multiplied to find the power associated with the negative-sequence current. Unless positive-sequence currents are changing in magnitude, they do not induce current in the field or damper windings, nor do they cause hysteresis or eddy-current losses in the rotor. Negative-sequence currents, however, by producing a flux rotating in the direction opposite to that of the rotor, induce currents of double frequency in the field and damper windings and cause eddy-current and hysteresis losses in the rotor iron. Although the windings keep the penetration of the flux from being very deep and thereby keep the losses small, the negative-sequence resistance is higher than the positive-sequence a-c resistance of the armature. Resistance is much smaller than reactance for negative-sequence currents, but the difference between resistance and reactance is less for negative-sequence currents than for positive-sequence currents.

The negative-sequence reactance of a synchronous machine may be computed from Eq. (12.1) after measuring X''_d and X''_q as described in Sec. 9.5. Negative-sequence impedance may be measured by driving the rotor at synchronous speed with the field short-circuited while applying negative-sequence voltage to the armature. Another convenient method is to measure the steady-state current in a line-to-line short circuit at the armature terminals while the machine is driven at synchronous speed with enough field excitation to give rated voltage at no load. At the same time, the voltage is measured between the shorted terminals and the open terminal, and a reading of a single-phase wattmeter with its current coil in one of the shorted lines and its potential coil between the shorted terminals and the open terminal is made. The reader may prove by symmetrical components that the negative-sequence impedance is equal to the ratio of the voltmeter reading to the ammeter reading, divided by 1.73. The negative-sequence *reactance* is equal to the impedance multiplied by the ratio of the wattmeter reading to the product of the voltmeter and ammeter readings.[2]

12.5 Zero-sequence Impedance of Synchronous Machines. When only zero-sequence current flows in the armature winding of a three-phase machine, the current and mmf of one phase are a maximum at the same time as the current and mmf of each of the other phases. The windings are so distributed around the circumference of the armature that the point of maximum mmf produced by one phase is displaced by 120 electrical degrees in space from the point of maximum mmf of each of the other phases. If the mmf produced by the current of each phase had a perfectly sinusoidal distribution in space, a plot of mmf around the armature for each phase would result in the three sinusoidal curves shown in Fig. 12.4 for a two-pole machine. The figure is drawn for one instant of time only. For other instants the pattern would be the same

[2] It is the purpose of this book to give the reader a qualitative concept of the various reactances encountered in a synchronous machine and to enable him to appreciate the reasons for the different values obtained for the various impedances. For a more complete discussion of machine impedances and methods of measurement, the reader should consult such books as C. Concordia, "Synchronous Machines," John Wiley & Sons, Inc., New York, 1951; C. F. Wagner and R. D. Evans, "Symmetrical Components," Chap. V, pp. 74–109, McGraw-Hill Book Company, Inc., New York, 1933; or Central Station Engineers of Westinghouse Electric Corporation, "Electrical Transmission and Distribution Reference Book," 4th ed., Chap. 6, pp. [illegible]–[illegible], East Pittsburgh, Pa., 1950.

except that the mmf of each phase would be altered in magnitude by the same amount. At any point on the armature of the machine the sum of the three mmfs shown in Fig. 12.4 is zero at all times. There would be no flux produced across the air gap, and the only reactance of any phase winding would be that due to leakage and end turns.

In an actual machine, the winding is not distributed to produce perfectly sinusoidal mmf. The flux resulting from the sum of the mmfs is very small but makes the zero-sequence reactance somewhat higher than in the ideal case where there is no air-gap flux due to zero-sequence armature current. Since there is some air-gap flux in the actual case, zero-sequence reactance is dependent on the resistance of the damper winding. The amount of flux depends on the breadth and chording

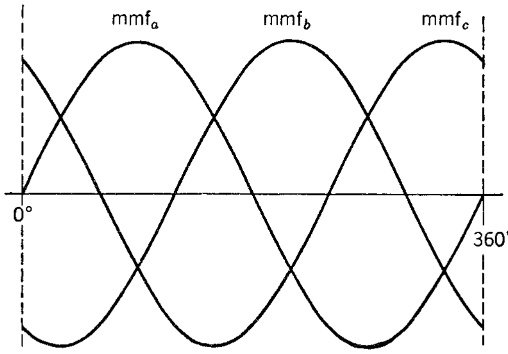


FIG. 12.4 The mmf produced at a given instant by the zero-sequence current in each phase of a three-phase armature winding as a function of position around the armature. Perfect sinusoidal distribution of each mmf is assumed.

factors of the winding. Because of the low flux, X_0 is always much lower than X_2 or X_d''. Core losses of the pulsating resultant flux cause the zero-sequence resistance to be slightly higher than positive-sequence resistance.

Since zero-sequence current means identical current in each phase winding, a convenient method of measuring zero-sequence impedance is to connect the three phase windings in series and to apply a single-phase voltage of the correct frequency. The d-c field winding should be short-circuited, and the rotor may be blocked or rotated at synchronous speed for the test. The measured impedance divided by 3 is the zero-sequence impedance per phase.

Another method of measuring zero-sequence impedance is by applying a line-to-line short circuit to the machine when driven at synchronous speed with the field energized to give rated voltage at no load. An ammeter is placed between the shorted terminals and the neutral of the machine. Voltage between the open terminal and neutral is read. This voltage divided by the ammeter reading is the zero-sequence impedance, as may be shown by symmetrical components. Components of the impedance may be found by inserting the current coils of a wattmeter in series with the ammeter and by placing the potential coils in parallel with the voltmeter. The zero-sequence *resistance* is the zero-sequence impedance multiplied by the ratio of the wattmeter reading to the product of the voltmeter and ammeter readings.

12.6 Sequence Impedances of Induction Motors. In so far as system short circuits are concerned, the most important difference between induction motors and synchronous motors is that synchronous motors receive their excitation from a d-c source which is considered to be constant and unaffected by the fault, whereas the excitation of induction motors comes from the a-c system. Therefore, when a short circuit occurs at the terminals of an induction motor, the current the motor supplies to the short circuit dies out in 1 or 2 cycles. Induction motors are often entirely neglected in studying faults on a large system, but their contribution to short-circuit current during the subtransient period should be included if they constitute a large part of the system. An average value for the subtransient reactance of large induction motors is 25%. The transient reactance of induction motors is infinite since current flows from the motor to the fault only during the subtransient period.

When making load studies, we can replace the induction motor by a series or parallel combination of resistance and inductive reactance as required by its rated voltage, horsepower, efficiency, and power factor. For steady-state operation we can use the equivalent circuit of the induction motor in our impedance diagram. It is convenient to look at the equivalent circuit now, for we can see from the diagram the relation between positive- and negative-sequence impedance. The equivalent circuit, whose development may be found in any textbook on a-c machinery,[3] is shown in Fig. 12.5. We will use the following nomenclature:

R_s = effective resistance of the stator per phase
X_s = stator leakage reactance per phase at supply frequency
R_r = effective resistance of the rotor per phase, referred to the stator
X_r = rotor leakage reactance per phase at supply frequency, referred to the stator
X_m = shunt magnetizing reactance
s = slip
R_1 = positive-sequence resistance of the equivale cir
R_2 = negative-sequence resistance of the equival rcuit
X_1 = positive-sequence reactance of the equi circuit
X_2 = negative-sequence reactance of the equi ircuit

[3] See, for instance, A. E. Fitzgerald and C. Kingsle "Elec," p. 397, McGraw-Hill Book Company, Inc., New York, 2

Slip is the ratio of the difference between the speed of the revolving magnetic field and the speed of the rotor to the speed of the revolving magnetic field. Positive-sequence current in the armature produces a field rotating in the same direction as the rotor. Slip is about 5% at full load. Negative-sequence current in the armature produces a field which rotates in the direction opposite to that of the rotor. If the motor

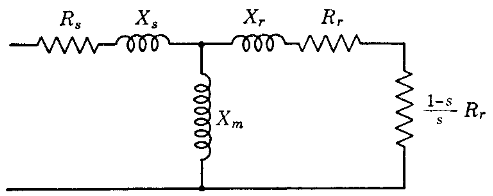


FIG. 12.5 Equivalent circuit of an induction motor.

were rotating at synchronous speed, the slip would be 200% for negative-sequence armature current. The slip for negative-sequence current at full load is about 200% − 5% = 195%. If we call s_1 the slip for positive-sequence current and s_2 the slip for negative-sequence current,

$$s_2 = 2 - s_1 \tag{12.2}$$

The input impedance of the equivalent circuit is the positive- or negative-sequence impedance of the induction motor, according to whether s_1 or s_2 is used to compute the resistance of the circuit. The circuit is solved more easily if it is simplified by placing the shunting reactance directly across the input. Such a simplification is often made in the equivalent circuit of an induction motor. If we let

$$R_t = R_s + \frac{R_r}{s} \tag{12.3}$$

and

$$X_t = X_s + X_r \tag{12.4}$$

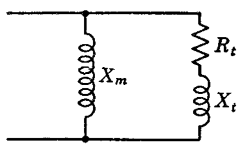


FIG. 12.6 Simplified equivalent circuit of an induction motor.

where s is s_1 for positive sequence and s_2 for negative sequence, the simplified circuit is that shown in Fig. 12.6. Reduction of the simplified circuit gives us the following equations for positive- and negative-sequence resistance and reactance:

$$R_{1,2} = \frac{X_m{}^2 R_t}{R_t{}^2 + (X_m + X_t)^2} \tag{12.5}$$

$$X_{1,2} = \frac{X_m(R_t{}^2 + X_m X_t + \text{[illegible]}}{R_t{}^2 + (X_m + X_t)\text{[illegible]}} \tag{12.6}$$

where $R_{1,2}$ = positive- or negative-sequence resistance, depending on whether s_1 or s_2 is used for slip to determine R_t by Eq. (12.3) for use in Eq. (12.5)

$X_{1,2}$ = positive- or negative-sequence reactance, depending on whether s_1 or s_2 is used for slip to determine R_t by Eq. (12.3) for use in Eq. (12.6)

The shunt reactance X_m is very much larger than X_t and R_t for either sequence. Therefore, X_1 and X_2 are nearly equal. Since positive-sequence R_t is larger than negative-sequence R_t and since X_m is very large, R_2 is less than R_1.

We saw in Sec. 12.5 that zero-sequence currents flowing in a three-phase armature winding produce very little flux, and as a result the zero-sequence impedance of synchronous machines is very low. Since the armatures of induction motors and synchronous machines are the same, the resultant flux for zero-sequence current in the armature of an induction motor is very small, and the zero-sequence impedance is correspondingly small.

12.7 Zero-sequence Impedance of Transmission Lines without Ground Wires. As stated in Sec. 12.2, we assumed balanced three-phase currents when we derived the equations for the inductance of a transposed three-phase transmission line. The resulting equations are valid only for balanced three-phase currents, including the positive- and negative-sequence components of unbalanced currents, but not including zero-sequence components. When only zero-sequence current flows in a transmission line, the current in each phase is identical. The current returns through the ground, through overhead ground wires, or through both. The total current in the return path is the sum of the zero-sequence current in the three phases, or three times the zero-sequence current in one phase. Because zero-sequence current is identical in magnitude and phase in each phase conductor rather than equal only in magnitude and displaced in phase by 120° from other phase currents, the magnetic field due to zero-sequence current is very different from the magnetic field caused by either positive- or negative-sequence current. The difference in magnetic field results in the zero-sequence reactance of a transmission line being 2 to 3.5 times as large as the positive-sequence reactance. The ratio is toward the higher portion of the specified range for double-circuit lines and lines without ground wires. We shall discuss the zero-sequence impedance of transmission lines without ground wires first, and later we shall include the effect of ground wires in our analysis.

The work of Carson[4] provides a convenient method for considering

[4] See J. R. Carson, "Wave Propagation in Overhead Wires with Ground Return," *Bell System Tech. J.*, vol. 5, pp. 539–554, October, 1926. Although based on a different

the effect of the return path through the ground. The method assumes the return current to be confined to a fictitious conductor which has a self GMD of 1 ft and which is directly below the conductors at a distance dependent on the resistivity of the earth. Actually the current in the earth spreads out over a wide area in order to follow the path of least impedance. The path followed by the current is governed by both resistance and reactance. In following the path of least impedance, the current is restrained from departing from the vicinity of the line by the increased reactance which would result therefrom, for the inductance of a circuit increases with the distance between its two sides.

Carson's method is convenient because it is adaptable to mathematical analysis and gives results which check closely with experimental data. In simplified form, Carson gives the resistance of the earth in ohms per mile as $1.588f \times 10^{-3}$, where f is frequency in cycles per second. Reactance is expressed in terms of the distance D_e from the overhead conductors to the fictitious conductor having a self GMD equal to 1 ft. The value of D_e may be computed by the following equation:

$$D_e^2 = 2{,}160 \sqrt{\frac{\rho}{f}} \tag{12.7}$$

where ρ is the resistivity of the earth in ohms per meter cube and D_e is expressed in feet. Often D_e is used to signify the quantity represented above by D_e^2. In the derivations which we are about to make, however, it seems more logical to define D_e^2 by Eq. (12.7) so that D_e is the separation between the overhead conductors and the fictitious return conductor. Table 12.1 gives values of D_e and D_e^2 obtained from Eq. (12.7) for $f = 60$ cps and for various values of ρ, depending upon the condition of the earth return.

TABLE 12.1 RESISTIVITIES AND D_e FOR $f = 60$ cps

Return circuit	ρ, ohms/meter^3	D_e^2, ft^2	D_e, ft
Sea water	1	279	16.7
Swampy ground	100	2,790	52.8
Dry earth	1,000	8,820	93.9

The concept of a fictitious return conductor is illustrated b[y Fig. 12.?], where the one overhead conductor is designated as a a[nd the] fictitious

assumption, equations developed by Rüdenberg give results differing only slightly from those obtained by Carson's equations. See R. Rüdenberg, "Transient Performance of Electric Power Systems," pp. 393–408, McGraw-Hill Book Company, Inc., New York, 1950.

return conductor is designated as e. The inductance of the circuit ae due only to the current in conductor a is, by Eq. (2.36),

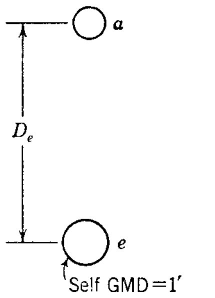


FIG. 12.7 Overhead conductor a and fictitious return conductor e.

$$L_a = 0.7411 \log \frac{D_e}{r'_a} \qquad \text{millihenrys/mile} \qquad (12.8)$$

and the inductance due to the current in conductor e is

$$L_e = 0.7411 \log \frac{D_e}{1} \qquad \text{millihenrys/mile} \qquad (12.9)$$

The inductance of the circuit ae, or the self-inductance of conductor a and its earth return, is

$$L_{aa} = L_a + L_e = 0.7411 \log \frac{D_e^2}{r'_a} \qquad \text{millihenrys/mile} \qquad (12.10)$$

The self-impedance of line a with its earth return is

$$Z_{aa} = R_a + 1.588f \times 10^{-3} + j4.657f \times 10^{-3} \log \frac{D_e^2}{r'_a} \qquad \text{ohms/mile} \qquad (12.11)$$

where R_a is the resistance of the overhead line a.

A single-circuit three-phase line consists of three conductors, each of which carries the same zero-sequence current. In so far as zero-sequence quantities are concerned, the three conductors may be considered as elements of a composite equivalent conductor, as discussed in Sec. 2.8. Therefore, we can find the inductive reactance of the circuit consisting of three overhead conductors, which are electrically in parallel, and the earth return by the method of GMD. We determine the self GMD of the three-element composite conductor and the GMD between the fictitious conductor and the elements of the composite overhead conductor. Since the distance from each conductor to the earth return is approximately the same, the GMD may be considered equal to D_e. The total zero-sequence impedance of the three overhead lines and the earth return is

$$Z_{aa} = \frac{R_a}{3} + 1.588f \times 10^{-3} + j4.657f \times 10^{-3} \log \frac{D_e^2}{D_{aa}} \qquad \text{ohms/mile} \qquad (12.12)$$

where R_a is the resistance of one of the three identical overhead conductors and D_{aa} is the self GMD of the equivalent composite conductor. For one of the three phases electrically in parallel for zero-sequence current, the zero-sequence impedance is

$$Z_0 = 3Z_{aa} = R_a + 4.764f \times 10^{-3} + j13.97f \times 10^{-3} \log \frac{D_e^2}{D_{aa}} \qquad \text{ohms/mile} \qquad (12.13)$$

The zero-sequence impedance of a multiple-circuit line consisting of identical conductors may be calculated in a similar manner since the zero-sequence current is always split equally between all the conductors which are treated as elements of one composite conductor.

Example 12.1

Find the positive- and zero-sequence impedance per mile at 60 cycles for a single-circuit three-phase line consisting of No. 2/0 hard-drawn copper conductors with flat, horizontal spacing of 12 ft between centers. Assume $\rho = 100$.

Solution

For No. 2/0 seven-strand hard-drawn copper, $r' = 0.01252$ ft

At 25°C, $R_a = 0.440$ ohm/mile

$$D_{eq} = \sqrt[3]{12 \times 12 \times 24} = 15.1 \text{ ft}$$

$$L = 0.7411 \log \frac{15.1}{0.0125} = 2.285 \text{ millihenrys/mile}$$

$$Z_1 = 0.440 + j2\pi 60 \times 2.285 \times 10^{-3} = 0.440 + j0.861 \text{ ohm/mile}$$

The self GMD of the equivalent composite conductor for zero sequence is

$$D_{aa} = \sqrt[9]{(0.01252)^3(12)^4(24)^2} = 1.42 \text{ f}$$

$$Z_0 = 0.440 + 4.764 \times 60 \times 10^{-3} + j13.97 \times 60 \times 10^{-3} \log \frac{2{,}790}{1.42}$$

$$= 0.727 + j2.76 \text{ ohms/mile}$$

The ratio of zero-sequence inductive reactance to positive-sequence inductive reactance is 3.21.

Example 12.1 shows the relation between the self GMD of the composite conductor used in the zero-sequence reactance formulas, the self GMD of one phase conductor, and the equivalent equilateral spacing D_{eq} of the three phases. The relation, as seen from the computation of D_{aa} in the example, is

$$D_{aa} = \sqrt[3]{r' D_{eq}^2} \qquad (12.14)$$

or

$$D_{aa} = \sqrt[3]{\text{self GMD}_{\text{one phase}}\, D_{eq}^2} \qquad (12.15)$$

12.8 Zero-sequence Impedance of Transmission Lines with Ground Wires. One or more conductors called ground wires are often placed above the circuit conductors of a transmission line. The ground wires are grounded at every tower. The connection to ground is through the tower footing resistance. The large number of such parallel connections to ground per mile allows us to assume that the ground wires and earth are electrically in parallel at the ends of the section of circuit being considered.

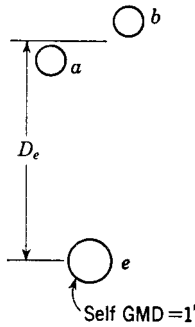


FIG. 12.8 Two overhead conductors a and b with earth return represented by e.

In order to study the effect of ground wires on zero-sequence impedance, it is necessary to determine the mutual impedance between two single-phase circuits having a common ground return. Let us consider the two conductors a and b shown in Fig. 12.8. These conductors carry currents I_a and I_b, which return through the earth represented by the fictitious conductor e. According to Eq. (2.26) the flux linkages between two points distant D_{be} and D_{ab} from conductor b due to I_b in that conductor are

$$\psi_{ae} = 2 \times 10^{-7} I_b \ln \frac{D_{be}}{D_{ab}} \qquad \text{weber-turns/meter} \qquad (12.16)$$

which must be equal to the flux linkages of circuit ae due to the current I_b in conductor b. Similarly the flux linkages of circuit ae due to the current I_b in the earth as represented by conductor e are

$$\psi_{ae} = 2 \times 10^{-7} I_b \ln \frac{D_{ae}}{1} \qquad \text{weber-turns/meter} \qquad (12.17)$$

The addition of Eqs. (12.16) and (12.17) gives the total flux linkages of circuit ae due to I_b in circuit be, namely,

$$\psi_{ae} = 2 \times 10^{-7} I_b \ln \frac{D_{ae} D_{be}}{D_{ab}} \qquad \text{weber-turns/meter} \qquad (12.18)$$

Since D_{ae} and D_{be} are almost equal to D_e, Eq. (12.18) may be simplified to give

$$\psi_{ae} = 2 \times 10^{-7} I_b \ln \frac{D_e^2}{D_{ab}} \qquad \text{weber-turns/meter} \qquad (12.19)$$

The mutual inductance between circuits ae and be is

$$M_{ab} = 0.7411 \log \frac{D_e^2}{D_{ab}} \qquad \text{millihenr[illegible]} \qquad (12.20)$$

The current I_b in circuit be causes a voltage d[illegible] in circuit ae equal to

$$I_b R_e + j2\pi f M_{ab} I_b \ [illegible]$$

where R_e is the resistance of conductor e, the conductor common to both circuits. This voltage divided by I_b is the mutual impedance between circuits ae and be, which is

$$Z_{ab} = R_e + j2\pi f M_{ab} = 1.588f \times 10^{-3} + 4.657f \times 10^{-3} \log \frac{D_e^2}{D_{ab}} \quad \text{ohms/mile} \qquad (12.21)$$

Of course, the above value of mutual impedance would be found also by determining the voltage induced in circuit be by the current I_a in circuit ae.

Now let us consider a 1-mile section of two single-phase circuits with a common earth return, as shown in Fig. 12.9. Examination of Fig. 12.9 shows that

$V_{ae} - V_{a'e'}$ = voltage drop per mile of circuit ae, including conductor a and the earth return.

$V_{be} - V_{b'e'}$ = voltage drop per mile of circuit be, including conductor b and the earth return.

Then

$$V_{ae} - V_{a'e'} = I_a Z_{aa} + I_b Z_{ab} \qquad (12.22)$$
$$V_{be} - V_{b'e'} = I_a Z_{ab} + I_b Z_{bb} \qquad (12.23)$$

where Z_{aa} = self-impedance of circuit ae
Z_{bb} = self-impedance of circuit be
Z_{ab} = mutual impedance between circuits ae and be

A slight modification of the circuit of Fig. 12.9 and of Eqs. (12.22) and (12.23) converts them to the case of a single-phase line with a ground

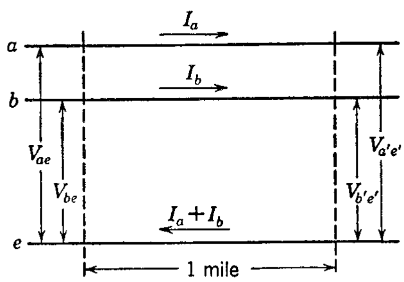


FIG. 12.9 Section of two single-phase circuits with common earth return.

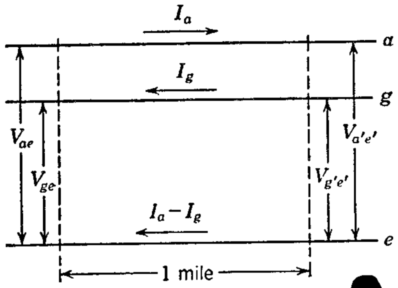


FIG. 12.10 Overhead cond[illegible]tor a [illegible] return circuit through the [illegible] g in parallel with th[illegible] presented by conductor e.

wire and earth return. If the circuit be is relabeled [illegible] the direction assumed to be positive for the flow of current in thi[illegible] is reversed, the conductor now labeled g can be considered [illegible] for conductor a, as shown in Fig. 12.10. The curr[illegible] returns through the parallel circuit which is composed of the ground wire and the earth. By comparison with Eqs. (12.22) and (12.23), since only the sign of I_b is changed when it becomes I_g,

$$V_{ae} - V_{a'e'} = I_a Z_{aa} - I_g Z_{ag} \tag{12.24}$$

$$V_{ge} - V_{g'e'} = I_a Z_{ag} - I_g Z_{gg} \tag{12.25}$$

Since we assumed that the ground wire g was electrically in parallel with the earth at the ends of the section, the voltage drop between the ground wire and earth is zero at the ends of the section; that is, $V_{ge} = 0$ and $V_{g'e'} = 0$. Substituting $V_{ge} - V_{g'e'} = 0$ in Eqs. (12.24) and (12.25) and solving for I_a by determinants, we obtain

$$I_a = \frac{-(V_{ae} - V_{a'e'})Z_{gg}}{-Z_{aa}Z_{gg} + Z_{ag}^2} \tag{12.26}$$

Since $V_{ae} - V_{a'e'}$ is the voltage drop in 1 mile of the circuit consisting of conductor a and the ground wire g in parallel with the earth return, the impedance per mile of the circuit is

$$Z'_a = \frac{V_{ae} - V_{a'e'}}{I_a} \tag{12.27}$$

$$Z'_a = \frac{Z_{aa}Z_{gg} - Z_{ag}^2}{Z_{gg}} \tag{12.28}$$

$$Z'_a = Z_{aa} - \frac{Z_{ag}^2}{Z_{gg}} \tag{12.29}$$

where the self-impedance Z_{aa} of the circuit consisting of the line and the earth and the self-impedance Z_{gg} of the circuit consisting of the ground wire and earth are computed by Eq. (12.11). The mutual impedance Z_{ag} is computed by Eq. (12.21).

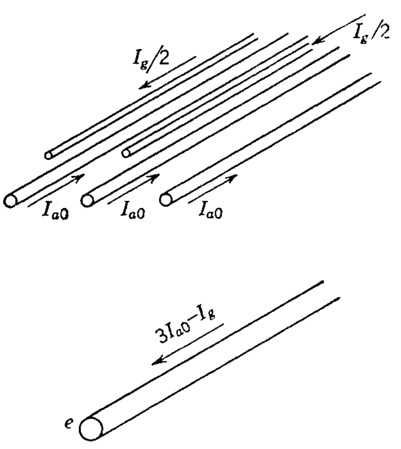


FIG. 12.11 Single-circuit three-phase line with two overhead ground wires and earth return.

In so far as the zero-sequence current of a three-phase line is concerned, the three line conductors, each carrying the current I_{a0} as shown in Fig. 12.11, may be considered as elements of the composite conductor designated a in Fig. 12.10. The grou[illegible] wi[illegible] each of which carries the cu[illegible] as shown in Fig. 12.1[illegible] be con-sidered as elemen[illegible] [illegible]posite conductor design[illegible] When the three-phase lin[illegible] treated, the zero-sequence voltage drop per mile of line, includi[illegible] earth return, is $V_{ae} - V_{a'e'}$ of the single-phase circuit of [illegible]g. 12.10. The zero-sequence current I_{a0} of the three-phase line is one third of the current of the line of Fig. 12.10. Therefore, the zero-sequence impedance of the three-phase line is

$$Z_0 = \frac{V_{a0}}{I_{a0}} = \frac{V_{ae} - V_{a'e'}}{I_a/3} \tag{12.30}$$

$$Z_0 = 3Z'_a \tag{12.31}$$

or

$$Z_0 = 3\left(Z_{aa} - \frac{Z_{ag}{}^2}{Z_{gg}}\right) \quad \text{ohms/mile} \tag{12.32}$$

where
$$Z_{aa} = \frac{R_a}{3} + 1.588f \times 10^{-3} + j4.657f \times 10^{-3} \log \frac{D_e{}^2}{D_{aa}} \quad \text{ohms/mile} \tag{12.33}$$

$$Z_{gg} = \frac{R_g}{n} + 1.588f \times 10^{-3} + j4.657f \times 10^{-3} \log \frac{D_e{}^2}{D_{gg}} \quad \text{ohms/mile} \tag{12.34}$$

$$Z_{ag} = 1.588f \times 10^{-3} + j4.657f \times 10^{-3} \log \frac{D_e{}^2}{D_{ag}} \quad \text{ohms/mile} \tag{12.35}$$

and

R_a = resistance in ohms per mile of one of the three identical line conductors
R_g = resistance in ohms per mile of one of n identical ground wires
n = number of ground wires
D_{aa} = self GMD of the composite line conductor
D_{gg} = self GMD of the composite ground wire
D_{ag} = GMD between line conductors and ground wires

For double-circuit lines the factor 3 is replaced by the factor 6 in Eqs. (12.30) to (12.33).

Example 12.2

Find the 60-cycle zero-sequence impedance per mile of the line of Example 12.1 when two ground wires are placed 10 ft above the horizontal plane of the line and 6 ft in from the outside conductors, as shown in Fig. 12.12. Each ground wire is three-strand No. 7 Copperweld having a self GMD of 0.00363 ft and a resistance of 3.07 ohms per mile at 25°C. Assume $\rho = 100$.

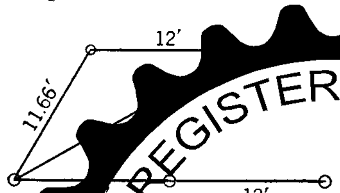


FIG. 12.12 Conductor arrangement for Example 12.2.

Solution

From Example 12.1,

$$R_a = 0.440 \text{ ohm/mile}$$
$$D_{aa} = 1.42 \text{ ft}$$

The self GMD of the composite ground wire is

$$D_{gg} = \sqrt{0.00363 \times 12} = 0.2085 \text{ ft}$$

and the GMD between line conductors and ground wires is

$$D_{ag} = \sqrt[6]{(11.66)^4(20.6)^2} = 14.1 \text{ ft}$$

From Eqs. (12.33) to (12.35),

$$Z_{aa} = \frac{0.440}{3} + 1.588 \times 60 \times 10^{-3} + j4.657 \times 60 \times 10^{-3} \log \frac{2{,}790}{1.42}$$
$$= 0.242 + j0.921 \text{ ohm/mile}$$
$$Z_{gg} = \frac{3.07}{2} + 1.588 \times 60 \times 10^{-3} + j4.657 \times 60 \times 10^{-3} \log \frac{2{,}790}{1.42}$$
$$= 1.630 + j1.155 = 2.00\underline{/35.40^\circ} \text{ ohms/mile}$$
$$Z_{ag} = 1.588 \times 60 \times 10^{-3} + j4.657 \times 60 \times 10^{-3} \log \frac{2{,}790}{14.1}$$
$$= 0.0953 + j0.642 = 0.65\underline{/81.56^\circ} \text{ ohm/mile}$$

From Eq. (12.32),

$$Z_0 = 3\left[0.242 + j0.921 - \frac{(0.65\underline{/81.56^\circ})^2}{2.00\underline{/35.40^\circ}}\right]$$
$$= 3(0.242 + j0.921 - 0.211\underline{/127.72^\circ}) = 3(0.370 + j0.753)$$
$$= 1.110 + j2.259 \text{ ohms/mile}$$

The ratio of zero-sequence to positive-sequence reactance is $\dfrac{2.259}{0.861} = 2.62$.

If resistance is neglected in computing the zero-sequence reactance in Example 12.2, the value found is $j1.692$ ohms, which is much smaller than $j2.259$ ohms, the value with resistance considered. Resistance is often neglected in setting up the sequence networks, but resistance must be included in the computations for the zero-sequence reactance of the network if the resistance of the ground wires is appreciable, as it is in the preceding example. It is particularly important to include the effect of resistance in determining zero-sequence reactance when the ground wires are steel.

Counterpoises are wires connected to each tower and buried in the earth a few feet. Their purpose is to improve lightning protection by reducing the tower footing resistance. Counterpoises may be a group of radial wires extending out from the base of each tower, or continuous wires connected to each tower, or a combination of the two arrangements. Continuous counterpoises provide a metallic return path for zero-sequence current and may be treated as additional ground wires. Because of their position, counterpoises do not carry the same value of current as the overhead ground wires and, therefore, cannot be included with the overhead ground wires as one composite conductor. The overhead ground wires should be treated as one conductor, and counterpoises consisting of more than one wire should be treated as another composite conductor. Analysis of more than one ground wire, where all the wires cannot be combined to form one composite conductor, is beyond the scope of this book.[5]

12.9 Zero-sequence Capacitance of Transmission Lines without Ground Wires. When precise results are desired for medium or long lines, the nominal- or the equivalent-π circuit must be used to represent the line. The circuit for zero-sequence currents is determined by the zero-sequence series impedance and shunt admittance per mile. The shunt admittance depends on the capacitance to neutral for zero-sequence currents and voltages.

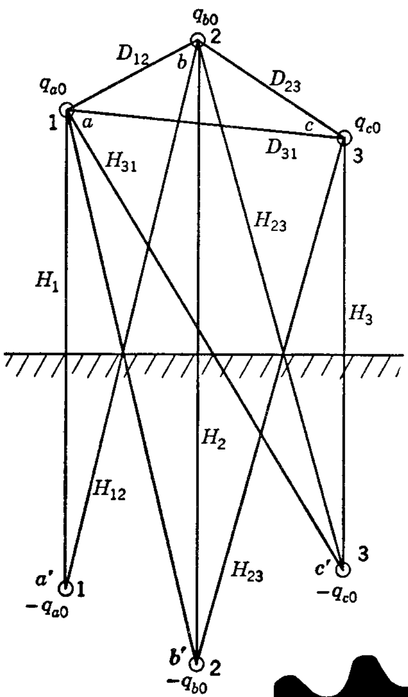


FIG. 12.13 Three-phase line with image charges.

Image charges were introduced in Sec. 3.8 to study the effect of earth on the capacitance of three-phase lines. The image-charge method may be applied to the study of zero-sequence capacitance to neutral. Figure 12.13 shows the image charges for a three-phase line. The charges produced on the lines by zero-sequence voltages are designated q_{a0}, q_{b0}, and q_{c0} to distinguish them from the charges produced by positive-sequence voltages. Equation (3.3) enables us to find the voltage between line a and its image a' with respect to all the line charges and their images. Since the voltage between a conductor and ground is half the voltage from the conductor to its image,

[5] See E. Clarke, "Circuit Analysis of A-C Power Systems," vol. I, pp. 392–400, John Wiley & Sons, Inc., New York, 1943.

$$V_{a0} = \frac{1}{2} V_{aa'} = \frac{1}{4\pi k}$$

$$\left(q_{a0} \ln \frac{H_1}{r_a} - q_{a0} \ln \frac{r_a}{H_1} + q_{b0} \ln \frac{H_{12}}{D_{12}} - q_{b0} \ln \frac{D_{12}}{H_{12}} + q_{c0} \ln \frac{H_{31}}{D_{31}} - q_{c0} \ln \frac{D_{31}}{H_{31}}\right)$$

$$= \frac{1}{2\pi k}\left(q_{a0} \ln \frac{H_1}{r_a} + q_{b0} \ln \frac{H_{12}}{D_{12}} + q_{c0} \ln \frac{H_{31}}{D_{31}}\right) \quad (12.36)$$

Similarly

$$V_{b0} = \frac{1}{2\pi k}\left(q_{a0} \ln \frac{H_{12}}{D_{12}} + q_{b0} \ln \frac{H_2}{r_b} + q_{c0} \ln \frac{H_{23}}{D_{23}}\right) \quad (12.37)$$

$$V_{c0} = \frac{1}{2\pi k}\left(q_{a0} \ln \frac{H_{31}}{D_{31}} + q_{b0} \ln \frac{H_{23}}{D_{23}} + q_{c0} \ln \frac{H_3}{r_c}\right) \quad (12.38)$$

Since the zero-sequence voltages of the lines to ground are identical according to the definition of zero sequence, the charges q_{a0}, q_{b0}, and q_{c0} must be unequal. For the usual spacing of transmission lines, however, the charges are very nearly equal. Little error and a much simpler expression is obtained by assuming the charges to be equal and the zero-sequence voltage to be the average of the resulting expressions for the three voltages. If we let

$$q_0 = q_{a0} = q_{b0} = q_{c0}$$

and

$$V_0 = \frac{V_{a0} + V_{b0} + V_{c0}}{3}$$

we obtain

$$V_0 = \frac{q_0}{6\pi k} \ln \left[\frac{H_1H_2H_3(H_{12}H_{23}H_{31})^2}{r_ar_br_c(D_{12}D_{23}D_{31})^2}\right] \quad (12.39)$$

$$V_0 = \frac{3q_0}{2\pi k} \ln \frac{\sqrt[9]{H_1H_2H_3(H_{12}H_{23}H_{31})^2}}{\sqrt[9]{r_ar_br_c(D_{12}D_{23}D_{31})^2}} \quad \text{volts} \quad (12.40)$$

The numerator of the logarithmic term in Eq. (12.40) is the GMD between the overhead conductors and their images. The denominator is the self GMD of the overhead lines considered as a composite conductor, except that the actual radius of each individual wire in the composite conductor replaces the self GMD of the wire in computing the self GMD of the composite conductor. If we let

D_{aa} = the self GMD of the composite line conductor (actual wire radius replacing r')

and

H_{aa} = GMD between line conductors and their images

Eq. (12.40) becomes

$$V_0 = \frac{3q_0}{2\pi k} \ln \frac{H_{aa}}{D_{aa}} \quad \text{volts} \quad (12.41)$$

and zero-sequence capacitance per phase to neutral is

$$C_0 = \frac{q_0}{V_0} = \frac{2\pi k}{3 \ln (H_{aa}/D_{aa})} \qquad \text{farads/meter} \tag{12.42}$$

$$C_0 = \frac{0.01294}{\log (H_{aa}/D_{aa})} \qquad \mu\text{f/mile} \tag{12.43}$$

12.10 Zero-sequence Capacitance of Transmission Lines with Ground Wires. To find the effect of ground wires on zero-sequence capacitance to neutral, we shall consider first a single conductor with ground-wire and earth return, as shown in Fig. 12.14, where the conductor is marked a and the ground wire is g, with corresponding charges q_a and q_g. Image conductors account for the earth in capacitance computations. The voltage from line a to ground is

$$V_a = \tfrac{1}{2} V_{aa'} = \frac{1}{2\pi k}\left(q_a \ln \frac{H_a}{r_a} + q_g \ln \frac{H_{ag}}{D_{ag}}\right) \tag{12.44}$$

and the voltage from wire g to ground is

$$V_g = \tfrac{1}{2} V_{gg'} = \frac{1}{2\pi k}\left(q_a \ln \frac{H_{ag}}{D_{ag}} + q_g \ln \frac{H_g}{r_g}\right) \tag{12.45}$$

Since wire g is grounded at every tower, $V_g = 0$. Solving Eqs. (12.44) and (12.45) for q_a gives

$$q_a = \frac{2\pi k V_a \ln \dfrac{H_g}{r_g}}{\ln \dfrac{H_a}{r_a} \ln \dfrac{H_g}{r_g} - \left(\ln \dfrac{H_{ag}}{D_{ag}}\right)^2} \tag{12.46}$$

Equation (12.46) can be applied to multicircuit lines with more than one ground wire by extending to it the principle of GMD, which was seen to apply to a circuit without ground wires. If we consider one composite conductor including all the line conductors and another including all the ground wires, Eq. (12.46) gives the total zero-sequence charge. The charge per phase is that given in Eq. (12.46) divided by 3. The voltage V_a is the zero-sequence voltage to neutral per phase V_{a0}. The distances in Eq. (12.46) must then be interpreted as follows:

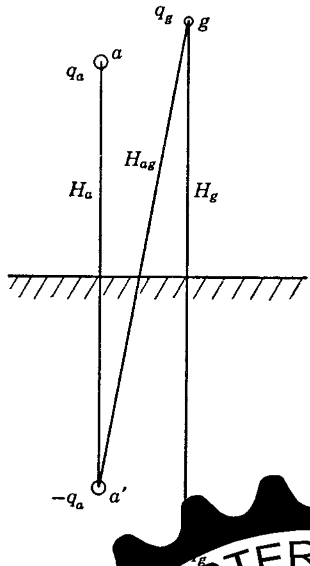


Fig. 12.14 Conductor a with ground wire g and image charges.

H_{ag} = GMD between the line conductors and the images of the ground wires

D_{ag} = GMD between the line conductors and the ground wires

H_a becomes H_{aa} = GMD between the line conductors and their images

H_g becomes H_{gg} = GMD between the ground wires and their images

r_a becomes D_{aa} = the self GMD of the line conductors considered as a composite conductor (except actual radius of each individual conductor is used in the computation)

r_g becomes D_{gg} = the self GMD of the ground wires considered as a composite conductor (except actual radius of each individual conductor is used in the computation)

Then,

$$q_{a0} = \frac{2\pi k}{3} \frac{V_{a0} \ln \dfrac{H_{gg}}{D_{gg}}}{\ln \dfrac{H_{aa}}{D_{aa}} \ln \dfrac{H_{gg}}{D_{gg}} - \left(\ln \dfrac{H_{ag}}{D_{ag}}\right)^2} \tag{12.47}$$

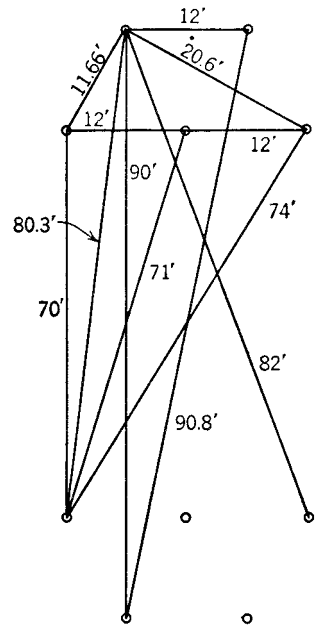


Fig. 12.15 Arrangement of conductors and image charges for Example 12.3.

and the zero-sequence capacitance to neutral is

$$C_0 = \frac{q_{a0}}{V_{a0}}$$

$$= \frac{0.01294 \log \dfrac{H_{gg}}{D_{gg}}}{\log \dfrac{H_{aa}}{D_{aa}} \log \dfrac{H_{gg}}{D_{gg}} - \left(\log \dfrac{H_{ag}}{D_{ag}}\right)^2} \quad \mu\text{f/mile} \tag{12.48}$$

Example 12.3

Find the zero-sequence capacitance of the line of Example 12.2 if the line conductors are 35 ft above the ground.

Solution

The line with its image conductors is shown in Fig. 12.15.

The values of D_{aa} and D_{gg} which [illegible]puted in Examples 12.1 and [illegible]re not valid for capacitance since th[illegible] actual outside radius of each conduc[illegible] must replace r' in capacitance calculation[illegible] Treating a three-strand conductor as a cylindrical conductor h[illegible]ng an outside radius equal to that of the three-strand conductor resu[illegible] in a slight error that may be neglected.

For No. 2/0 seven-strand hard-drawn copper, $r = \dfrac{0.414}{2 \times 12} = 0.0172$ ft, and for No. 7 three-strand Copperweld, $r = \dfrac{0.311}{2 \times 12} = 0.0130$ ft.

$$
\begin{aligned}
D_{aa} &= \sqrt[9]{(0.0172)^3(12)^4(24)^2} = 1.58 \text{ ft} \\
D_{gg} &= \sqrt[4]{(0.0130)^2(12)^2} = 0.395 \text{ ft} \\
D_{ag} &= 14.1 \text{ ft (from Example 12.2)} \\
H_{aa} &= \sqrt[9]{(70)^3(71)^4(74)^2} = 71.4 \text{ ft} \\
H_{gg} &= \sqrt[4]{(90)^2(90.8)^2} = 90.3 \text{ ft} \\
H_{ag} &= \sqrt[6]{(80.3)^4(82)^2} = 81.0 \text{ ft} \\
C_0 &= \frac{0.01294 \log \dfrac{90.3}{0.395}}{\log \dfrac{71.4}{1.58} \log \dfrac{90.3}{0.395} - \left(\log \dfrac{81.0}{14.1}\right)^2} \\
&= 0.00918 \ \mu\text{f/mile}
\end{aligned}
$$

12.11 Positive- and Negative-sequence Networks. As was pointed out in Chap. 11, an independent network of impedances of one sequence only can be synthesized for the flow of current of that sequence. Such sequence networks were discussed in Chap. 11 for one unloaded generator with an impedance in the neutral. Simple interconnections of the sequence networks were made to determine the amount of current of each sequence during different kinds of faults.

The object of obtaining the values of the sequence impedances of a power system is to enable us to construct the sequence networks for the complete system. The network of a particular sequence for a power system shows all the paths for the flow of current of that sequence in the system.

We discussed the construction of some rather complex positive-sequence networks in Chap. 8. The transition from a positive-sequence network to a negative-sequence network is simple. Three-phase synchronous generators and motors have internal voltages of positive-sequence only, since they are designed to generate balanced voltages. Since the positive- and negative-sequence impedances are the same in a static symmetrical system, conversion of a positive-sequence network to a negative-sequence network is accomplished by changing, if necessary, only the impedances which represent rotating machinery and by omitting the emfs. Electromotive forces are omitted on the assumption of balanced generated voltages and the absence of negative-sequence voltages induced from outside sources.

Since all the neutral points of a symmetrical three-phase system are at the same potential when balanced three-phase currents are flowing, all the neutral points must be at the same potential for either positive-or negative-sequence currents. Therefore the neutral of a symmetrical three-phase system is the logical reference potential for specifying positive- and negative-sequence voltage drops and is the reference bus of the positive- and negative-sequence networks. Impedance connected between the neutral of a machine and ground is not a part of either the positive- or negative-sequence network because neither positive- nor negative-sequence current can flow in an impedance so connected.

Negative-sequence networks, like the positive-sequence networks of Chap. 8, may contain the exact equivalent circuits of parts of the system or may be simplified by omitting series resistance and shunt admittance.

Example 12.4

Draw the negative-sequence network for the system described in Example 8.2. Assume that the negative-sequence reactance of each machine is equal to its subtransient reactance. Omit resistance.

Solution

Since all the negative-sequence reactances of the system are equal to the positive-sequence reactances, the negative-sequence network is iden-

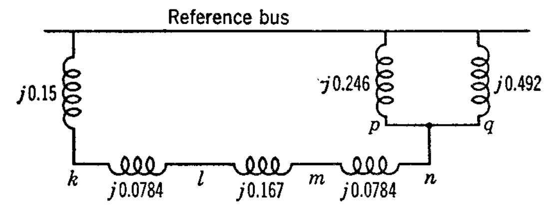


Fig. 12.16 Negative-sequence network for Example 12.4.

tical to the positive-sequence network of Fig. 8.8 except for the omission of emfs from the negative-sequence network. The required network is drawn in Fig. 12.16.

12.12 Zero-sequence Networks. A three-phase system operates single phase in so far as the zero-sequence currents are concerned, for the zero-sequence currents are the same in magnitude and phase at any point in all the phases of the system. Therefore zero-sequence currents will flow only if a return path exists through which a completed circuit is provided. The reference for zero-sequence voltages is the potential of the ground at the point in the system at which any particular voltage is specified. Since zero-sequence currents may be flowing in the ground, the ground is not necessarily at the same potential at all points, and the reference bus of the zero-sequence network does not represent a ground of uniform potential. The impedance of the ground and ground wires is included in the zero-sequence impedance of the transmission line, and the return circuit of the zero-sequence network is a conductor of zero impedance, which is the reference bus of the system. It is because the impedance of the ground is included in the zero-sequence impedance that voltages measured to the reference bus of the zero-sequence network give the correct voltage to ground.

If a circuit is Y-connected, with no connection from the neutral to ground or to another neutral point in the circuit, the sum of the currents flowing into the neutral in the three phases is zero. Since currents whose

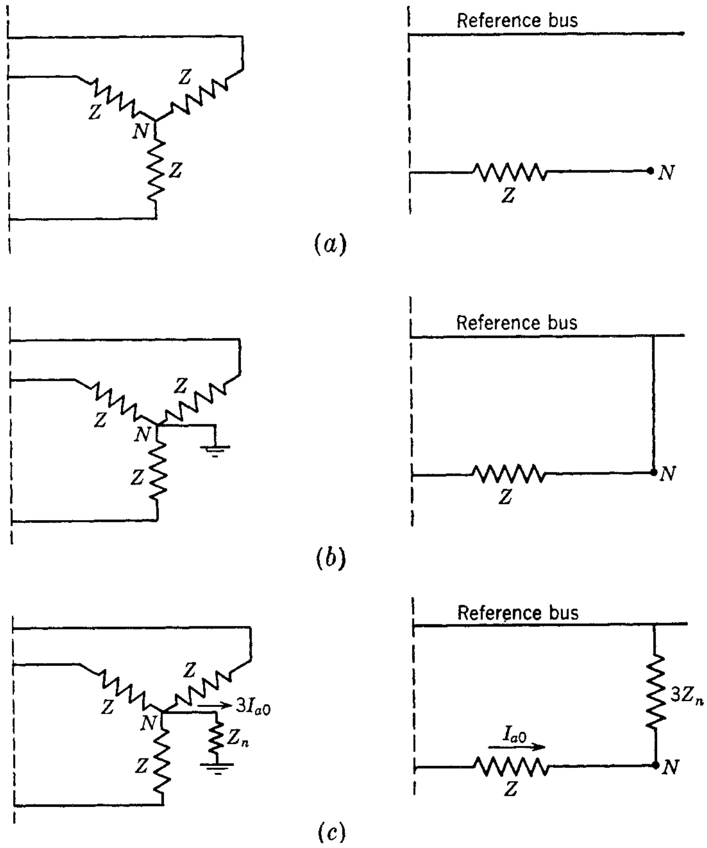


FIG. 12.17 Zero-sequence networks for Y-connected loads.

sum is zero have no zero-sequence components, the impedance to zero-sequence current is infinite beyond the neutral point, which fact is indicated by an open circuit in the zero-sequence network between the neutral of the Y-connected circuit and the reference bus, as shown in Fig. 12.17a.

If the neutral of a Y-connected circuit is grounded through zero impedance, a zero-impedance connection is inserted to connect the neutral point and the reference bus of the zero-sequence network, as shown in Fig. 12.17b.

If the impedance Z_n is inserted between the neutral and ground of a Y-connected circuit, an impedance of $3Z_n$ must be placed between the neutral and reference bus of the zero-sequence network, as shown in Fig. 12.17*c*. As explained in Sec. 11.2, the zero-sequence voltage drop caused in the zero-sequence network by I_{a0} flowing through $3Z_n$ is the same as in the actual system where $3I_{a0}$ flows through Z_n. Impedance consisting of a resistor or reactor is usually connected between the neutral of a generator and ground to limit the zero-sequence current during a fault. The impedance of such a current-limiting resistor or reactor is represented in the zero-sequence network in the manner described.

A Δ-connected circuit, since it can provide no return path, offers infinite impedance to zero-sequence line currents. The zero-sequence network is open at the Δ-connected circuit. Zero-sequence currents may circulate inside the Δ circuit since the Δ is a closed series circuit for circulating single-phase currents. Such currents would have to be produced in the

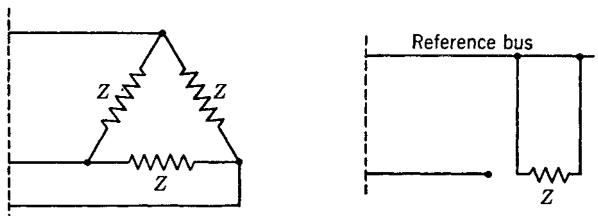


FIG. 12.18 Δ-connected load and its zero-sequence network.

Δ, however, by induction from an outside source or by zero-sequence generated voltages. A Δ circuit and its zero-sequence network are shown in Fig. 12.18. Even when zero-sequence voltages are generated in the phases of the Δ, no zero-sequence voltage exists between the Δ terminals, for the rise in voltage in each phase of the generator is matched by the voltage drop in the zero-sequence impedance of each phase.

The zero-sequence equivalent circuits of three-phase transformers deserve special attention. The various possible combinations of the primary and secondary windings in Y and Δ alter the zero-sequence network. Transformer theory enables us to construct the equivalent circuit for the zero-sequence network. We remember that no current flows in the primary of a transformer unless current flows in the secondary, if we neglect the relatively small magnetizing current. We know, also, that the primary current is determined by the secondary current and the turns ratio of the windings, again with magnetizing current neglected. These principles guide us in the analysis of individual cases. Five possible connections of two-winding transformers will be discussed. These connections are shown in Fig. 12.19. The arrows on the connection diagrams show the possible paths for the flow of zero-sequence current. Absence of an arrow indicates that the transformer connection is such that zero-sequence current cannot flow. The zero-sequence approximately equivalent circuit, with resistance and a path for magnetizing current omitted, is shown in Fig. 12.19 for each connection. The letters P and Q identify corresponding points on the connection diagram

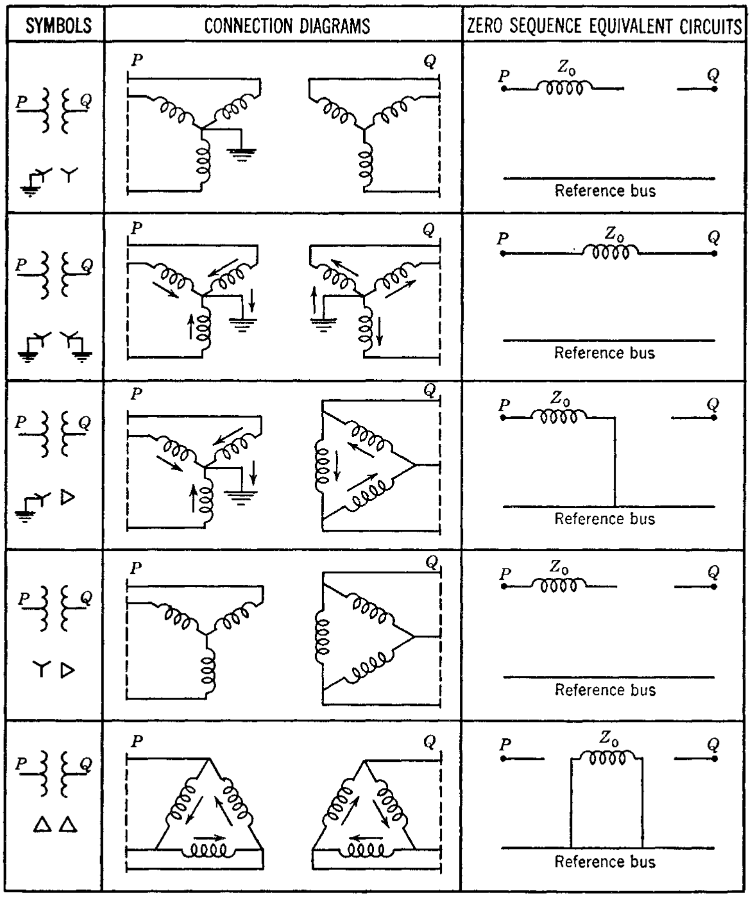


FIG. 12.19 Zero-sequence equivalent circuits of three-phase transformer banks, together with diagrams of connections and the symbols for one-line diagrams.

and equivalent circuit. The reasoning to justify the equivalent circuit for each connection follows.

Case 1. Y-Y *Bank, One Neutral Grounded.* If either one of the neutrals of a Y-Y bank is ungrounded, zero-sequence current cannot flow in either winding. The absence of a path through one winding prevents current in the other. An open circuit exists for zero-sequence current between the two parts of the system connected by the transformer.

Case 2. Y-Y *Bank, Both Neutrals Grounded.* Where both neutrals of a Y-Y bank are grounded, a path through the transformer exists for zero-sequence currents in both windings. Provided the zero-sequence current can follow a complete circuit outside the transformer on both sides, it can flow in both windings of the transformer. In the zero-sequence network, points on the two sides of the transformer are connected by the zero-sequence impedance of the transformer in the same manner as was followed in the positive- and negative-sequence networks.

Case 3. Y-Δ *Bank, Grounded* Y. If the neutral of a Y-Δ bank is grounded, zero-sequence currents have a path to ground through the Y because corresponding induced currents can circulate in the Δ. The zero-sequence current circulating in the Δ to balance the zero-sequence

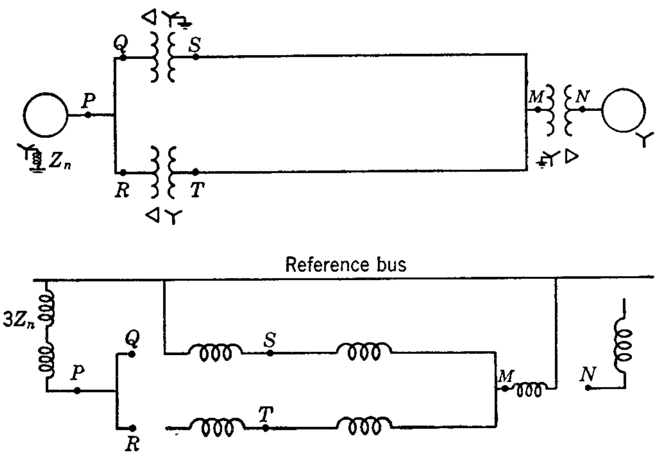


FIG. 12.20 One-line diagram of a small power system and the corresponding zero-sequence network.

current in the Y cannot flow in the lines connected to the Δ. The equivalent circuit must provide for a path from the line on the Y side through the equivalent resistance and leakage reactance of the transformer to the reference bus. An open circuit must exist between the line and the reference bus on the Δ side. If the connection from neutral to ground contains an impedance Z_n, the zero-sequence equivalent circuit must have an impedance of $3Z_n$ in series with the equivalent resistance and leakage reactance of the transformer to connect the line on the Y side to ground.

Case 4. Y-Δ *Bank, Ungrounded* Y. An ungrounded Y is the case where the impedance Z_n between neutral and ground is infinite. The impedance $3Z_n$ in the equivalent circuit of Case 3 for zero-sequence impedance becomes infinite. Zero-sequence current cannot flow in the transformer windings.

Case 5. Δ-Δ *Bank.* Since a Δ circuit provides no return path for zero-sequence current, no zero-sequence current can flow into a Δ-Δ bank, although it can circulate within the Δ windings.

Zero-sequence equivalent circuits determined for various parts of the system separately are readily combined to form the complete zero-sequence network. Figures 12.20 and 12.21 show one-line diagrams of

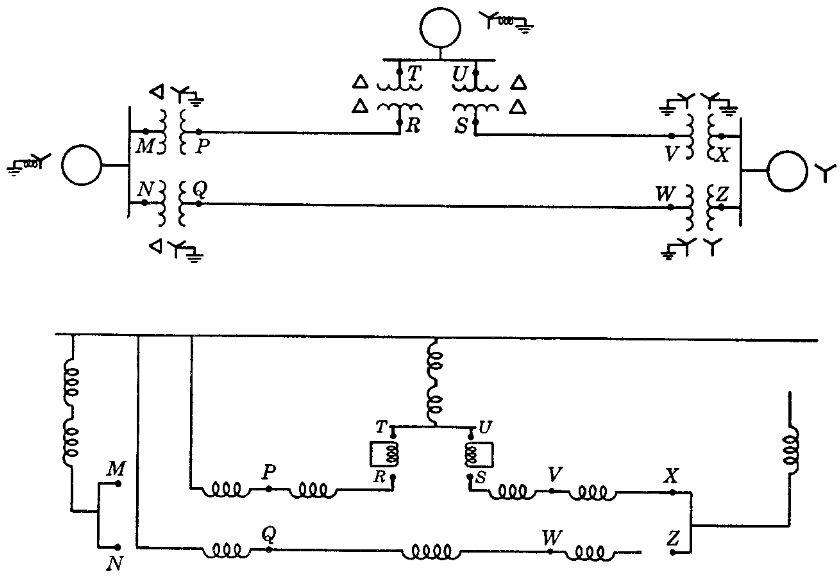


FIG. 12.21 One-line diagram of a small power system and the corresponding zero-sequence network.

two small power systems and their corresponding zero-sequence networks simplified by omitting resistances and shunt admittances.

Example 12.5

Draw the zero-sequence network for the system described in Example 8.2. Assume zero-sequence reactances for the generator and motors of 0.05 per unit. Current-limiting reactors of 2.0 ohms each are in the neutral of the generator and the larger motor. The zero-sequence reactance of the transmission line is 250 ohms.

Solution

The zero-sequence leakage reactance of transformers is equal to the positive-sequence reactance. So, for the transformers, $X_0 = 0.07$[illegible] per unit.

Zero-sequence reactances of the generator and moto[illegible]

Generator: $X_0 = 0.05$ per unit

Motor 1: $X_0 = 0.05 \times \dfrac{30{,}000}{20{,}000} \times \left(\dfrac{12.5}{13.8}\right)^2 = 0.061$ per unit

Motor 2: $X_0 = 0.05 \times \dfrac{30{,}000}{10{,}000} \times \left(\dfrac{12.5}{13.8}\right)^2 = 0.123$ per unit

For the current-limiting reactors,

$$\text{Base } Z \text{ in the low-tension circuits} = \frac{13.8^2 \times 1{,}000}{30{,}000} = 6.35 \text{ ohms}$$

$$\text{Reactance} = \frac{2.0}{6.35} = 0.315 \text{ per unit}$$

In the impedance network: $3Z_n = 3 \times 0.315 = 0.945$ per unit

For the transmission line,

$$X_0 = {}^{250}\!/_{480} = 0.52 \text{ per unit}$$

The zero-sequence network is shown in Fig. 12.22.

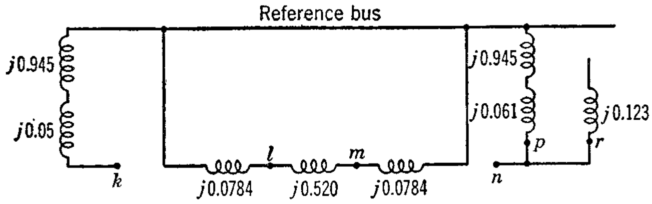


Fig. 12.22 Zero-sequence network for Example 12.5.

12.13 Conclusions. A knowledge of the positive-sequence network is necessary for load studies on power systems, for fault calculations, and for stability studies. If the fault calculations or stability studies involve unsymmetrical faults on otherwise symmetrical systems, the negative- and zero-sequence networks are also needed. Synthesis of the zero-sequence network requires particular care, because the zero-sequence network may differ considerably from the positive- and negative-sequence networks.

All the impedances of a transmission line can be calculated from the physical dimensions of the line. Impedances of transformers, synchronous machines, and induction motors are determined by test or from tables of average per-unit values.

PROBLEMS

12.1 The negative-sequence impedance of an alternator is measured with two terminals short-circuited. Meters are connected as described in Sec. 12.4. The ammeter, voltmeter, and wattmeter read, respectively, 2,150 amp, 415 volts, and 825 kw. The alternator is rated 5,000 kva, 2.4 kv. Find the nega[illegible]ce resistance and reactance in per unit. Prove by symmetrical compon[illegible] method yields the desired quantities.

12.2 The zero-sequence impedance of an alternator is measure[illegible] terminals connected to ground through a common ammeter. Meters [illegible]ected as described in Sec. 12.5. The ammeter, voltmeter, and wattmeter read, [illegible]ely, 2,400 amp, 132 volts, and 38 kw. The alternator is rated 5,000 [illegible]4 kv. Find the zero-sequence resistance and reactance in per unit. Prove [illegible] that the method yields the desired quantities.

12.3 For a certain induction motor, the values of the equivalent circuit shown in Fig. 12.5 are as follows:

$$R_s = 0.055 \text{ per unit} \qquad X_s + X_r = 0.16 \text{ per unit}$$
$$R_r = 0.050 \text{ per unit} \qquad X_m = 3.00 \text{ per unit}$$

Find the per-unit resistance and reactance to positive- and negative-sequence current when the motor is operating at a slip of 5%.

12.4 A single-circuit 60-cycle three-phase transmission line is constructed with flat, horizontal spacing. Adjacent conductors are spaced 15.5 ft apart. The conductors are No. 4/0 hard-drawn copper with seven strands. Determine the zero-sequence series impedance per mile per phase for the line without ground wires. Assume damp earth beneath the line and a wire temperature of 25°C.

12.5 If the line described in Prob. 12.4 has two overhead ground wires, compute the zero-sequence series impedance per mile per phase. Each overhead ground conductor is placed 7.75 ft in a horizontal direction from the center conductor of the three-phase line and 10 ft above the plane of the line conductors. Each ground wire has an outside diameter of 0.360 in., a self GMD of 2.64×10^{-4} ft at 60 cps, and a resistance of 6.00 ohms/mile. Assume damp earth beneath the line and a wire temperature of 25°C.

12.6 If the conductors of the line described in Prob. 12.4 are 37 ft above the ground, find the zero-sequence capacitance to neutral per mile without ground wires.

12.7 If the conductors of the line described in Prob. 12.4 are 37 ft above ground and the ground wires described in Prob. 12.5 are installed, find the zero-sequence capacitance to neutral per mile.

12.8 Find the ratios of the zero-sequence reactances found in Probs. 12.4 and 12.5 to the positive-sequence inductive reactance of the line.

12.9 Find the ratios of the zero-sequence capacitances found in Probs. 12.6 and 12.7 to the positive-sequence capacitance to neutral of the line.

12.10 Draw the negative- and zero-sequence impedance networks for the power system of Prob. 8.5. Mark the values of all reactances in per unit on a base of 30,000 kva, 6.9 kv in the circuit of generator 1. Letter the networks to correspond to the one-line diagram. The neutrals of generators 1 and 2 are connected to ground through current-limiting reactors having a reactance of 5%, each on the base of the machine to which it is connected. Each generator has negative- and zero-sequence reactances of 15% and 5%, respectively, on its own rating as base. The zero-sequence reactance of the transmission line is 250 ohms from B to C and 210 ohms from C to E.

12.11 Draw the negative- and zero-sequence impedance networks for the power system of Prob. 8.8. Choose a base of 50,000 kva, 138 kv in the 40-ohm transmission line, and mark all reactances in per unit. The negative-sequence reactance of each synchronous machine is equal to its subtransient reactance. The zero-sequence reactance of each machine is 8% based on its own rating. The neutrals of the machines are connected to ground through current-limiting reactors having a reactance of 5%, each on the base of the machine to which it is connected. Assume that the zero-sequence reactances of the transmission lines are 300% of their positive-sequence reactances.

CHAPTER 13

UNSYMMETRICAL FAULTS ON POWER SYSTEMS

13.1 Fundamental Relations. In Chap. 11 we derived equations and made interconnections of sequence networks for various types of faults on isolated generators at no load. Now that we understand the synthesis of sequence networks for complete power systems, we can derive the equations and sequence-network connections for faults which occur on balanced systems of any degree of complexity and on systems containing any number of synchronous machines, induction motors, and symmetrical loads.

In the derivation of equations for the symmetrical components of currents and voltages in a general network during a fault, we will designate as I_a, I_b, and I_c the currents flowing out of the original balanced system at the fault from phases a, b, and c, respectively. We can visualize the currents I_a, I_b, and I_c by referring to Fig. 13.1, which shows the three lines of the three-phase system at the part of the network where the fault occurs. The flow of current from each line into the fault is indicated by arrows shown on the diagram beside hypothetical stubs connected to each line at the fault location. Appropriate connections of the stubs represent various types of faults. For instance, connecting stubs b and c produces a line-to-line fault through zero impedance. The current in stub a is then zero, and I_b is equal to $-I_c$.

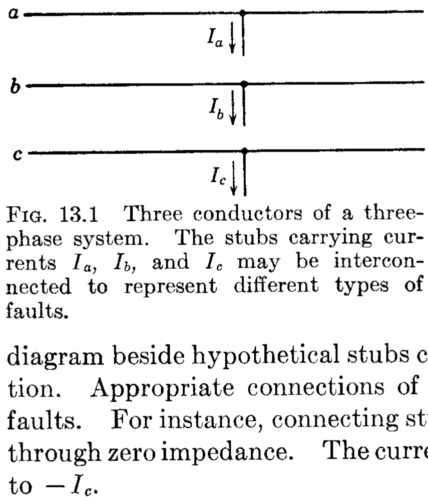


FIG. 13.1 Three conductors of a three-phase system. The stubs carrying currents I_a, I_b, and I_c may be interconnected to represent different types of faults.

The line-to-ground voltages at the fault will be designated V_a, V_b, and V_c. Before the fault occurs, the line-to-neutral voltage of phase a at the fault will be called V_f, which is a positive-sequence voltage since the system is assumed to be balanced. We met the prefault voltage V_f previously in Chap. 9 in calculations to determine the currents in a power system when a symmetrical three-phase fault occurred.

A single-line diagram of a power system containing three synchronous machines is shown in Fig. 13.2. Such a system is sufficiently general that equations derived therefrom are applicable to any balanced system regardless of the complexity. Figure 13.2 also shows the sequence networks of the system. The point where a fault is assumed to occur is

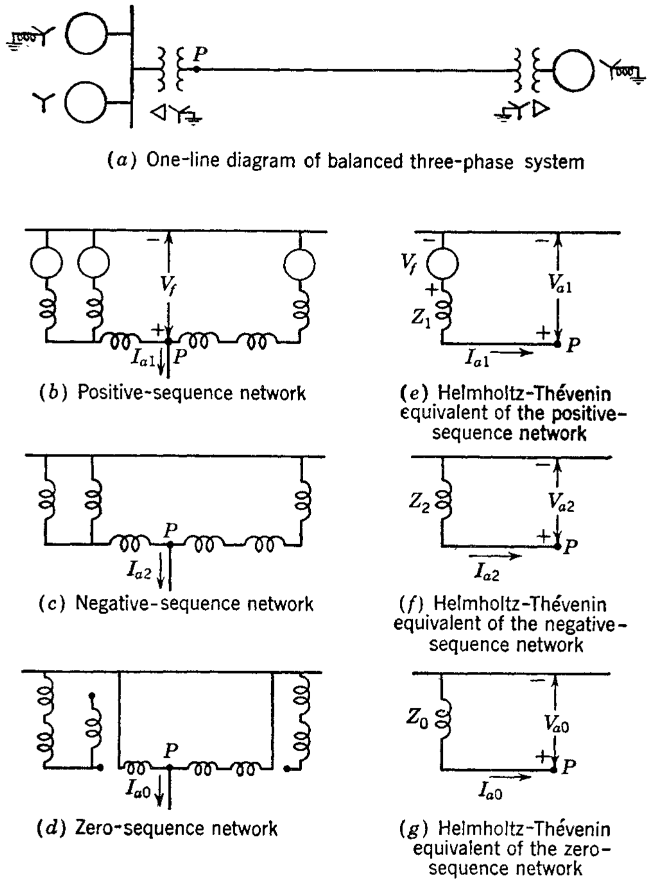


FIG. 13.2 One-line diagram of a three-phase system, the three sequence networks of the system, and the Helmholtz-Thévenin equivalent of each network for a fault at P.

marked P on the single-line diagram and on the sequence networks. As we saw in Chap. 9, the load current flowing in the positive-sequence network is the same, and the voltages to ground external to the machines are the same, regardless of whether the machines are represented by their voltages behind subtransient reactance and their subtransient reactances, or by their voltages behind transient reactances and their transient reactances, or by their voltages behind synchronous reactances and their synchronous reactances.

Since linearity is assumed in drawing the sequence networks, each of the networks can be replaced by its Helmholtz-Thévenin equivalent between the two terminals composed of its reference bus and the point of application of the fault.[1] The Helmholtz-Thévenin equivalent circuit of each sequence network is shown adjacent to the diagram of the corresponding network in Fig. 13.2. The internal voltage of the single generator of the equivalent circuit for the positive-sequence network is V_f, the prefault voltage to neutral at the point of application of the fault. The impedance Z_1 of the equivalent circuit is the impedance measured between point P and the reference bus of the positive-sequence network with all the internal emfs short-circuited. The value of Z_1 is dependent on whether subtransient, transient, or synchronous reactance is used in the sequence network, which is, in turn, dependent on whether subtransient, transient, or steady-state currents are being computed.

Since no negative- or zero-sequence currents are flowing before the fault occurs, the prefault voltage between point P and the reference bus is zero in the negative- and zero-sequence networks. Therefore, no emfs appear in the equivalent circuits of the negative- and zero-sequence networks. The impedances Z_2 and Z_0 are measured between point P and the reference bus in their respective networks and depend on the location of the fault.

Since I_a is the current flowing from the system into the fault, its components I_{a1}, I_{a2}, and I_{a0} flow out of their respective sequence networks and the equivalent circuits of the networks at P, as shown in Fig. 13.2. Examination of the equivalent circuits of the sequence networks shows that the voltages V_{a1}, V_{a2}, and V_{a0} at point P are expressed by the following equations:

$$V_{a1} = V_f - I_{a1}Z_1 \tag{13.1}$$
$$V_{a2} = -I_{a2}Z_2 \tag{13.2}$$
$$V_{a0} = -I_{a0}Z_0 \tag{13.3}$$

The only differences between Eqs. (13.1) to (13.3) and Eqs. (11.2) to (11.4) are the substitution of V_f for E_a and the interpretation of Z_1, Z_2, and Z_0. For a fault at the terminals of an isolated generator at no load, E_a and V_f are equal, and Eqs. (13.1) to (13.3) reduce to Eqs. (11.2) to (11.4).

13.2 Single Line-to-ground Fault on a Power System. [illegible] line-to-ground fault, the hypothetical stubs on the th[illegible] are connected as shown in Fig. 13.3. The following relations [illegible]st at the fault:

$$I_b = 0 \qquad I_c = 0 \qquad V_a = 0$$

The three equations above are the same as tho[illegible] apply to a line-

[1] For a statement of the Helmholtz-Thévenin theorem, see Sec. 8.9.

to-ground fault on a single generator. These equations with the general equations (13.1) to (13.3) and the relations of symmetrical components must have the same solutions as are found for similar equations in Sec. 11.3, except that V_f replaces E_a. Thus, for a line-to-ground fault,

$$I_{a1} = I_{a2} = I_{a0} \qquad (13.4)$$

and

$$I_{a1} = \frac{V_f}{Z_1 + Z_2 + Z_0} \qquad (13.5)$$

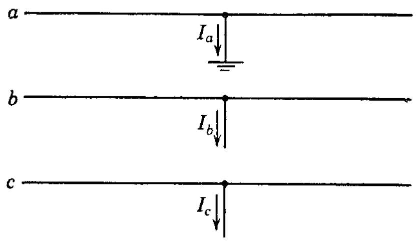


FIG. 13.3 Connection diagram of the hypothetical stubs for a single line-to-ground fault.

Equations (13.4) and (13.5) indicate that the three sequence networks should be connected in series through the fault point in order to simulate a single line-to-ground fault.

13.3 Line-to-line Fault on a Power System. For a line-to-line fault, the hypothetical stubs on the three lines at the fault are connected as shown in Fig. 13.4. The following relations exist at the fault:

$$V_b = V_c \qquad I_a = 0 \qquad I_b = -I_c$$

The above equations are identical in form to those which apply to a line-to-line fault on an isolated generator. Their solution in the manner

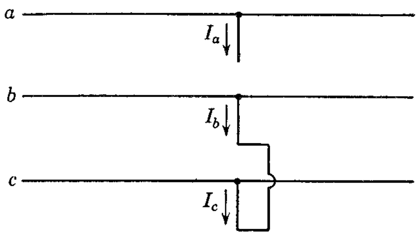


FIG. 13.4 Connection diagram of the hypothetical stubs for a line-to-line fault.

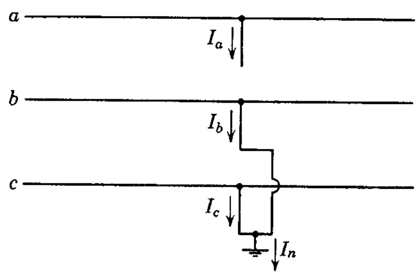


FIG. 13.5 Connection diagram of the hypothetical stubs for a double line-to-ground fault.

of Sec. 11.4, with Eqs. (13.1) to (13.3) replacing Eqs. (11.2) to (11.4), yields

$$V_{a1} = V_{a2} \qquad (13.6)$$

$$I_{a1} = \frac{V_f}{Z_1 + Z_2} \qquad (13.7)$$

Equations (13.6) and (13.7) indicate that the positive- and negative-sequence networks should be connected in parallel at the fault point in order to simulate a line-to-line fault.

13.4 Double Line-to-ground Fault on a Power System. For a double line-to-ground fault, the stubs are connected as shown in Fig. 13.5. The following relations exist at the fault:

$$V_b = V_c = 0$$
$$I_a = 0$$

By comparison with the derivation made in Sec. 11.5,

$$V_{a1} = V_{a2} = V_{a0} \tag{13.8}$$

$$I_{a1} = \frac{V_f}{Z_1 + Z_2Z_0/(Z_2 + Z_0)} \tag{13.9}$$

Equations (13.8) and (13.9) indicate that the three sequence networks should be connected in parallel at the fault point in order to simulate a double line-to-ground fault.

13.5 Interpretation of the Interconnected Sequence Networks. In the preceding sections we have seen that the sequence networks of a

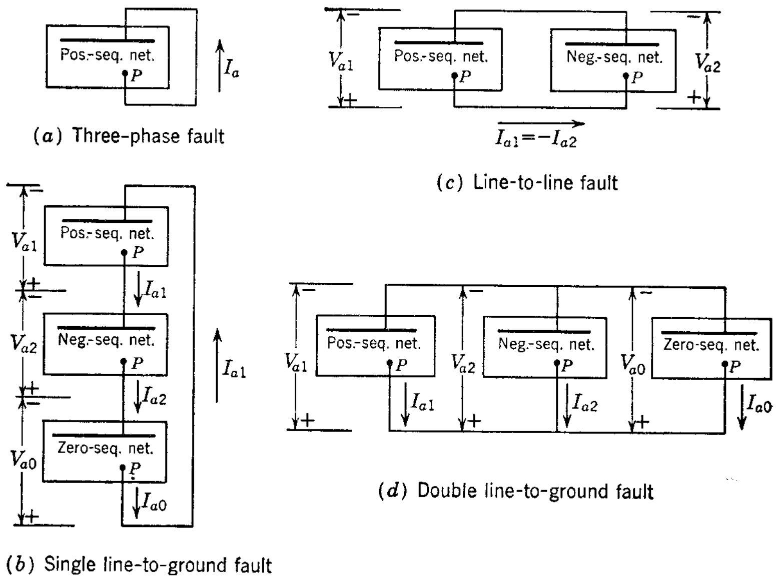
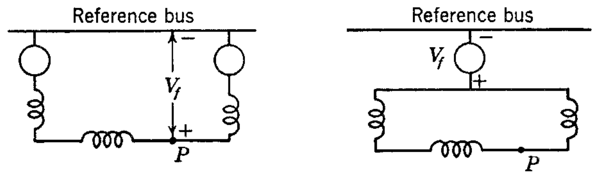


Fig. 13.6 Connections of the sequence networks to simulate various types of faults. The sequence networks are indicated by rectangles. The point at which the fault occurs is P.

power system can be so interconnected that solving the resulting network yields the symmetrical components of current and voltage at the fault. The connections of the sequence networks to simulate various types of faults, including a symmetrical three-phase fault, are shown in Fig. 13.6. The sequence networks are indicated schematically by rectangles enclosing a heavy line to represent the reference bus of the network and a point marked P to represent the point in the network where the fault occurs. The positive-sequence network contains emfs which represent the internal voltages of the machines.

If the sequence networks are set up on a calculating board and interconnected as shown in Fig. 13.6, the current measured in per unit in a particular branch of one of the sequence networks is a symmetrical component of the per-unit current of phase a of the corresponding part of the power system. A current measured in a certain branch of the positive-sequence network is the positive-sequence component of the current of phase a. Currents measured in the same branch of the negative- and zero-sequence networks when added to the positive-sequence component give the total current for phase a in that branch. The symmetrical components of the currents in phases b and c in any branch of a system can be determined from the symmetrical components of the current in phase a of the same branch. To express the per-unit currents



(a) Positive-sequence network

(b) Helmholtz-Thévenin equivalent of the positive-sequence network

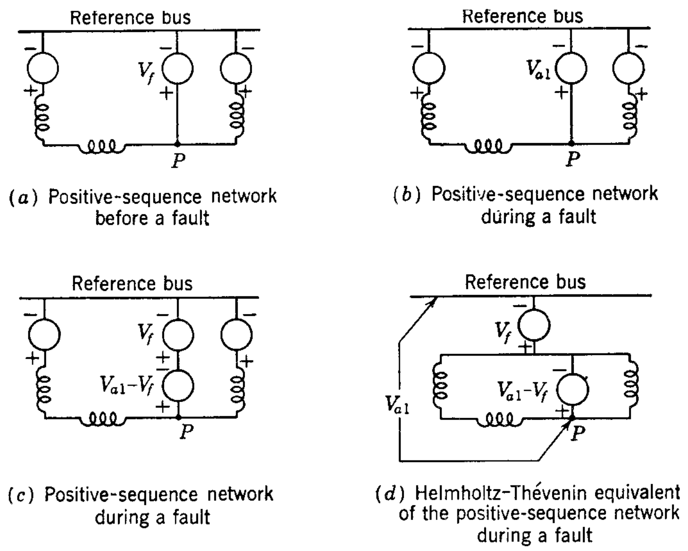
FIG. 13.7 Positive-sequence network and its Helmholtz-Thévenin equivalent.

read on a calculating board as per-unit currents in the actual power system with the proper phase relation to other currents in the system, it may be necessary to account for the phase shift in Y-Δ transformers.

If the emfs in a positive-sequence network such as that shown in Fig. 13.7a are replaced by short circuits, the impedance between the fault point P and the reference bus is the positive-sequence impedance Z_1 in the equations developed for faults on a power system and is the series impedance of the Helmholtz-Thévenin equivalent of the circuit between P and the reference bus. If the voltage V_f is connected in series with this modified positive-sequence network, the resulting circuit, shown in Fig. 13.7b, is the Helmholtz-Thévenin equivalent of the original positive-sequence network. The circuits shown in Fig. 13.7 are equivalent only in their effect on any external connections made between P and the reference bus of the original networks. We can easily see that no current flows in the branches of the equivalent circuit in the absence of an external connection, but current will flow in the branches of the original positive-sequence network if any difference exists in the phase or magnitude of the two emfs in the network. In Fig. 13.7a the current flowing in the branches in the absence of an external connection is the prefault load current.

When other sequence networks are interconnected with the positive-sequence network of Fig. 13.7a or its equivalent shown in Fig. 13.7b, the current flowing out of the network or its equivalent is I_{a1}, and the voltage between P and the reference bus is V_{a1}. With such an external connection, the current in any branch of the positive-sequence network is the positive-sequence current in phase a of that branch during the fault. The current in any branch of the Helmholtz-Thévenin equivalent of Fig. 13.7b, however, is that portion of the component I_{a1} of the fault current determined by the apportioning of I_{a1} as determined by the impedances of the branches.

Figure 13.8a shows an emf equal to the prefault voltage V_f connected between P and the reference bus of the positive-sequence network.



(a) Positive-sequence network before a fault

(b) Positive-sequence network during a fault

(c) Positive-sequence network during a fault

(d) Helmholtz-Thévenin equivalent of the positive-sequence network during a fault

FIG. 13.8 Diagrams to explain the adding of the prefault current in any branch of a circuit to the component of fault current flowing in that branch.

Obviously the addition of this emf has no effect on the positive-sequence network before the fault occurs, and the current in the network is the prefault current only. Figure 13.8b shows an emf equal to V_{a1} connected between P and the reference bus of the positive-sequence network. Since V_{a1} is the voltage across the positive-sequence network after a fault occurs, the circuit of Fig. 13.8b must represent the condition of the positive-sequence network during a fault. The current in each branch of the positive-sequence network under this condition is the current in the branch during a fault.

Another method of obtaining the voltage V_{a1} between P and the reference bus is shown in Fig. 13.8c where two emfs, one equal to V_f and one equal to $V_{a1} - V_f$, are connected in series. Short-circuiting $V_{a1} - V_f$ in the circuit of Fig. 13.8c produces the condition shown in Fig. 13.7a, and the prefault current flows in all branches of the circuit. Short-circuiting all emfs except $V_{a1} - V_f$ results in the application of

$V_{a1} - V_f$ between P and the common bus of the modified positive-sequence network (the positive-sequence network with all emfs shorted). The resulting condition is exactly that produced when the equivalent circuit shown in Fig. 13.7*b* is interconnected with other networks to simulate a fault, for then V_{a1} appears between P and the reference bus and $V_{a1} - V_f$ appears across the modified positive-sequence network. Figure 13.8*d* illustrates the condition of the Helmholtz-Thévenin equivalent of the positive-sequence network during a fault.

By the principle of superposition, the current in a branch of the positive-sequence network during a fault as simulated by Fig. 13.8*c* is the sum of the current in the branch with $V_{a1} - V_f$ shorted and the current in the branch with all emfs except $V_{a1} - V_f$ shorted. We conclude that the current in a branch of the positive-sequence network connected to simulate a fault is equal to the prefault current in the branch plus that portion of the positive-sequence component of fault current I_{a1} which is apportioned to that branch as one of several series and parallel impedances through which parts of I_{a1} flow.

When a calculating board is available, it is convenient to set up the positive-sequence network with the generator units on the board representing the emfs in the positive-sequence network. The current measured in per unit in any branch of the positive-sequence network without external connections is the per-unit value of the prefault current in phase *a* of the corresponding branch of the system. After interconnection of the positive-sequence network with the negative- and zero-sequence networks, the currents measured in the branches of the positive-sequence network include the prefault current and the additional amount due to the fault. When a calculating board is not available, the best approach is to convert the positive-sequence network to its Helmholtz-Thévenin equivalent by shorting the emfs in the network and placing V_f in series with the network modified by shorting the emfs. Upon interconnecting the various sequence networks to simulate a fault, the currents calculated for the branches of the modified positive-sequence network include only the additional currents due to the fault to which must be added the prefault currents in the particular branches being analyzed.

Example 13.1

A 7,500-kva, 4.16-kv generator is supplying a group [illegible]tical synchronous motors through a transformer bank composed [illegible] three single-phase units, each of which is rated 2,400–600 vo[illegible]500 kva. The leakage reactance of each transformer is 10%. [illegible] 600-volt windings are connected in Δ to the motors, and the [illegible]0-volt windings are connected in Y to the generator. The transfo[illegible]er neutral is solidly grounded. The reactances of the generator a[illegible]′ = 10%, X_2 = [illegible] and $X_0 = 5\%$. The motors are rated 600 volts, and they operate at 89.5% efficiency when carrying full load at unity power factor and rated voltage. The sum of their output ratings is 6,000 hp. The reactances of each motor based on its own rating are $X'' = 20\%$, $X_2 = 20\%$, $X_0 = 4\%$, and each is grounded through a reactance of 2%. Each of the identical motors is supplying an equal share of a total load of 5,000 hp and is operating at rated voltage, 85% power factor lag, and 88% efficiency when a single line-to-ground fault occurs on the low-tension side of the transformer bank. Treat the group of motors as a single equivalent motor. Specify completely the sequence networks to simulate the fault on a calculating board. Compute the voltages behind subtransient reactance for the generator and motor. Determine the subtransient line currents in all parts of the system and the line-to-line voltages at the motor and generator terminals.

Solution

The one-line diagram of the system is shown in Fig. 13.9.

Choose the generator rating as base: 7,500 kva, 4.16 kv at the generator.

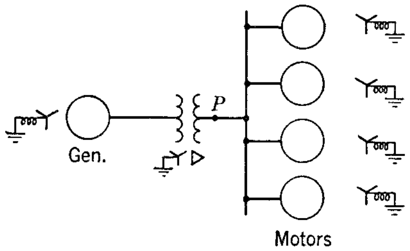


FIG. 13.9 One-line diagram of the system of Example 13.1.

The transformer three-phase rating is identical to the base selected, as shown below:

$$\sqrt{3} \times 2{,}400 = 4{,}160 \text{ volts, or } 4.16 \text{ kv}$$
$$3 \times 2{,}500 = 7{,}500 \text{ kva}$$

The base for the motor circuit is 7,500 kva, 600 volts. The individual motors are identical and operating under identical conditions. Therefore, they are equivalent to one large motor rated 6,000 hp, 600 volts. The input rating of the single equivalent motor is

$$\frac{6{,}000 \times 0.746}{0.895} = 5{,}000 \text{ kva}$$

and the reactances of the equivalent motor in per cent are the same on the base of the combined rating as the reactances of the individual motors on the base of the rating of an individual motor. The reactances of the equivalent motor on the selected base are

$$X'' = 0.2 \times \frac{7{,}500}{5{,}000} = 0.3 \text{ per unit}$$
$$X_2 = 0.2 \times \frac{7{,}500}{5{,}000} = 0.3 \text{ per unit}$$
$$X_0 = 0.04 \times \frac{7{,}500}{5{,}000} = 0.06 \text{ per unit}$$

The reactance in the zero-sequence network to account for the current-limiting reactance is

$$3X_n = 3 \times 0.02 \times \frac{7{,}500}{5{,}000} = 0.09 \text{ per unit}$$

Figure 13.10 shows the connection of the sequence networks on a calculating board.

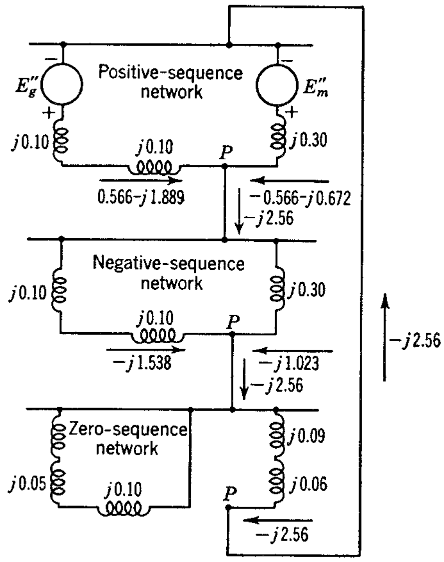


FIG. 13.10 Connection of the sequence networks of Example 13.1. Subtransient currents are marked in per unit for a single line-to-ground fault at P.

Since the motors are operating at rated voltage equal to the base voltage of the motor circuit, the prefault voltage of phase a at the fault is

$$V_f = 1.0 \text{ per unit}$$

Base current for the motor circuit is

$$\frac{7{,}500{,}000}{\sqrt{3} \times 600} = 7{,}220 \text{ amp}$$

and the actual motor current is

$$\frac{746 \times 5{,}000}{0.88 \times \sqrt{3} \times 600 \times 0.85} = 4{,}810 \text{ amp}$$

The per-unit current drawn by the motor through [illegible] before the fault occurs is

$$\frac{4{,}810}{7{,}220} = 0.667\underline{/-31.8^\circ} = 0.566 - j0.351 \text{ per unit}$$

The voltages behind subtransient reactance are computed as follows: For the motor,

$$\begin{aligned} E''_m &= 1.0 - j0.3(0.566 - j0.351) \\ &= 0.895 - j0.17 \\ &= 0.912\underline{/-10.8^\circ} \text{ per unit} \end{aligned}$$

and for the generator,

$$\begin{aligned} E''_g &= 1.0 + (j0.1 + j0.1)(0.566 - j0.351) \\ &= 1.07 + j0.1132 \\ &= 1.075\underline{/6.03^\circ} \text{ per unit} \end{aligned}$$

For an analytical solution the positive-sequence network is replaced

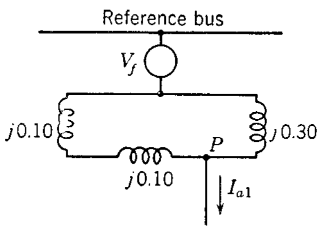


FIG. 13.11 Helmholtz-Thévenin equivalent of the positive-sequence network of Example 13.1.

by its Helmholtz-Thévenin equivalent circuit which is shown in Fig. 13.11. The computations follow.

$$\begin{aligned} Z_1 &= \frac{(j0.1 + j0.1)(j0.3)}{j(0.1 + 0.1 + 0.3)} = j0.12 \text{ per unit} \\ Z_2 &= \frac{(j0.1 + j0.1)(j0.3)}{j(0.1 + 0.1 + 0.3)} = j0.12 \text{ per unit} \\ Z_0 &= j0.15 \text{ per unit} \\ I_{a1} &= \frac{V_f}{Z_1 + Z_2 + Z_0} = \frac{1.0}{j0.12 + j0.12 + j0.15} = \frac{1.0}{j0.39} = -j2.56 \\ I_{a2} &= I_{a1} = -j2.56 \\ I_{a0} &= I_{a1} = -j2.56 \end{aligned}$$

Current in the fault $= 3I_{a0} = 3(-j2.56) = -j7.68$ per unit

$$\begin{aligned} I_{b1} &= a^2 I_{a1} = (-0.5 - j0.866)(-j2.56) = -2.22 + j1.28 \\ I_{b2} &= aI_{a2} = (-0.5 + j0.866)(-j2.56) = 2.2 + j1.28 \\ I_{b0} &= I_{a0} = -j2.56 \\ I_b &= I_{b1} + I_{b2} + I_{b0} = 0 \\ I_{c1} &= aI_{a1} = (-0.5 + j0.866)(-j2.56) = 2.22 + j1.28 \\ I_{c2} &= a^2 I_{a2} = (-0.5 - j0.866)(-j2.56) = -2.22 + j1.28 \\ I_{c0} &= I_{a0} = -j2.56 \\ I_c &= I_{c1} + I_{c2} + I_{c0} = 0 \end{aligned}$$

The component of I_{a1} flowing toward P from the transformer is

$$-j2.56 \times \frac{j0.3}{j0.2 + j0.3} = -j2.56 \times 3/5 = -j1.538$$

and the component of I_{a1} flowing toward P from the motor is

$$-j2.56 \times \frac{j0.2}{j0.2 + j0.3} = -j2.56 \times 2/5 = -j1.023$$

Similarly the component of I_{a2} from the transformer is $-j1.538$, and the component of I_{a2} from the motor is $-j1.023$. All of I_{a0} flows toward P from the motor. Components of I_{b1}, I_{b2}, I_{b0}, I_{c1}, I_{c2}, and I_{c0} flowing toward the fault point from the transformer and from the motor are found from the corresponding components of I_{a1}, I_{a2}, and I_{a0}. Current in a line is determined by adding the prefault current to the symmetrical components of the current due to the fault. All the prefault current is of positive sequence if the circuit is balanced. The computations are as follows:

From the transformer:

$$\begin{array}{lll}
\text{Due to the fault:} & I_{a1} = & 0 \qquad - j1.538 \\
\text{Prefault:} & I_{a1} = & \underline{0.566 - j0.351} \\
\quad\text{Total:} & I_{a1} = & 0.566 - j1.889 \\
 & I_{a2} = & 0 \qquad - j1.538 \\
 & I_{a0} = & 0
\end{array}$$

The above per-unit currents in the sequence networks are shown in Fig. 13.10. The line currents from the transformer to the fault are

$$\begin{aligned}
I_a &= I_{a1} + I_{a2} + I_{a0} = 0.566 - j3.427 \text{ per unit} \\
I_b &= a^2 I_{a1} + a I_{a2} + I_{a0} \\
&= (-0.5 - j0.866)(0.566 - j1.889) + (-0.5 + j0.866)(-j1.538) + 0 \\
&= -0.588 + j1.223 \text{ per unit} \\
I_c &= a I_{a1} + a^2 I_{a2} + I_{a0} \\
&= (-0.5 + j0.866)(0.566 - j1.889) + (-0.5 - j0.866)(-j1.538) + 0 \\
&= 0.022 + j2.204 \text{ per unit}
\end{aligned}$$

From the motor:

$$\begin{array}{lll}
\text{Due to the fault:} & I_{a1} = & 0 \qquad - j1.023 \\
\text{Prefault:} & I_{a1} = & \underline{-0.566 + j0.351} \\
\quad\text{Total:} & I_{a1} = & -0.566 - j0.672 \\
 & I_{a2} = & 0 \qquad - j1.023 \\
 & I_{a0} = & 0 \qquad - j2.56
\end{array}$$

The above per-unit currents in the sequence networks are shown in Fig. 13.10. The line currents from the motor to the fault are

$$I_a = I_{a1} + I_{a2} + I_{a0} = -0.566 - j4.255 \text{ per unit}$$

$$\begin{aligned} I_b &= a^2 I_{a1} + a I_{a2} + I_{a0} \\ &= (-0.5 - j0.866)(-0.566 - j0.672) + (-0.5 + j0.866)(-j1.023) \\ &\qquad - j2.56 \\ &= 0.588 - j1.223 \text{ per unit} \end{aligned}$$

$$\begin{aligned} I_c &= a I_{a1} + a^2 I_{a2} + I_{a0} \\ &= (-0.5 + j0.866)(-0.566 - j0.672) + (-0.5 - j0.866)(-j1.023) \\ &\qquad - j2.56 \\ &= -0.022 - j2.204 \text{ per unit} \end{aligned}$$

Some checks on the numerical work may be applied at this point. They are:

1. Since the Δ-connected windings of the transformer have no ground, the sum of all the line currents from the transformer to the fault at P must be zero. Thus,

$$\begin{aligned} 0.566 - j3.427 - 0.588 + j1.223 + 0.022 + j2.204 &= 0 \\ 0 &= 0 \end{aligned}$$

2. The sum of the currents flowing in line a from the transformer and from the motor toward the fault must equal the current in the fault. Thus,

$$\begin{aligned} 0.566 - j3.427 - 0.566 - j4.255 &= -j7.68 \\ -j7.682 &\cong -j7.68 \end{aligned}$$

3. No current flows into the fault from lines b and c, and the current flowing toward P from the transformer must be the negative of the current flowing toward P from the motor in these lines. Thus,

$$0.588 - j1.223 = -(-0.588 + j1.223)$$

and

$$0.022 + j2.204 = -(-0.022 - j2.204)$$

The line currents in per unit are shown in Fig. 13.12.

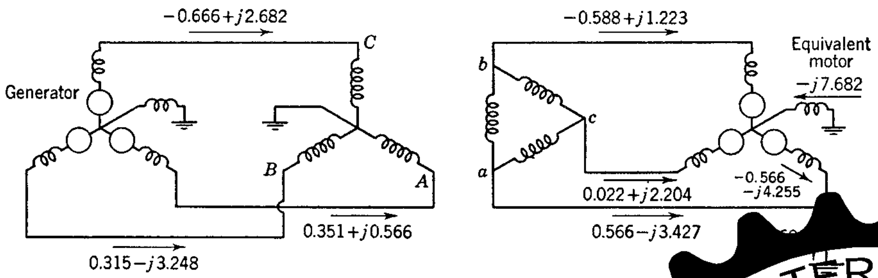


FIG. 13.12 Per-unit values of subtransient line currents in all pa[rts of] the system of Example 13.1.

Because of the phase shift in the Y-Δ transform[er] ba[nk] the line currents on the Y side must be found by Eqs. (1[0.]) an[d] (1[0.]). Since lower-case letters have been used for lines on the Δ [si]de of the transformer in the present example, upper-case letters will be used for lines on the Y side to agree with Fig. 13.12. Then, on the generator side of the transformer, in per unit of base line current,

$$
\begin{aligned}
I_{A1} &= jI_{a1} = j(0.566 - j1.889) = 1.889 + j0.566 \\
I_{A2} &= -jI_{a2} = -j(-j1.538) = -1.538 \\
I_{A0} &= 0 \text{ (since there are no zero-sequence currents on the generator side of the transformer)} \\
I_A &= I_{A1} + I_{A2} = 1.889 + j0.566 - 1.538 \\
&= 0.351 + j0.566 \text{ per unit} \\
I_{B1} &= a^2 I_{A1} = (-0.5 - j0.866)(1.889 + j0.566) = -0.454 - j1.918 \\
I_{B2} &= aI_{A2} = (-0.5 + j0.866)(-1.538) = 0.769 - j1.33 \\
I_B &= I_{B1} + I_{B2} = 0.315 - j3.248 \text{ per unit} \\
I_{C1} &= aI_{A1} = (-0.5 + j0.866)(1.889 + j0.566) = -1.435 + j1.352 \\
I_{C2} &= a^2 I_{A2} = (-0.5 - j0.866)(-1.538) = 0.769 + j1.33 \\
I_C &= I_{C1} + I_{C2} = -0.666 + j2.682 \text{ per unit}
\end{aligned}
$$

Voltages at the motor terminals are:

$$
\begin{aligned}
V_{a1} &= V_f - I_{a1}Z_1 = 1.0 - (-j2.56)(j0.12) = 0.692 \\
V_{a2} &= -I_{a2}Z_2 = -(-j2.56)(j0.12) = -0.308 \\
V_{a0} &= -I_{a0}Z_0 = -(-j2.56)(j0.15) = -0.384 \\
V_a &= V_{a1} + V_{a2} + V_{a0} = 0.692 - 0.308 - 0.384 = 0 \\
V_{b1} &= a^2 V_{a1} = (-0.5 - j0.866)(0.692) = -0.346 - j0.599 \\
V_{b2} &= aV_{a2} = (-0.5 + j0.866)(-0.308) = 0.154 - j0.267 \\
V_{b0} &= V_{a0} = -0.384 \\
V_b &= V_{b1} + V_{b2} + V_{b0} = -0.576 - j0.866 \text{ per unit} \\
V_{c1} &= aV_{a1} = (-0.5 + j0.866)(0.692) = -0.346 + j0.599 \\
V_{c2} &= a^2 V_{a2} = (-0.5 - j0.866)(-0.308) = 0.154 + j0.267 \\
V_{c0} &= -0.384 \\
V_c &= V_{c1} + V_{c2} + V_{c0} = -0.576 + j0.866 \text{ per unit} \\
V_{ab} &= V_a - V_b = 0 - (-0.576 - j0.866) = 0.576 + j0.866 \text{ per unit} \\
V_{bc} &= V_b - V_c = -0.576 - j0.866 - (-0.576 + j0.866) \\
&= -j1.732 \text{ per unit} \\
V_{ca} &= V_c - V_a = -0.576 + j0.866 \text{ per unit}
\end{aligned}
$$

The above voltages are in per unit of the base *voltage to neutral* of the motor circuit.

Voltages at the generator terminals, *without regard to phase shift in the transformer*, are found by adding the voltage drop between the generator terminals and the fault to the voltage at the fault in each sequence network. At the generator terminals, with phase shift disregarded,

$$
\begin{aligned}
V_{a1} &= 0.692 + j0.1(0.566 - j1.889) = 0.881 + j0.0566 \\
V_{a2} &= -0.308 + j0.1(-j1.538) = -0.154 \\
V_{a0} &= 0 \text{ (since the generator neutral is grounded through reactance and no zero-sequence current is flowing)}
\end{aligned}
$$

An alternate method of finding the voltage at the generator terminals from the sequence networks is to calculate the voltage drop from the generator terminals to the reference bus by the equations

$$
\begin{aligned}
V_{a1} &= E_g'' - I_{a1}Z_1 \\
V_{a2} &= -I_{a2}Z_2 \\
V_{a0} &= -I_{a0}Z_0
\end{aligned}
$$

where Z_1, Z_2, and Z_0 are the sequence impedances of the generator and I_{a1}, I_{a2}, and I_{a0} are the symmetrical components of the current in the generator. As a check on the values previously found,

$$
\begin{aligned}
V_{a1} &= 1.07 + j0.1132 - j0.1(0.566 - j1.889) = 0.881 + j0.0566 \\
V_{a2} &= -(-j1.538)(j0.1) = -0.154 \\
V_{a0} &= 0(j0.05) = 0
\end{aligned}
$$

The voltages at the generator terminals, when phase shift is taken into account, are found as follows:

$$
\begin{aligned}
V_{A1} &= jV_{a1} = j(0.881 + j0.0566) = -0.0566 + j0.881 \\
V_{A2} &= -jV_{a2} = -j(-0.154) = 0 + j0.154 \\
V_A &= V_{A1} + V_{A2} = -0.0566 + j1.035 \text{ per unit} \\
V_{B1} &= a^2V_{A1} = (-0.5 - j0.866)(-0.0566 + j0.881) \\
&= 0.791 - j0.392 \\
V_{B2} &= aV_{A2} = (-0.5 + j0.866)(j0.154) = -0.133 - j0.077 \\
V_B &= V_{B1} + V_{B2} = 0.658 - j0.469 \text{ per unit} \\
V_{C1} &= aV_{A1} = (-0.5 + j0.866)(-0.0566 + j0.881) \\
&= -0.735 - j0.490 \\
V_{C2} &= a^2V_{A2} = (-0.5 - j0.866)(j0.154) = 0.133 - j0.077 \\
V_C &= V_{C1} + V_{C2} = -0.602 - j0.567 \text{ per unit} \\
V_{AB} &= V_A - V_B = -0.0566 + j1.035 - 0.658 + j0.469 \\
&= -0.715 + j1.504 \text{ per unit} \\
V_{BC} &= V_B - V_C = 0.658 - j0.469 + 0.602 + j0.567 \\
&= 1.260 + j0.098 \text{ per unit} \\
V_{CA} &= V_C - V_A = -0.602 - j0.567 + 0.0566 - j1.035 \\
&= -0.545 - j1.602 \text{ per unit}
\end{aligned}
$$

The above line-to-line voltages are in per unit of the base *line-to-neutral* voltage.

When the values of the current and voltage bases in the various parts of the system are determined, we can convert the per-unit currents and voltages to amperes and volts.

Base current on the motor side of the transformer

$$= \frac{7,500,000}{\sqrt{3} \times 600} = 7,220 \text{ amp}$$

Base current on the generator side of the transformer

$$= \frac{7{,}500{,}000}{\sqrt{3} \times 4{,}160} = 1{,}040 \text{ amp}$$

Base voltage to neutral in the motor circuit $= \dfrac{600}{\sqrt{3}} = 346$ volts

Base voltage to neutral in the generator circuit $= \dfrac{4{,}160}{\sqrt{3}} = 2{,}400$ volts

The subtransient currents and voltages in the various parts of the system are:

Fault current $= 7{,}220(-j7.68) = -j55{,}550 = 55{,}550\underline{/-90^\circ}$ amp

Currents from the transformer to the fault:

In line a: $7{,}220(0.566 - j3{,}427) = 4{,}090 - j24{,}750 = 25{,}000\underline{/-80.6^\circ}$ amp

In line b: $7{,}220(-0.588 + j1.223) = -4{,}250 + j8{,}850 = 9{,}850\underline{/115.6^\circ}$ amp

In line c: $7{,}220(0.022 + j2.204) = 159 + j15{,}900 = 16{,}000\underline{/264.3^\circ}$ amp

Currents from the motor to the fault:

In line a: $7{,}220(-0.566 - j4.255) = -4{,}090 - j30{,}800 = 31{,}100\underline{/262.5^\circ}$ amp

In line b: $7{,}220(0.588 - j1.223) = 4{,}250 - j8{,}850 = 9{,}850\underline{/-64.4^\circ}$ amp

In line c: $7{,}220(-0.022 - j2.204) = -159 - j15{,}900 = 16{,}000\underline{/264.3^\circ}$ amp

Currents from the generator to the transformer:

In line A: $1{,}040(0.351 + j0.566) = 365 + j590 = 695\underline{/58.3^\circ}$ amp

In line B: $1{,}040(0.315 - j3.248) = 328 - j3{,}380 = 3{,}400\underline{/-84.4^\circ}$ amp

In line C: $1{,}040(-0.666 + j2.682) = -693 + j2{,}790 = 2{,}870\underline{/103.9^\circ}$ amp

Voltages at the motor terminals:

$V_{ab} = 346(0.576 + j0.866) = 200 + j300 = 360\underline{/56.3^\circ}$ volts

$V_{bc} = 346(-j1.732) = 0 - j600 = 600\underline{/-90^\circ}$ volts

$V_{ca} = 346(-0.576 + j0.866) = -200 + j300 = 360\underline{/123.}$[illegible] volts

Voltages at the generator terminals:

$V_{AB} = 2{,}400(-0.715 + j1.504) = -1{,}720 + j3{,}$[illegible] $= 4{,}000\underline{/115.4^\circ}$ volts

$V_{BC} = 2{,}400(1.260 + j0.098) = 3{,}030 + j235$ [illegible] $030\underline{/4.45^\circ}$ volts

$V_{CA} = 2{,}400(-0.545 - j1.602) = -1{,}310 -$ [illegible]$50 = 4{,}060\underline{/251.2^\circ}$ volts

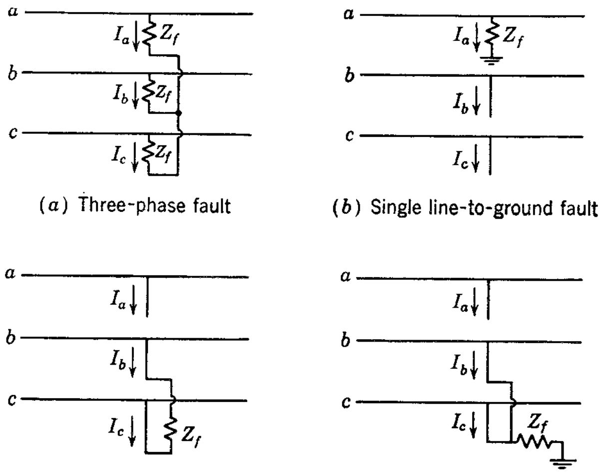
In the preceding example the synchronous motors are operating at a power factor of 85% lag. Ordinarily, synchronous motors are designed to operate at unity power factor or at a leading power factor of 80%. The motors specified in the example were assumed to be operating at a lagging power factor in order to be more illustrative of the over-all load on a power system, which is usually lagging.

When the operating voltage is not specified, it is convenient to assume that the prefault voltage V_f is equal to the rated voltage of the part of the system in which the fault occurs. Then, if the base voltage is equal to the rated voltage, V_f is 1.0 per unit. If the voltage is specified at a certain point in the system, the voltage V_f at the fault must be calculated.

The work involved in solving Example 13.1 would have been somewhat simplified if the system had not been carrying current before the fault. When resistance is neglected, the fault current is purely reactive and in any case is usually considerably larger than the prefault current. If the prefault current is nearly in phase with the voltage, the addition of the fault current to the prefault current does not result in a current magnitude greatly in excess of the component due to the fault current alone. The addition of the prefault current becomes important as the prefault current lags the voltage by larger amounts.

The method of finding the subtransient current for a fault on a power system containing more than one group of synchronous machines is not applicable to finding the steady-state current for a sustained fault. Substitution of synchronous reactances and voltages behind synchronous reactance in the positive-sequence network would give the steady-state current only if the phase relations between the internal voltages of the synchronous machines remained the same after the fault as they were before the fault. Voltages behind subtransient reactance have the same phase before and after a fault because the rotors have not had time to shift their relative positions during the very short interval between the instant before the occurrence of the fault and the subtransient period. Angular positions of the rotors of synchronous machines depend on the loads and on the impedances between machines. Since a fault changes the impedance between machines, the phase angles between various voltages behind synchronous reactance before a fault differ greatly from the phase angles between the same voltages in the steady-state period of a sustained fault. In fact the machines of the system may [illegible] in synchronism unless the fault is removed quickly. The problem of determining the rapidity with which a fault must be removed in order to prevent the loss of synchronism is discussed in Chap. 15. The determination of current for a sustained fault is of much less importance than the determination of subtransient and transient currents because faults are usually isolated in a few cycles.

13.6 Faults through Impedance. All the faults discussed in the preceding sections consisted of direct short circuits between lines and from one or two lines to ground. Although such direct short circuits result in the highest value of fault current and are, therefore, the most conservative values to use when determining the effects of anticipated faults, the fault impedance is seldom zero. Most faults are the result of insulator flashovers, where the impedance between the line and ground depends on the resistance of the arc, of the tower itself, and of the tower footing if ground wires are not used. Tower footing resistances form the



(*a*) Three-phase fault (*b*) Single line-to-ground fault

(*c*) Line-to-line fault (*d*) Double line-to-ground fault

FIG. 13.13 Connection diagrams of the hypothetical stubs for various faults through impedance.

major part of the resistance between line and ground and depend on the soil conditions. The resistance of dry earth is 10 to 100 times the resistance of swampy ground. The effect of impedance in the fault is found by deriving equations similar to those for faults through zero impedance. Connections of the hypothetical stubs for faults through an impedance are shown in Fig. 13.13.

A system which includes the fault remains symmetrical after the occurrence of a *three-phase fault* having the same impedance between [illegible]h line and a common point. Only positive-sequence currents f[illegible] the fault impedance Z_f equal in all phases, as shown in [illegible] 13.13*a*, the voltage at the fault is

$$V_a = I_a Z_f$$

and, since only positive-sequence currents flow, [illegible]

$$V_{a1} = I_{a1}Z_f = V_f - I_{a1}Z_{\ldots}$$

and

$$I_{a1} = \frac{V_f}{Z_1 + Z_f} \tag{13.10}$$

The sequence-network connection is shown in Fig. 13.14*a*.

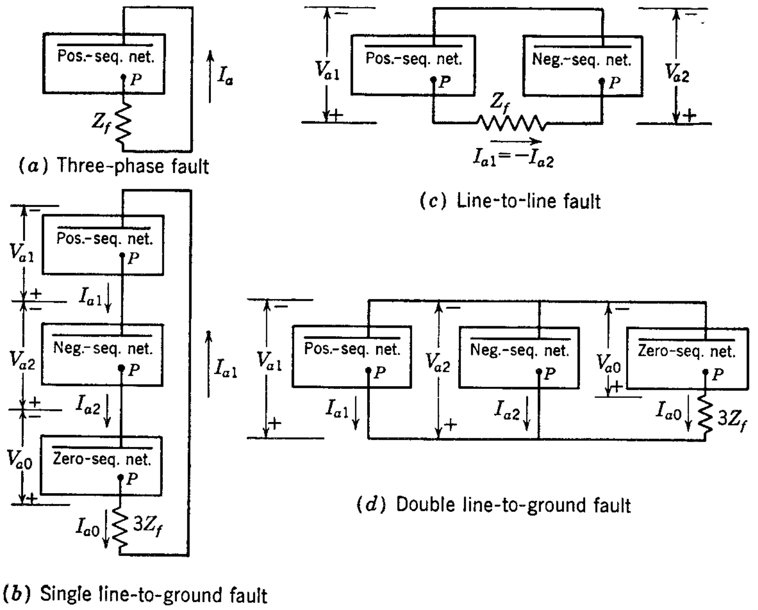


FIG. 13.14 Connections of the sequence networks to simulate various types of faults through impedance at point P.

A *single line-to-ground fault* through impedance is shown in Fig. 13.13*b*. The conditions at the fault are

$$I_b = 0 \qquad I_c = 0 \qquad V_a = I_a Z_f$$

From Eqs. (10.19) to (10.21), since I_b and I_c equal zero, we obtain

$$I_{a1} = I_{a2} = I_{a0} = \frac{I_a}{3}$$

$$I_a = 3I_{a1}$$

and from Eq. (10.5),

$$V_a = V_{a1} + V_{a2} + V_{a0} = 3I_{a1}Z_f$$

Replacing V_{a1} by $V_f - I_{a1}Z_1$, and replacing V_{a2} and V_{a0} by $-I_{a2}Z_2$ and $-I_{a0}Z_0$, respectively, yields

$$V_f - I_{a1}Z_1 - I_{a2}Z_2 - I_{a0}Z_0 = 3I_{a1}Z_f$$

Solving for I_{a1}, after substituting I_{a1} for I_{a2} and I_{a0}, we obtain

$$I_{a1} = \frac{V_f}{Z_1 + Z_2 + Z_0 + 3Z_f} \tag{13.11}$$

The sequence-network connection for a single line-to-ground fault through impedance is shown in Fig. 13.14*b*.

A *line-to-line fault* through impedance is shown in Fig. 13.13*c*. The total impedance between the faulted lines is Z_f. The conditions at the fault are

$$I_a = 0 \qquad I_b = -I_c \qquad V_b - V_c = I_b Z_f$$

From Eqs. (10.6) and (10.7),

$$V_b - V_c = (a^2 - a)V_{a1} - (a^2 - a)V_{a2} = I_b Z_f$$
$$-j\sqrt{3}\,V_{a1} + j\sqrt{3}\,V_{a2} = I_b Z_f \tag{13.12}$$

Substituting $I_a = 0$ and $I_c = -I_b$ in Eqs. (10.19) and (10.20), we obtain

$$I_{a1} = \tfrac{1}{3}(a - a^2)I_b = j\frac{I_b}{\sqrt{3}}$$

and

$$I_{a2} = \tfrac{1}{3}(a^2 - a)I_b = -j\frac{I_b}{\sqrt{3}}$$

from which

$$I_{a1} = -I_{a2}$$

and

$$I_b = -j\sqrt{3}\,I_{a1}$$

Substituting the expressions of Eqs. (13.1) and (13.2) for V_{a1} and V_{a2} in Eq. (13.12), and substituting I_{a1} for $-I_{a2}$ and $-j\sqrt{3}\,I_{a1}$ for I_b, we obtain

$$-j\sqrt{3}\,(V_f - I_{a1}Z_1) + j\sqrt{3}\,I_{a1}Z_2 = -j\sqrt{3}\,I_{a1}Z_f$$

and

$$I_{a1} = \frac{V_f}{Z_1 + Z_2 + Z_f} \tag{13.13}$$

The sequence-network connection for a line-to-line fault through impedance is shown in Fig. 13.14*c*.

During a *double line-to-ground* fault through impedance there may be impedance between each line and a common point such as a tower, and there may be additional impedance between the tower and true ground. The most common double line-to-ground fault is a flashover from the insulators of two phases to the tower, where the impedance of the tower footing is considerably more than the arc impedance. This type of fault, for most practical purposes, can be calculated with zero impedance between lines and with the fault impedance Z_f between the short-circuited lines and ground, as shown in Fig. 13.13*d*. The conditions at the fault are

$$V_b = V_c = (I_b + I_c)Z_f \qquad I_a = 0$$

Since the current flowing in the ground is $3I_{a0}$,

$$I_b + I_c = 3I_{a0}$$
$$V_b = 3I_{a0}Z_f$$

Substituting V_b for V_c in Eqs. (10.9), (10.13), and (10.15) gives

$$V_{a1} = \tfrac{1}{3}[V_a + (a + a^2)V_b] = \tfrac{1}{3}(V_a - V_b) \tag{13.14}$$
$$V_{a2} = \tfrac{1}{3}[V_a + (a + a^2)V_b] = \tfrac{1}{3}(V_a - V_b) \tag{13.15}$$
$$V_{a0} = \tfrac{1}{3}(V_a + 2V_b) \tag{13.16}$$

Therefore, by Eqs. (13.14) and (13.15),

$$V_{a1} = V_{a2}$$

and, upon subtracting Eq. (13.14) from Eq. (13.16), we obtain

$$V_{a0} - V_{a1} = V_b = 3I_{a0}Z_f$$
$$V_{a0} = V_{a1} + 3I_{a0}Z_f$$

Substitution of the expressions of Eqs. (13.1) and (13.3) for V_{a1} and V_{a0} gives

$$-I_{a0}Z_0 = V_{a1} + 3I_{a0}Z_f$$
$$I_{a0} = -\frac{V_f - I_{a1}Z_1}{Z_0 + 3Z_f}$$

Noting that $V_{a1} = V_{a2}$, we obtain from Eqs. (13.1) and (13.2)

$$I_{a2} = -\frac{V_{a1}}{Z_2} = -\frac{V_f - I_{a1}Z_1}{Z_2}$$

Since $I_a = 0$, we have

$$I_{a1} + I_{a2} + I_{a0} = 0$$
$$I_{a1} - \frac{V_f - I_{a1}Z_1}{Z_2} - \frac{V_f - I_{a1}Z_1}{Z_0 + 3Z_f} = 0$$
$$I_{a1}Z_2(Z_0 + 3Z_f) - V_f(Z_0 + 3Z_f) + I_{a1}Z_1(Z_0 + 3Z_f) - V_fZ_2 + I_{a1}Z_1Z_2 = 0$$
$$I_{a1} = \frac{V_f[Z_2 + (Z_0 + 3Z_f)]}{Z_1[Z_2 + (Z_0 + 3Z_f)] + Z_2(Z_0 + 3Z_f)}$$
$$I_{a1} = \frac{V_f}{Z_1 + Z_2(Z_0 + 3Z_f)/(Z_2 + Z_0 + 3Z_f)} \tag{13.17}$$

For a double line-to-ground fault which consists of a sh[...] between lines b and c and which has the impedance Z_f bet[...] shorted point and ground, Eq. (13.17) indicates that the se[...]ce networks should be connected as shown in Fig. 13.14d.

All the relations derived for faults through impeda[...]e become identical to equations for faults through zero impedance [...] is zero. Problems involving faults through impedance are solved [...] a manner similar to that followed in the illustrative examples for faults that do not contain any impedance.

13.7 Open Conductors. One or two open conductors constitute a fault which is somewhat different from those which have been discussed previously. Such faults are very easily represented on the calculating board. Since the result of such a fault is an unbalanced condition, the use of symmetrical components is indicated. If simple relations can be found for the symmetrical components of the voltage across the open circuit, the connection of the sequence networks will be obvious.

One open conductor results when only one fuse is blown in one of the lines of a three-phase system or when one conductor is broken. The same condition exists with single-pole, high-speed, reclosing breakers during the time when one phase is open. If the voltage on one side of the open circuit occurring in phase a is V_a and on the other side it is $V_{a'}$, the voltage between the open points is $V_{aa'}$. The symmetrical components of $V_{aa'}$ will be found, and they will indicate how the sequence networks should be connected. Since only conductor a is open, the voltages at the fault location in phases b and c are equal on either side of the fault point—that is, $V_b = V_{b'}$ and $V_c = V_{c'}$. The lines at the fault are shown in Fig. 13.15. The conditions at the fault are

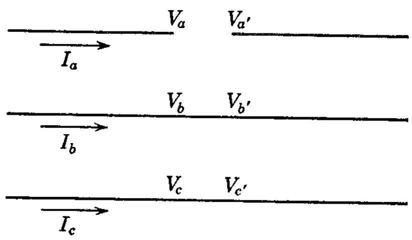


FIG. 13.15 Section of a three-phase line with line a open.

$$I_a = 0 \qquad V_b - V_{b'} = V_{bb'} = 0 \qquad V_c - V_{c'} = V_{cc'} = 0$$

Then,

$$V_{aa'1} = \tfrac{1}{3}(V_{aa'} + aV_{bb'} + a^2V_{cc'}) = \frac{V_{aa'}}{3}$$

$$V_{aa'2} = \tfrac{1}{3}(V_{aa'} + a^2V_{bb'} + aV_{cc'}) = \frac{V_{aa'}}{3}$$

$$V_{aa'0} = \tfrac{1}{3}(V_{aa'} + V_{bb'} + V_{cc'}) = \frac{V_{aa'}}{3}$$

so,

$$V_{aa'1} = V_{aa'2} = V_{aa'0} \tag{13.18}$$

and, since $I_a = 0$, we obtain

$$I_{a1} + I_{a2} + I_{a0} = 0 \tag{13.19}$$

Equations (13.18) and (13.19) show that the positive-, negative-, and zero-sequence networks should be connected in parallel at the points on either side of the open circuit. Figure 13.16 shows the connection of the sequence networks for the system of Fig. 13.2 with [illegible] open at point [illegible]. An analytic solution of the problem is obtained by applying the principle of superposition with each generated voltage shorted in turn. Note that the impedance met by each sequence current is the impedance of the sequence network measured across the open points.

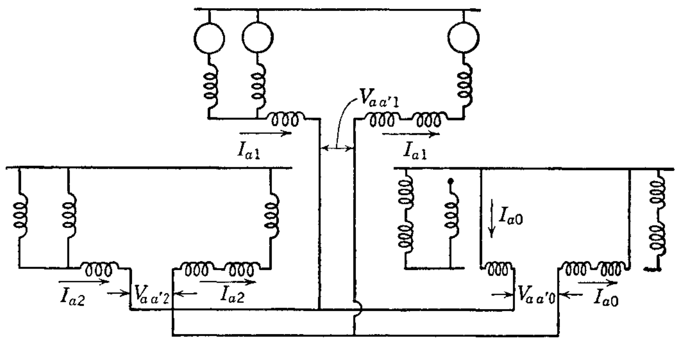


FIG. 13.16 Connection of the sequence networks for the system of Fig. 13.2 with conductor a open at point P.

Example 13.2

Find the subtransient currents flowing in lines b and c at point P of Example 13.1 when line a is opened at that point.

Solution

The connection of the sequence networks to find subtransient current with one open conductor is shown in Fig. 13.17. The internal voltages

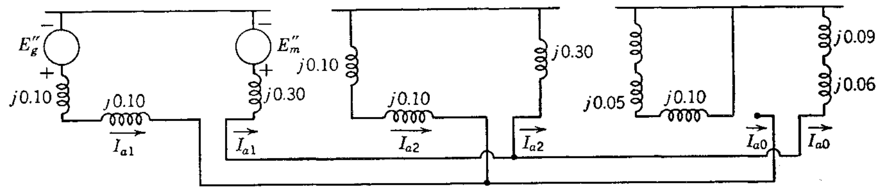


FIG. 13.17 Connection of the sequence networks for Example 13.2.

are voltages behind subtransient reactances. No zero-sequence current can flow, because the zero-sequence network is open-circuited for the connection shown. The sequence impedances measured between the open points of each network are

$$Z_1 = j(0.1 + 0.1 + 0.3) = j0.5$$
$$Z_2 = j(0.1 + 0.1 + 0.3) = j0.5$$
$$Z_0 = \infty$$

From the solution of Example 13.1 the voltages be[hind] s[ubt]ransient reactance for the motor and generator are, respect[ively],

$$E''_m = 0.912\underline{/-10.8^\circ} \text{ per unit}$$
$$E''_g = 1.075\underline{/6.03^\circ} \text{ per unit}$$

When the internal voltage of the motor is short-circuited, the current I_{a1}, as determined from Fig. 13.17, is

$$I_{a1} = \frac{1.075\angle 6.03^\circ}{j0.5 + j0.5} = 1.075\angle{-83.97^\circ} = 0.113 - j1.068$$

Equation (13.19) shows that $I_{a2} = -I_{a1}$, since $I_{a0} = 0$. Therefore

$$I_{a2} = -0.113 + j1.068 = 1.075\angle 96.03^\circ$$

When the internal voltage of the generator is short-circuited, the current I_{a1} is

$$I_{a1} = -\frac{0.912\angle{-10.8^\circ}}{j0.5 + j0.5} = -0.912\angle{-100.8^\circ} = 0.171 + j0.897$$

and

$$I_{a2} = -0.171 - j0.897$$

By the principle of superposition, due to the internal voltages of both the motor and the generator,

$$\begin{aligned}
I_{a1} &= 0.113 - j1.068 + 0.171 + j0.897 = 0.284 - j0.171 \\
&= 0.331\angle{-31.1^\circ} \\
I_{a2} &= -0.284 + j0.171 = 0.331\angle 148.9^\circ \\
I_a &= I_{a1} + I_{a2} + I_{a0} = 0 \\
I_{b1} &= a^2 I_{a1} = 0.331\angle 208.9^\circ = -0.290 - j0.160 \\
I_{b2} &= a I_{a2} = 0.331\angle 268.9^\circ = -0.006 - j0.331 \\
I_b &= -0.290 - j0.160 - 0.006 - j0.331 = -0.296 - j0.491 \\
&= 0.573\angle 238.9^\circ \text{ per unit} \\
I_{c1} &= a I_{a1} = 0.331\angle 88.9^\circ = 0.006 + j0.331 \\
I_{c2} &= a^2 I_{a2} = 0.331\angle 28.9^\circ = 0.290 + j0.160 \\
I_c &= 0.006 + j0.331 + 0.290 + j0.160 = 0.296 + j0.491 \\
&= 0.573\angle 58.9^\circ \text{ per unit}
\end{aligned}$$

In amperes, at the motor,

$$\begin{aligned}
I_a &= 0 \\
I_b &= 7{,}220 \times 0.573\angle 238.9^\circ = 4{,}140\angle 238.9^\circ \text{ amp} \\
I_c &= 7{,}220 \times 0.573\angle 58.9^\circ = 4{,}140\angle 58.9^\circ \text{ amp}
\end{aligned}$$

In Example 13.2, the principle of superposition was used to demonstrate the method to be followed in more complicated problems. [illegible] simple network of this example, use of the principle [illegible] superposition has no advantage over the solution by Kirchhoff-law equations. Subtransient quantities were used in the positive-sequence network, and subtransient currents were found.

To find the steady-state currents in the network with or [illegible] the magnitudes of the voltages behind synchronous reactance must be found together with the angle between the voltages under steady-state conditions with one line open. Very often the two remaining conductors are incapable of transmitting all the load that was previously carried by three conductors. The following example indicates the method of finding the angle between the voltages behind synchronous reactance.

Example 13.3

Determine whether the equivalent motor of Example 13.1 can continue to carry its load of 5,000 hp at 88% efficiency under steady-state conditions after one line is opened at the motor terminals. Assume that both the motor and generator have synchronous reactances of 100% on the chosen base.

Solution

If we assume that the excitation of the machines does not change, the magnitudes of the voltages behind synchronous reactance are unchanged by opening one line. Their computation is similar to the computation of voltages behind subtransient reactance in Example 13.1, except that synchronous reactances replace the subtransient reactances in the positive sequence network of Fig. 13.10. For the motor,

$$\begin{aligned} E_m &= 1.0 - j1.0(0.566 - j0.351) = 1.0 - j0.566 - 0.351 \\ &= 0.649 - j0.566 = 0.860\underline{/-41.1^\circ} \text{ per unit} \end{aligned}$$

and, for the generator,

$$\begin{aligned} E_g &= 1.0 + j1.1(0.566 - j0.351) = 1.0 + j0.623 + 0.386 \\ &= 1.386 + j0.623 = 1.52\underline{/24.2^\circ} \text{ per unit} \end{aligned}$$

The per-unit power drawn by the motor, on the assumption of the same efficiency as in Example 13.1, is

$$P_m = \frac{5{,}000 \times 0.746/0.88}{7{,}500} = 0.565 \text{ per unit}$$

According to Eq. (7.6) the power received at the end of a four-terminal network is

$$P_R = \frac{|V_S| \cdot |V_R|}{|B|} \cos(\beta - \delta) - \frac{|A| \cdot |V_R|^2}{|B|} \cos(\beta - \alpha)$$

The connection of the sequence networks shown in F[illegible] indicates that we are dealing with two emfs applied to a ci[illegible] consisting of series impedance only. Equations (6.18) give t[illegible]owing values for the $ABCD$ constants of such a circuit:

$$\begin{aligned} A &= 1 \qquad & C &= [illegible] \\ B &= R + jX \qquad & D &= [illegible] \end{aligned}$$

Neglecting resistance and substituting the values of the $ABCD$ constants in the equation for power received gives

$$P_R = \frac{|V_S| \cdot |V_R|}{|X|} \sin \delta$$

If we consider the reactance of each machine as part of the series impedance X, we have for the sending- and receiving-end voltages

$$|V_S| = |E_g| \qquad \text{and} \qquad |V_R| = |E_m|$$

The series impedance is the sum of the impedances of the positive- and negative-sequence networks. So,

$$X = j1.0 + j0.1 + j0.1 + j0.1 + j0.3 + j1.0 = j2.6$$

The equation for power received becomes

$$P_R = \frac{1.52 \times 0.860}{2.6} \sin \delta = 0.503 \sin \delta$$

Since $P_R = P_m$,

$$0.503 \sin \delta = 0.565$$
$$\sin \delta = 1.12$$

Since the sine of an angle cannot exceed 1.0, the motor cannot carry the specified load if one line is open. The motor can carry some smaller load with one line open. A new angle δ found in the same manner as that followed in the preceding solution is the angle between E_m and E_g for the smaller load. After the value of δ is determined, the circuit can be solved for the line currents.

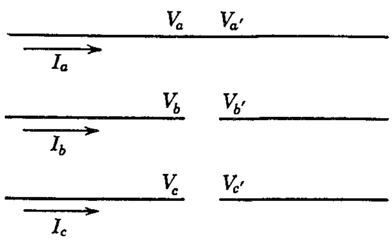


FIG. 13.18 Section of a three-phase line with lines b and c open.

Sequence networks can be interconnected to simulate *two open conductors* in an otherwise symmetrical three-phase system. Figure 13.18 shows lines b and c open. The voltages are designated V_b and V_c on one side of the open conductors and $V_{b'}$ and $V_{c'}$ on the other side. Since line a is not open, V_a is equal to $V_{a'}$. At the fault

$$I_b = 0 \qquad I_c = 0 \qquad V_a - V_{a'} = V_{aa'} = 0$$

Then,

$$I_{a1} = \tfrac{1}{3}(I_a + aI_b + a^2I_c) = \frac{I_a}{3}$$

$$I_{a2} = \tfrac{1}{3}(I_a + a^2I_b + aI_c) = \frac{I_a}{3}$$

$$I_{a0} = \tfrac{1}{3}(I_a + I_b + I_c) = \frac{I_a}{3}$$

So,

$$I_{a1} = I_{a2} = I_{a0} \tag{13.20}$$

and, since $V_{aa'} = 0$, we obtain

$$V_{aa'1} + V_{aa'2} + V_{aa'0} = 0 \tag{13.21}$$

Equations (13.20) and (13.21) are satisfied by opening each sequence network at the fault point and connecting the three networks in series as shown in Fig. 13.19 for the system of Fig. 13.2 with the open-conductor fault in the transmission line. Problems involving two open

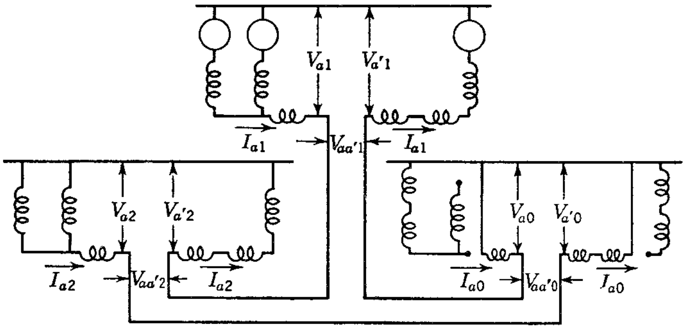


FIG. 13.19 Connection of the sequence networks for the system of Fig. 13.2 with conductors b and c open at point P.

conductors are solved in a manner similar to that followed in Example 13.2. No current will flow in the sequence networks of the system of Fig. 13.2 if they are connected to simulate two open conductors at the motor terminals rather than in the transmission line, because the zero-sequence network offers infinite impedance to the current in that case. Examination of the system of Fig. 13.2 shows us that no current could flow into the motor through line a from the transformer when lines b and c are open, because the Δ-connected secondary of the transformer does not provide a return path through the ground for the current. If the transformer secondary were Y-connected with the neutral grounded, current could flow in line a with lines b and c open.

PROBLEMS

13.1 The reactances of a generator rated 10,000 kva, 6.9 kv are $X'' = X_2 = 15\%$ and $X_0 = 5\%$. The neutral of the generator is grounded through a reactor of [illegible]1 ohm. The generator is connected to a Δ-Y transformer rated 10,000 [illegible] 44Y kv with a reactance of 7.5%. The neutral of the transformer [illegible] grounded. The terminal voltage of the generator is 6.9 kv when a single li[illegible]-ground fault occurs on the open-circuited, high-tension side of the transf[illegible] Find the initial symmetrical rms current in all phases of the generator.

13.2 A 5,000-kva, 13.8-kv generator is Y-connected a[illegible] grounded through a reactance of 2.5%. The reactances of the generator [illegible] X'' [illegible] X_2 [illegible] and $X_0 = 2.5\%$. The generator supplies a Δ-connected moto[illegible] rated 2,500 kva, 13.8 kv with reactances of $X'' = X_2 = 20\%$ and $X_0 = 10\%$. T[illegible] motor is drawing rated current at rated voltage and 0.8 power factor lagging when a single line-to-ground fault occurs on the line of negligible impedance connecting the generator and motor. Find the initial symmetrical rms current in the fault and in the faulted line at the generator and at the motor.

13.3 A double line-to-ground fault occurs on lines b and c at the point P in the circuit whose one-line diagram is shown in Fig. 13.20. Find the subtransient current in phase b of machine 1. Machine 2 is operating as a synchronous motor and drawing 800 kw at rated voltage and 0.8 power factor leading, when the fault occurs. Both

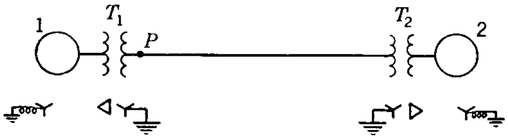


FIG. 13.20 One-line diagram for Prob. 13.3.

machines are rated 1,250 kva, 600 volts with reactances of $X'' = X_2 = 10\%$ and $X_0 = 4\%$. Each three-phase transformer is rated 1,250 kva, 600Δ–4,160Y volts with leakage reactance of 5%. The reactances of the transmission line are $X_1 = X_2 = 15\%$ and $X_0 = 50\%$ on a base of 1,250 kva, 4.16 kv.

13.4 Find the initial symmetrical rms current in each phase of the generator and equivalent motor of the system of Example 13.1 for a single line-to-ground fault at the motor terminals when the generator is operating at rated voltage and the motors are unloaded. Assume that the motors draw negligible current at no load.

13.5 Find the initial symmetrical rms current in the faulted phase of the generator of the system of Example 13.1 for a single line-to-ground fault at the terminals of the generator when it is delivering rated current with rated voltage and unity power factor at its terminals. The reactance in the neutral of the generator is 5%.

13.6 A single line-to-ground fault occurs at the middle of a transmission line. The one-line diagram is the same as that of Fig. 13.20 except that T_2 is ungrounded. Find the subtransient current in the fault and in each line on both sides of the fault. Draw a sketch of the lines at the fault and mark the subtransient current carried by each line. Express each current as a phasor. Select as reference the prefault voltage from line a to neutral at the fault. Line-to-line voltage at the fault is 110 kv before the fault occurs, and the line current is 78.7 amp at unity power factor. Machine 2 is a motor. Both machines are rated 15,000 kva, 6.6 kv, $X'' = 30\%$ $X_2 = 40\%$, and $X_0 = 5\%$. The line is rated 15,000 kva, 110 kv, $X_1 = X_2 = 8\%$, $X_0 = 24\%$. The transformers are composed of single-phase units, each of which is rated 5,000 kva, 6.6–63.5 kv with $X = 10\%$.

13.7 Line a is suddenly opened at the point P in the circuit described in Prob. 13.3 and shown in the one-line diagram of Fig. 13.20. At the time the line is opened, machine 2 is operating as a motor drawing 1,250 kva at 600 volts, 0.8 power factor leading. Find the subtransient current in amperes in each phase of the generator and motor.

13.8 Lines b and c are suddenly opened at the point P in the circuit described in Prob. 13.3 and shown in the one-line diagram of Fig. 13.20. At the time the lines are opened, machine 2 is operating as a motor drawing 1,250 kva at 600 volts, 0.8 power factor leading. Find the subtransient current in amperes in each phase of the generator and motor.

13.9 Find the maximum power that can be transferred from the generator to the motor of Prob. 13.3, first with one line open at P, then with two lines open at P. Assume that the d-c excitation of the machines remains the same as in Prob. 13.3. Synchronous reactances of the generator and motor are [illegible] and 100%, respectively.

CHAPTER 14

UNSYMMETRICAL SYSTEMS

14.1 The Occurrence of Unsymmetrical Systems. The preceding chapters have discussed systems which are normally symmetrical. Each phase of a symmetrical system offers the same impedance as any other phase to the flow of current of a particular sequence. Complete transpositions of transmission lines have been assumed in order to achieve such a symmetrical system. If a line is not transposed, the dissymmetry is so slight that the resulting unbalance is neglected without greatly affecting the calculations. The only dissymmetries which we have considered are the abnormal ones introduced by unsymmetrical faults consisting of open conductors or by short circuits including faults through impedance.

Let us turn our attention to unsymmetrical three-phase systems where the impedance is not the same in all three phases under normal conditions of operation. We shall restrict our discussion, however, to systems which do not involve coupling between phases. In such systems, current flowing in one phase does not induce a voltage in any other phase. Because of the way in which the reactances are calculated, the restriction does not apply to three-phase transmission lines even though current in a conductor of one phase induces a voltage in the conductors of the other phases. The calculation of inductance on the assumption of complete transposition of a line results in a value of self-inductance which includes the effect of mutual inductance. When inductance is so calculated, correct results are achieved by considering no coupling to exist between phases.

Unsymmetrical circuits arise when the system is serving unbalanced loads or when unequal impedances are in series with the system in each phase. Unbalanced series impedances exist when transformers are connected open-Δ, when current-limiting impedances of unequal value are installed in each phase, or when transformer banks are composed of dissimilar units. Open-conductor faults and the operation of single-pole reclosing breakers are special cases of unsymmetrical series impedances where one or two of the series impedances are infinite. The general unbalanced three-phase load, not involving coupling between phases, will be discussed, as will single-phase loads from line to neutral and from line to line.

14.2 The Effects of Unsymmetrical Circuits. In the symmetrical systems discussed in previous chapters the currents of any particular sequence produce voltage drops of like sequence only. Therefore, the current flowing in any sequence network for such a system is the current of that particular sequence only. We have not encountered coupling between the sequence networks. In unsymmetrical three-phase circuits we shall see that current of a given sequence, in general, produces voltage drops of all three sequences. Positive-sequence current, for instance, may produce negative- and zero-sequence voltage drops in addition to positive-sequence voltage drops. Therefore any possible connection of the sequence networks to represent the unbalanced system must have positive-sequence current flowing through impedances in a portion of the negative- and zero-sequence networks to give the required voltage drops in those networks. Thus, there is a mutual impedance between networks. Such coupling between sequence networks is not to be confused with coupling between phases of the actual circuit. Coupling between sequence networks exists even though there may be no coupling between phases.

In addition to understanding the effect of an unbalanced system in causing coupling between sequence networks, the engineer must know the effects of the unbalance on the operation of the system in order to calculate the amount of unbalance which can be tolerated. One disadvantage of unbalanced systems is the negative-sequence current which flows because of the negative-sequence voltage induced by the coupling between sequence networks. Negative-sequence current flowing in the armature of a three-phase machine causes a magnetic field which revolves at synchronous speed opposite to the direction of rotation of the rotor. Currents of double the frequency of the system are thereby induced in the rotors of generators and synchronous motors. The negative-sequence current in the armature of an induction motor induces rotor currents of $(2 - s)$ times the frequency of the supply, where s is the slip of the motor. The negative-sequence current causes a reduction in the torque of the induction motor because the oppositely revolving magnetic field causes torque in the reverse direction. The reverse torque is small [illegible] because of the high reactance of the rotor to the high-frequency rotor currents. More important than reduced torque is the increased heating due to the negative-sequence current. In large systems the unbalance must be kept within reasonable limits to prevent overheating of the solid rotors of the turbogenerators. Turbogenerator[illegible] designed to carry safely negative-sequence current not exceeding 15 [illegible] of rated current.

When the impedances of a system are unsymmetrical under normal operating conditions, the resulting unbalance may cause the voltage between two of the lines to exceed the maximum allowable when normal voltage is maintained between another pair of lines. Thus, allowable unbalance of voltage rather than the heating of negative-sequence current may be the limiting factor in determining the amount of unbalance to be tolerated.

14.3 General Equations Involving Unsymmetrical Series Impedances. Some of the types of unbalanced systems previously enumerated can best be analyzed by developing a set of equations for the positive-, negative-, and zero-sequence voltage drops between two points on either side of the unsymmetrical part of the system. The three sequence voltage drops will be derived in terms of the symmetrical components of the current and the series impedances between the points on either side of the dissymmetry in each phase.

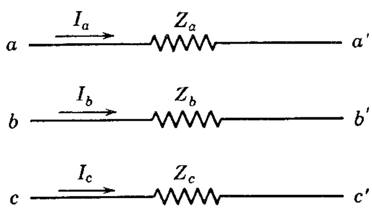


FIG. 14.1 Portion of a three-phase system showing three unequal series impedances.

Figure 14.1 shows the unsymmetrical part of a system with three unequal series impedances Z_a, Z_b, and Z_c. No coupling exists between the phases of the system. The voltage drops across the part of the system shown are

$$\begin{aligned} V_{aa'} &= I_a Z_a \\ V_{bb'} &= I_b Z_b \\ V_{cc'} &= I_c Z_c \end{aligned} \tag{14.1}$$

Replacing I_a, I_b, and I_c in Eqs. (14.1) by their symmetrical components yields

$$\begin{aligned} V_{aa'} &= (I_{a1} + I_{a2} + I_{a0})Z_a \\ V_{bb'} &= (a^2 I_{a1} + aI_{a2} + I_{a0})Z_b \\ V_{cc'} &= (aI_{a1} + a^2 I_{a2} + I_{a0})Z_c \end{aligned} \tag{14.2}$$

The symmetrical components of each voltage drop are

$$\begin{aligned} V_{aa'1} &= \tfrac{1}{3}(V_{aa'} + aV_{bb'} + a^2 V_{cc'}) \\ V_{aa'2} &= \tfrac{1}{3}(V_{aa'} + a^2 V_{bb'} + aV_{cc'}) \\ V_{aa'0} &= \tfrac{1}{3}(V_{aa'} + V_{bb'} + V_{cc'}) \end{aligned} \tag{14.3}$$

Substituting Eqs. (14.2) in (14.3) and collecting terms yields

$$\begin{aligned} V_{aa'1} &= \tfrac{1}{3}I_{a1}(Z_a + Z_b + Z_c) + \tfrac{1}{3}I_{a2}(Z_a + a^2 Z_b + aZ_c) \\ &\qquad + \tfrac{1}{3}I_{a0}(Z_a + aZ_b + a^2 Z_c) \\ V_{aa'2} &= \tfrac{1}{3}I_{a1}(Z_a + aZ_b + a^2 Z_c) + \tfrac{1}{3}I_{a2}(Z_a + Z_b + Z_c) \\ &\qquad + \tfrac{1}{3}I_{a0}(Z_a + a^2 Z_b + aZ_c) \\ V_{aa'0} &= \tfrac{1}{3}I_{a1}(Z_a + a^2 Z_b + aZ_c) + \tfrac{1}{3}I_{a2}(Z_a + aZ_b + a^2 Z_c) \\ &\qquad + \tfrac{1}{3}I_{a0}(Z_a + Z_b + Z_c) \end{aligned} \tag{14.4}$$

Equations (14.4) are the general equations for the symmetrical components of the voltage drop from point a to point a' in terms of the symmetrical components of the current I_a and the unbalanced series impedances between the points.

Equations (14.4) can be written more concisely and in a form which is more easily remembered by letting

$$\begin{aligned} Z_S &= \tfrac{1}{3}(Z_a + Z_b + Z_c) \\ Z_{M1} &= \tfrac{1}{3}(Z_a + aZ_b + a^2Z_c) \\ Z_{M2} &= \tfrac{1}{3}(Z_a + a^2Z_b + aZ_c) \end{aligned} \tag{14.5}$$

Then

$$\begin{aligned} V_{aa'1} &= I_{a1}Z_S + I_{a2}Z_{M2} + I_{a0}Z_{M1} \\ V_{aa'2} &= I_{a1}Z_{M1} + I_{a2}Z_S + I_{a0}Z_{M2} \\ V_{aa'0} &= I_{a1}Z_{M2} + I_{a2}Z_{M1} + I_{a0}Z_S \end{aligned} \tag{14.6}$$

The impedance Z_S is called the *sequence self-impedance.* The product of a current of any sequence and the sequence self-impedance yields a voltage drop of the same sequence as that of the current. For instance, the product of the positive-sequence current I_{a1} and Z_S is a component of the positive-sequence voltage drop $V_{aa'1}$.

The impedances Z_{M1} and Z_{M2} are called *sequence mutual impedances.* The product of a current of one sequence and a sequence mutual impedance yields a voltage drop of a sequence different from that of the current. For instance, the product of the positive-sequence current I_{a1} and the sequence mutual impedance Z_{M1} is a component of the negative-sequence voltage drop $V_{aa'2}$. Similarly, the product of the positive-sequence current I_{a1} and the sequence mutual impedance Z_{M2} is a component of the zero-sequence voltage drop $V_{aa'0}$.

The equations for the sequence self-impedance and mutual impedances in terms of the impedances of the phases of the circuit are similar in form to Eqs. (10.9), (10.13), and (10.15), and Eqs. (10.19) to (10.21), respectively, for the symmetrical components of voltage and current. The equation for the sequence self-impedance Z_S corresponds to the equations for zero-sequence components of voltage and current. The sequence mutual impedances Z_{M1} and Z_{M2} correspond to the equations for positive- and negative-sequence components of voltage and current, respectively. Thus, the sequence self-impedance Z_S might have been called the zero-sequence component of the phase impedances, and the mutual impedances Z_{M1} and Z_{M2} might have been called the positive- and negative-sequence components of the phase impedances. Positive-, negative-, and zero-sequence impedances are names which have been used previously, however, for describing the impedances of a symmetrical circuit to the flow of positive-, negative-, and zero-sequence currents and have been represented by Z_1, Z_2, and Z_0. The similarity between the sequence self-impedance and mutual-impedance equations and the equations for the symmetrical components of voltage and current is an aid in remembering Eqs. (14.5). The numerical subscripts for the sequence mutual impedances have been chosen to emphasize this similarity, and the absence of a numerical subscript on the self-impedance term may be interpreted as a numerical subscript of zero.

A further aid in remembering the equations is the fact that the sum of the numerical subscripts of the symbols for current and impedance in each term composing the positive-sequence voltage drop is 1 or 4. The sum is 2 for the negative-sequence voltage drop and 0 or 3 for the zero-sequence voltage drop.

14.4 Unsymmetrical Three-phase Load. An unsymmetrical three-phase load is a special case of three unsymmetrical series impedances

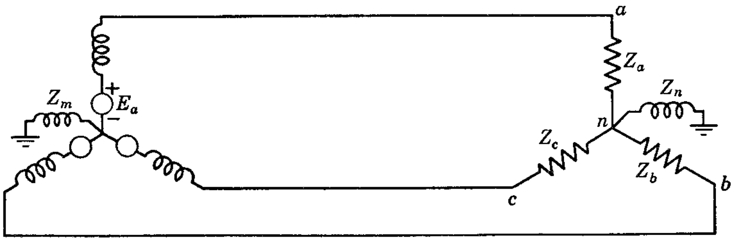


FIG. 14.2 Generator supplying an unsymmetrical Y load grounded through Z_n.

where the ends of the three series impedances are connected to a common point. If the points a', b', and c' of Fig. 14.1 are connected together, and the common point is grounded through Z_n, the impedances Z_a, Z_b, and Z_c become the impedances of a grounded, unbalanced three-phase load. With the addition of a generator grounded through a reactor to represent the system to the left of points a, b, and c of Fig. 14.1, the circuit becomes that shown in Fig. 14.2.

The sequence components of voltage of the unbalanced load from line a to neutral are, from Eqs. (14.6),

$$\begin{aligned} V_{an1} &= I_{a1}Z_S + I_{a2}Z_{M2} + I_{a0}Z_{M1} \\ V_{an2} &= I_{a1}Z_{M1} + I_{a2}Z_S + I_{a0}Z_{M2} \\ V_{an0} &= I_{a1}Z_{M2} + I_{a2}Z_{M1} + I_{a0}Z_S \end{aligned} \tag{14.7}$$

where Z_S, Z_{M1}, and Z_{M2} are determined from Eqs. (14.5).

From Eqs. (11.2) to (11.4) the sequence components of voltage from point a to ground are

$$\begin{aligned} V_{a1} &= E_a - I_{a1}Z_1 \\ V_{a2} &= -I_{a2}Z_2 \\ V_{a0} &= -I_{a0}Z_0 \end{aligned} \tag{14.8}$$

where Z_1 and Z_2 are the positive- and negative-sequence impedances of the generator and connecting lines, and Z_0 is the zero-sequence impedance of the generator and connecting lines plus $3Z_m$ to account for the impedance in the generator neutral.

Since the voltage from point a to ground is independent of the path, V_{a1} and V_{a2} are equal, respectively, to V_{an1} and V_{an2}. V_{a0} is equal to $V_{an0} + 3I_{a0}Z_n$. Therefore, from Eqs. (14.7) and (14.8),

$$\begin{aligned} E_a &= I_{a1}(Z_1 + Z_S) + I_{a2}Z_{M2} + I_{a0}Z_{M1} \\ 0 &= I_{a1}Z_{M1} + I_{a2}(Z_2 + Z_S) + I_{a0}Z_{M2} \\ 0 &= I_{a1}Z_{M2} + I_{a2}Z_{M1} + I_{a0}(Z_0 + Z_S + 3Z_n) \end{aligned} \tag{14.9}$$

Equations (14.9) can be solved for the symmetrical components of current in terms of E_a and the impedances.

If the unsymmetrical load is connected to a very large system, addition of the load has negligible effect on the system. With balanced voltages at the load and a very large system, Eqs. (14.9) are simplified by substituting for E_a the positive-sequence component of the voltage from a to n, V_{a1}, and by letting Z_1, Z_2, and Z_0 be zero. Then

$$\begin{aligned} V_{a1} &= I_{a1}Z_S + I_{a2}Z_{M2} + I_{a0}Z_{M1} \\ 0 &= I_{a1}Z_{M1} + I_{a2}Z_S + I_{a0}Z_{M2} \\ 0 &= I_{a1}Z_{M2} + I_{a2}Z_{M1} + I_{a0}(Z_S + 3Z_n) \end{aligned} \tag{14.10}$$

In such a system, if the neutral of the load is solidly grounded, Z_n is zero. Equations (14.10) are not needed to compute the load currents, however, when the neutral is solidly grounded since the voltages to neutral are then balanced, and thus the currents can be easily determined by ordinary methods. In fact, a solution by Kirchhoff's law is simpler than using Eqs. (14.10) whenever the voltages applied to the grounded-Y load are balanced, regardless of the impedance in the neutral. Both methods require the simultaneous solution of three equations, but the solution by Kirchhoff-law equations is less involved than the solution by symmetrical components. If the supply is not an infinite bus, Eqs. (14.9) must be used since they account for the different values of impedance of the system to positive-, negative-, and zero-sequence currents.

If the unbalanced-Y load is not connected to ground at the neutral, zero-sequence current cannot flow. With zero-sequence currents absent, Eqs. (14.9) become

$$E_a = I_{a1}(Z_1 + Z_S) + I_{a2}Z_{M2} \tag{14.11}$$

$$0 = I_{a1}Z_{M1} + I_{a2}(Z_2 + Z_S) \tag{14.12}$$

$$0 = I_{a1}Z_{M2} + I_{a2}Z_{M1} + V_n \tag{14.13}$$

where V_n is the voltage from the neutral of the load to ground. Equation (14.13) contains the voltage V_n since the zero-sequence voltage from point a to ground is $V_{a0} = V_{an0} + V_n = 0$. Equations (14.11) and (14.12) are solved by determinants with the following results:

$$I_{a1} = \frac{E_a(Z_2 + Z_S)}{(Z_1 + Z_S)(Z_2 + Z_S) - Z_{M1}Z_{M2}} \tag{14.14}$$

$$I_{a2} = \frac{-E_a Z_{M1}}{(Z_1 + Z_S)(Z_2 + Z_S) - Z_{M1}Z_{M2}} \tag{14.15}$$

Example 14.1

A generator has a synchronous reactance of 1.0 per unit. Negative- and zero-sequence reactances are 0.3 and 0.05 per unit, respectively. The resistance of the generator may be neglected. The load is Y-connected and ungrounded, with pure resistance of 1.0 per unit in phase a, and impedances in phases b and c of $0 + j1.0$ per unit. Find the voltage in per unit across each impedance of the load and from the neutral of the load to the neutral of the generator. The no-load voltage of the generator is 1.3 per unit.

Solution

$$Z_a = 1 + j0 \qquad Z_1 = 0 + j1$$
$$Z_b = 0 + j1 \qquad Z_2 = 0 + j0.3$$
$$Z_c = 0 + j1 \qquad E_a = 1.3 + j0$$

$$Z_S = \tfrac{1}{3}(1 + j2) = 0.333 + j0.667$$

$$Z_{M1} = \tfrac{1}{3}[1 + j(-0.5 + j0.866) + j(-0.5 - j0.866)]$$
$$= 0.333 - j0.333 = 0.471\underline{/-45^\circ}$$

$$Z_{M2} = \tfrac{1}{3}[1 + j(-0.5 - j0.866) + j(-0.5 + j0.866)]$$
$$= 0.333 - j0.333 = 0.471\underline{/-45^\circ}$$

$$I_{a1} = \frac{1.3(j0.3 + 0.333 + j0.667)}{(j1.0 + 0.333 + j0.667)(j0.3 + 0.333 + j0.667) - (0.471\underline{/-45^\circ} \times 0.471\underline{/-45^\circ})}$$

$$= \frac{0.433 + j1.259}{-1.510 + j1.097} = \frac{1.33\underline{/71.0^\circ}}{1.86\underline{/144.0^\circ}} = 0.715\underline{/-73.0^\circ} = 0.209 - j0.684$$

$$I_{a2} = \frac{-1.3 \times 0.471\underline{/-45^\circ}}{1.86\underline{/144^\circ}} = 0.329\underline{/-9.0^\circ} = 0.325 - j0.052$$

$$I_a = 0.209 - j0.684 + 0.325 - j0.052$$
$$= 0.534 - j0.736 = 0.907\underline{/-54.0^\circ} \text{ per unit}$$

$$I_{b1} = 0.715\underline{/240^\circ - 73.0^\circ} = 0.715\underline{/167.0^\circ} = -0.697 + j0.161$$
$$I_{b2} = 0.329\underline{/120^\circ - 9.0^\circ} = 0.329\underline{/111.0^\circ} = -0.118 + j0.307$$
$$I_b = -0.815 + j0.46[illegible]$$
$$= 0.941\underline{/150.[illegible]} \text{ per unit}$$

$$I_{c1} = 0.715\underline{/120^\circ - 73.0^\circ} = 0.715\underline{/47.0^\circ} = 0.488 + [illegible]523$$
$$I_{c2} = 0.329\underline{/240^\circ - 9.0^\circ} = 0.329\underline{/231.0^\circ} = -0.20[illegible]\ j0.25[illegible]$$
$$I_c = 0.28[illegible] + j0.268$$
$$= 0.3[illegible]3[illegible] \text{ per unit}$$

$$
\begin{aligned}
V_{an} &= 1 \times (0.534 - j0.736) = 0.534 - j0.736 \\
&= 0.907\angle{-54.0} \text{ per unit (line-to-neutral base)} \\
V_{bn} &= j1 \times (-0.815 + j0.468) = -0.468 - j0.815 \\
&= 0.941\angle{240.1^\circ} \text{ per unit (line-to-neutral base)} \\
V_{cn} &= j1 \times (0.281 + j0.268) = -0.268 + j0.281 \\
&= 0.388\angle{113.7^\circ} \text{ per unit (line-to-neutral base)}
\end{aligned}
$$

From Eq. (14.13),

$$
\begin{aligned}
V_n &= -(I_{a1}Z_{M2} + I_{a2}Z_{M1}) \\
&= -(0.715\angle{-73.0^\circ} \times 0.471\angle{-45^\circ} + 0.329\angle{-9.0^\circ} \times 0.471\angle{-45^\circ}) \\
&= -(0.337\angle{-118.0^\circ} + 0.155\angle{-54.0^\circ}) \\
&= 0.067 + j0.424 = 0.430\angle{81.2^\circ} \text{ per unit (line-to-neutral base)}
\end{aligned}
$$

If the unbalanced load is Δ-connected, it can be converted to its equivalent Y. The neutral of the equivalent Y is ungrounded, and zero-sequence current is absent. The components of the line current are found by Eqs. (14.14) and (14.15) with self and mutual impedances to correspond to the equivalent Y.

In the derivation of equations for unsymmetrical three-phase loads a simple system of one generator and one load was assumed. Analysis in terms of one generator is sufficient if all the unbalanced loads are at one location, for a symmetrical system containing more than one emf can be reduced to a single emf and series impedance by the Helmholtz-Thévenin theorem. The voltage E_a in the equations is replaced by the voltage from line a to ground at the load point with the unsymmetrical loads disconnected. The impedances Z_1, Z_2, and Z_0 are the positive-, negative-, and zero-sequence impedances looking into the network at the load point with the unsymmetrical loads disconnected. Synchronous reactance rather than subtransient or transient reactance determines Z_1 since steady-state values are desired.

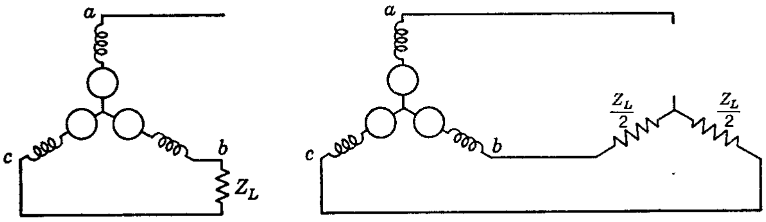
14.5 Single-phase Line-to-line Load. A single-phase load connected from line to line is identical to an ungrounded Y-connected load having infinite impedance in one phase. If the load is on lines b and c, the impedance in phase a of the Y is infinite, as shown in Fig. 14.3 where the impedance Z_L is divided equally between phases b and c. The Y load of Fig. 14.3b is identical to the single-phase load of Fig. 14.3a.

Since there is no ground connection, only positive- and negative-sequence currents will flow in the circuit, and Eqs. (14.14) and (14.15) give the values of I_{a1} and I_{a2}. Since $Z_b = Z_c$,

$$Z_S = \tfrac{1}{3}(Z_a + 2Z_b) \tag{14.16}$$

$$Z_{M1} = Z_{M2} = \tfrac{1}{3}(Z_a - Z_b) \tag{14.17}$$

The impedance terms in the numerator and denominator of Eq. (14.14)



(*a*) Single-phase load between lines *b* and *c*

(*b*) Unbalanced-wye load equivalent to single-phase load Z_L between lines *b* and *c*

FIG. 14.3 Generator supplying a single-phase line-to-line load Z_L.

are

$$Z_2 + Z_S = Z_2 + \frac{Z_a + 2Z_b}{3} \tag{14.18}$$

$$(Z_1 + Z_S)(Z_2 + Z_S) = Z_1Z_2 + \frac{Z_1Z_a + 2Z_1Z_b}{3} + \frac{Z_2Z_a + 2Z_2Z_b}{3} + \frac{Z_a^2 + 4Z_aZ_b + 4Z_b^2}{9} \tag{14.19}$$

$$Z_{M1}Z_{M2} = \frac{Z_a^2 - 2Z_aZ_b + Z_b^2}{9} \tag{14.20}$$

After substitution of these terms in Eq. (14.14), the numerator and denominator of the right-hand side of the equation are divided by Z_a, and as Z_a approaches infinity

$$I_{a1} = \frac{E_a/3}{(Z_1 + Z_2)/3 + (4Z_b + 2Z_b)/9} \tag{14.21}$$

$$I_{a1} = \frac{E_a}{Z_1 + Z_2 + 2Z_b} \tag{14.22}$$

and, since $Z_b = Z_L/2$

$$I_{a1} = \frac{E_a}{Z_1 + Z_2 + Z_L} \tag{14.23}$$

Similarly, from Eq. (14.15),

$$I_{a2} = \frac{-E_a}{Z_1 + Z_2 + Z_L} \tag{14.24}$$

The single-phase load Z_L between phases *b* and *c* could have been treated as a line-to-line fault through impedance with the same result. Comparison of Eqs. (14.23) and (13.13) shows the similarity. The currents I_{a1} and I_{a2} are equal in magnitude and 180° out of phase in both cases. Therefore, the positive- and negative-sequence networks may be interconnected as shown in Fig. 14.4 to represent the line-to-line load on an otherwise symmetrical system in the same manner as was shown in Fig. 13.14*c*, to represent a line-to-line fault through the impedance Z_f.

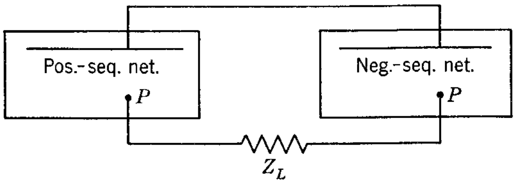


FIG. 14.4 Connection of the sequence networks for a single-phase line-to-line load Z_L.

Example 14.2

A 1,000-kva, 240-volt turbine generator has a synchronous reactance of 100% and a negative-sequence reactance of 10%. At its terminals, the generator is supplying a balanced three-phase load of 900 kw at 240 volts. The load is pure resistance. If a single-phase load of 100 kw, purely resistive, is applied between two of the terminals of the generator, find the negative-sequence current in the generator in per cent of rated current and the line voltages in per cent of rated voltage.

Solution

Choose as base 1,000 kva, 240 volts.

$$\text{Base impedance} = \frac{(0.24)^2 \times 1{,}000}{1{,}000} = 0.0576 \text{ ohm}$$

$$\text{Impedance of the three-phase load} = \frac{(240/\sqrt{3})^2}{900{,}000/3} = 0.064 \text{ ohm}$$

$$= \frac{0.064}{0.0576} = 1.11 \text{ per unit}$$

$$\text{Impedance of the single-phase load} = \frac{(240)^2}{100{,}000} = 0.576 \text{ ohm}$$

$$= \frac{0.576}{0.0576} = 10.0 \text{ per unit}$$

Figure 14.5 shows the connection of the sequence networks. The Helmholtz-Thévenin equivalent of the positive-sequence network has a gen-

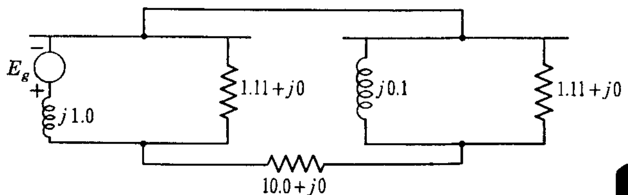


FIG. 14.5 Diagram of the sequence networks for Exam

erated voltage of 1.0 per unit, since the terminal vo the generator is 240 volts before the single-phase load is added, and series impedance is

$$Z_1 = \frac{j1.0 \times 1.11}{1.11 + j1.0} = 0.743\underline{/48.0^\circ} = 0. \quad j0.5$$

For the negative-sequence network

$$Z_2 = \frac{j0.1 \times 1.11}{1.11 + j0.1} = 0.099\underline{/84.8^\circ} = 0.009 + j0.099$$

Then, from Eqs. (14.23) and (14.24),

$$I_{a1} = \frac{1.0}{0.498 + j0.552 + 0.009 + j0.099 + 10.0}$$
$$= \frac{1.0}{10.507 + j0.651} = 0.095\underline{/-3.6^\circ}$$

and

$$I_{a2} = -I_{a1} = 0.095\underline{/176.4^\circ}$$

The negative-sequence current divides between the two branches of the network inversely as the impedances. The magnitude of the portion of the negative-sequence current through the generator is

$$0.095 \frac{1.11}{|1.11 + j0.1|} \cong 0.095 \text{ per unit}$$

Since the rated current is 1.0 per unit, the negative-sequence current in the generator in terms of rated current is 9.5%.

The voltages are calculated as follows:

$$V_{a1} = 1.0 - 0.095\underline{/-3.6^\circ} \times 0.743\underline{/48.0^\circ} = 1.0 - 0.051 - j0.049$$
$$= 0.949 - j0.049 = 0.949\underline{/-3.0^\circ}$$
$$V_{a2} = -0.095\underline{/176.4^\circ} \times 0.099\underline{/84.8^\circ} = 0.0094\underline{/81.2^\circ} = 0.0014 + j0.0093$$
$$V_a = V_{a1} + V_{a2} = 0.950 - j0.040 \text{ per unit (line-to-neutral base)}$$
$$V_{b1} = 0.949\underline{/237.0^\circ} = -0.516 - j0.796$$
$$V_{b2} = 0.0094\underline{/201.2^\circ} = -0.009 - j0.003$$
$$V_b = -0.525 - j0.799 \text{ per unit (line-to-neutral base)}$$
$$V_{c1} = 0.949\underline{/117.0^\circ} = -0.432 + j0.845$$
$$V_{c2} = 0.0094\underline{/321.2^\circ} = 0.007 - j0.006$$
$$V_c = -0.425 + j0.839 \text{ per unit (line-to-neutral base)}$$

In per unit of the base line-to-line voltage,

$$V_{ab} = \frac{V_a - V_b}{\sqrt{3}} = \frac{1}{\sqrt{3}}(0.950 - j0.040 + 0.525 + j0.799)$$
$$= 0.958\underline{/27.2^\circ} \text{ per unit}$$

$$V_{bc} = \frac{V_b - V_c}{\sqrt{3}} = \frac{1}{\sqrt{3}}(-0.525 - j0.799 + 0.425 - j0.839)$$
$$= 0.94[illegible]/26[illegible] per unit

$$V_{ca} = \frac{V_c - V_a}{\sqrt{3}} = \frac{1}{\sqrt{3}}(-0.425 + j0.839 - 0.950 + j0.040)$$
$$= 0.941\underline{/147.4^\circ} \text{ per unit}$$

If V_{bc} is raised to 240 volts by increasing the excitation of the generator, all the above voltages and currents are increased by a factor of

1.0/0.945 = 1.06. Then the negative-sequence current in the generator in terms of rated current is 1.06 × 9.5% = 10.1%, and

$$V_{ab} = 1.06 \times 0.958 = 1.016 \text{ per unit} = 101.6\% \text{ of rated voltage}$$
$$V_{bc} = 100\% \text{ of rated voltage}$$
$$V_{ca} = 1.06 \times 0.941 = 0.999 \text{ per unit} = 99.9\% \text{ of rated voltage}$$

14.6 Single-phase Line-to-neutral Load. A simple method of finding the relations between the sequence networks for a single-phase line-to-neutral load is to recognize that the line-to-neutral load has the same effect as a sustained line-to-ground fault through impedance. Instead of deriving the relations between the sequence networks from Eqs. (14.9), let us refer to the derivation of Eq. (13.11) and note that the load impedance Z_L in Fig. 14.6 corresponds to a fault impedance Z_f.

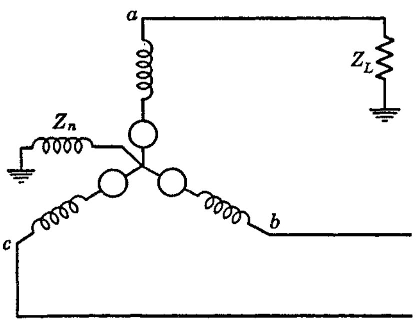


FIG. 14.6 Generator supplying a single-phase line-to-neutral load Z_L.

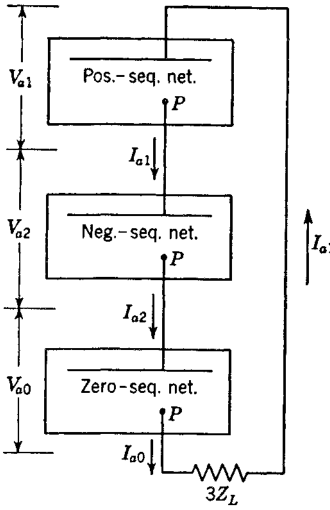


FIG. 14.7 Connection of the sequence networks for a single-phase line-to-neutral load Z_L.

Then, for a single generator supplying a line-to-neutral load only,

$$I_{a1} = I_{a2} = I_{a0} \tag{14.25}$$

and

$$I_{a1} = \frac{E_a}{Z_1 + Z_2 + Z_0 + 3Z_L} \tag{14.26}$$

where E_a is the no-load generated voltage of phase a. The zero-sequence impedance Z_0 includes $3Z_n$ to account for the impedance in the neutral of the generator.

If the line-to-neutral load occurs in a system which is otherwise symmetrical, the system is replaced by its Helmholtz-Thévenin equivalent with the impedances handled in the usual manner. By comparison with Fig. 13.14*b* the sequence networks must be connected as shown in Fig. 14.7. The solution of problems proceeds in a manner similar to that discussed for the single-phase line-to-line load.

14.7 The Special Case of Two Equal Series Impedances. When two of three unsymmetrical series impedances are equal, an interconnection of sequence networks can be made to represent the unsymmetrical circuit. An example of such an unsymmetrical series circuit, as will be shown later, is the equivalent circuit of an open-Δ transformer bank. Other examples occur when one of three reactances (such as a series capacitor or current-limiting reactor) differs from the other two, or when an impedance is inserted in series with only one of the lines. The operation of a single-pole reclosing breaker during a fault inserts an infinite impedance between two points in one phase while the impedance is zero between corresponding points of the other two phases. Another instance of an unsymmetrical series circuit where the impedances are equal in two of the phases is the line having one open conductor, which was discussed in Chap. 13. Lines having two open conductors are in the same category.

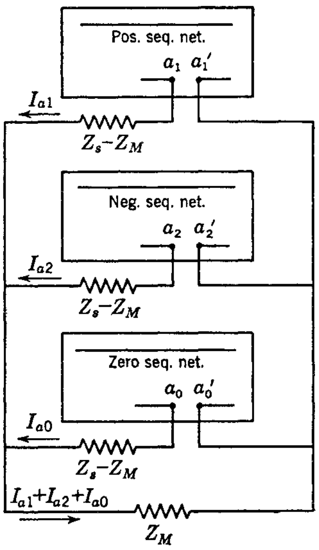


FIG. 14.8 Connection of the sequence networks for equal series impedances in two of the three lines.

When the two impedances Z_b and Z_c are equal in an unsymmetrical series circuit composed of the impedances Z_a, Z_b, and Z_c in phases a, b, and c, respectively, Z_{M1} is equal to Z_{M2}. Upon substitution of Z_b for Z_c in Eqs. (14.5), we have

$$\begin{aligned} Z_{M1} &= \tfrac{1}{3}(Z_a + aZ_b + a^2Z_b) = \tfrac{1}{3}(Z_a - Z_b) \\ Z_{M2} &= \tfrac{1}{3}(Z_a + a^2Z_b + aZ_b) = \tfrac{1}{3}(Z_a - Z_b) \end{aligned} \tag{14.27}$$

and, letting $Z_{M1} = Z_{M2} = Z_M$, Eqs. (14.6) become

$$\begin{aligned} V_{aa'1} &= I_{a1}Z_S + I_{a2}Z_M + I_{a0}Z_M \\ V_{aa'2} &= I_{a1}Z_M + I_{a2}Z_S + I_{a0}Z_M \\ V_{aa'0} &= I_{a1}Z_M + I_{a2}Z_M + I_{a0}Z_S \end{aligned} \tag{14.28}$$

The interconnection of the sequence networks shown in Fig. 14.8 satisfies Eqs. (14.28). The impedance $Z_S - Z_M$ is inserted in each network between points a and a' in series with the mutual impedance Z_M, which is common to all three sequence networks between the points a and a'.

Now let us evaluate the impedance terms Z_M and $Z_S - Z_M$ for some special circuits in all of which $Z_b = Z_c$ and the connection of Fig. 14.8 is valid. We shall call Case I the condition having only the restriction that $Z_b = Z_c$. Conditions for the four cases that we shall investigate are:

Case I: Z_a arbitrary, Z_b arbitrary, $Z_c = Z_b$
Case II: Z_a arbitrary, $Z_b = Z_c = 0$
Case III: $Z_a = 0$, Z_b arbitrary, $Z_c = Z_b$
Case IV: $Z_a = \infty$, $Z_b = Z_c = 0$

The values of Z_M and $Z_S - Z_M$ computed for the above four cases are shown in Table 14.1.

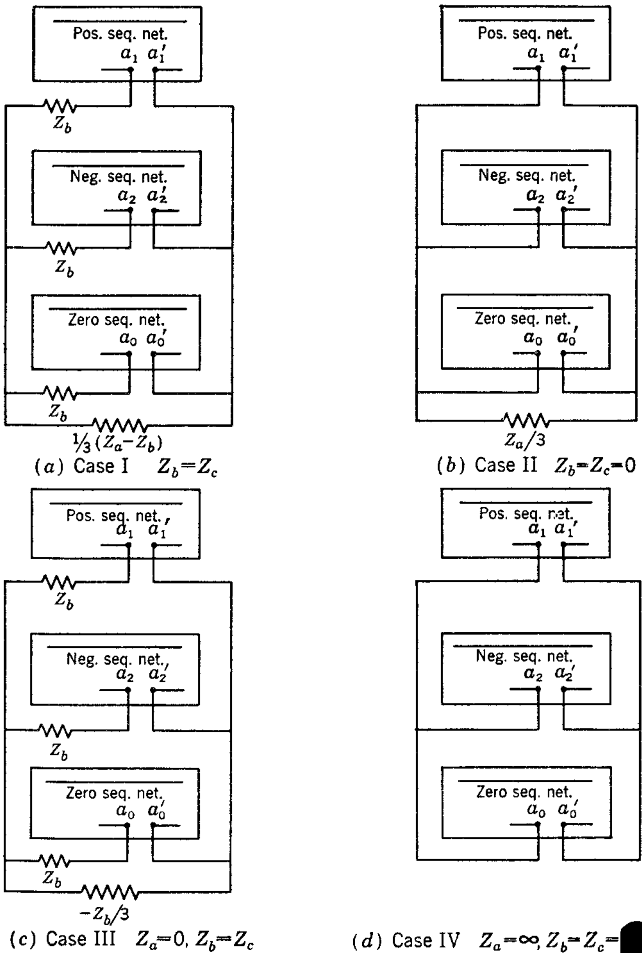


FIG. 14.9 Connections of the sequence networks for special cases of nonsymmetrical series circuits, in all of which $Z_b = Z_c$.

When the impedances of Table 14.1 are substituted for Z_M and $Z_S - Z_M$ in Fig. 14.8, the sequence network connections for the four cases are as shown in Fig. 14.9. Case IV is that for one open conductor, and has would be expected, the interconnection of the networks is identical to that shown in Fig. 13.16.

TABLE 14.1 IMPEDANCE TERMS FOR FIG. 14.8

Case	Series impedance			$Z_S - Z_M$	Z_M
	Phase a	Phase b	Phase c		
I	Z_a	Z_b	Z_b	Z_b	$\frac{1}{3}(Z_a - Z_b)$
II	Z_a	0	0	0	$Z_a/3$
III	0	Z_b	Z_b	Z_b	$-Z_b/3$
IV	∞	0	0	0	∞

If there is no path for zero-sequence current, a simple relation exists between I_{a1} and I_{a2} in an unsymmetrical series circuit with $Z_b = Z_c$. The connection of the sequence networks is shown in Fig. 14.10, from which, by recognizing that I_{a1} splits inversely as the impedances in the two parallel branches, we obtain

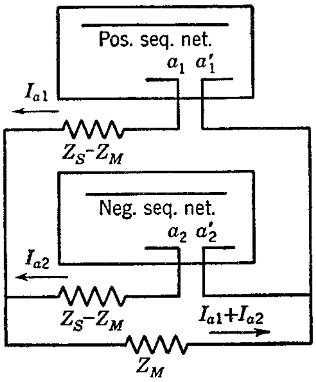


FIG. 14.10 Connection of the sequence networks for equal series impedances in two lines of a three-phase system, zero-sequence currents absent.

$$I_{a2} = -I_{a1}\frac{Z_M}{Z_M + (Z_S - Z_M) + Z_2}$$

$$I_{a2} = -I_{a1}\frac{Z_M}{Z_S + Z_2} \tag{14.29}$$

where Z_2 is the series impedance of the negative-sequence network measured between points a and a'. Equation (14.29) provides a convenient method of determining the negative-sequence current as a percentage of the positive-sequence current when examining the possibility of overheating due to negative-sequence current.

If two of three series impedances are infinite, the proper interconnection of the networks cannot be determined from Fig. 14.8 because Eqs. (14.28) are indeterminate in such a case. If $Z_a = 0$ and $Z_b = Z_c = \infty$,

$$\begin{aligned} Z_S &= \tfrac{1}{3}(Z_a + Z_b + Z_c) = \frac{2Z_b}{3} \\ Z_{M1} &= \tfrac{1}{3}(Z_a + aZ_b + a^2Z_c) = -\frac{Z_b}{3} \\ Z_{M2} &= \tfrac{1}{3}(Z_a + a^2Z_b + aZ_c) = -\frac{Z_b}{3} \end{aligned} \tag{14.30}$$

and Eqs. (14.28) become

$$
\begin{aligned}
V_{aa'1} &= \tfrac{1}{3}(2I_{a1}Z_b - I_{a2}Z_b - I_{a0}Z_b) = \tfrac{1}{3}Z_b(2I_{a1} - I_{a2} - I_{a0}) \\
V_{aa'2} &= \tfrac{1}{3}(-I_{a1}Z_b + 2I_{a2}Z_b - I_{a0}Z_b) \\
&\qquad = \tfrac{1}{3}Z_b(-I_{a1} + 2I_{a2} - I_{a0}) \qquad (14.31) \\
V_{aa'0} &= \tfrac{1}{3}(-I_{a1}Z_b - I_{a2}Z_b + 2I_{a0}Z_b) \\
&\qquad = \tfrac{1}{3}Z_b(-I_{a1} - I_{a2} + 2I_{a0})
\end{aligned}
$$

Since it was shown in Chap. 13 that $I_{a1} = I_{a2} = I_{a0}$ when two conductors are open, Eqs. (14.31) are indeterminate, because the voltages are the product of zero times infinity. The proper connection of the networks for two open conductors is discussed in Chap. 13 and shown in Fig. 13.19.

14.8 The Open-Δ Transformer Bank. One reason some power companies prefer to use banks of single-phase transformers rather than a single three-phase unit is that a bank can be operated open-Δ in the

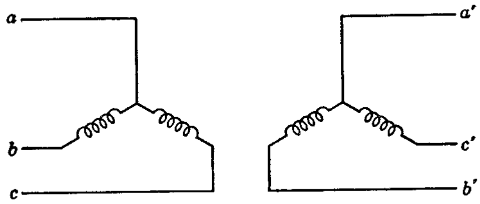


FIG. 14.11 Wiring diagram of an open-Δ transformer bank.

event of damage to the windings of one phase. On a growing system open-Δ banks are sometimes installed at first, and the additional unit is added when the load requirements increase beyond the rating of the open Δ.

The open-Δ transformer bank constitutes an unbalanced series impedance in the system. If the two units are identical, the equivalent circuit, with magnetizing current disregarded, comprises equal impedances in series in two phases and zero impedance in the third phase. Figure 14.11 shows the connections of an open-Δ transformer bank, and the circuit has been drawn deliberately to show that the circuit can be treated as a Y connection with one phase short-circuited. Thus, the equivalent circuit of the open-Δ transformer bank is the same as that of a Y-Y transformer with one phase short-circuited, as shown in Fig. 14.12.

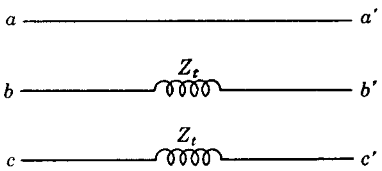


FIG. 14.12 Equivalent circuit of an open-Δ transformer bank, magnetizing current neglected.

The proper value of per-unit impedance for the equivalent circuit must be determined. The leakage impedance of the individual transformer will usually be given in per cent or per unit based on its rated kv and kva.

Let

Z_T = leakage impedance of each transformer in ohms
Z_t = per-unit leakage impedance of each transformer on the three-phase base of the system
$Z_{t'}$ = per-unit leakage impedance of each transformer based on the rating of the transformer

Then

$$Z_{t'} = \frac{Z_T(\text{kva})}{(\text{kv})^2 \times 1{,}000} \tag{14.32}$$

where (kv) and (kva) are rated values for the transformer. If the rated kv of each transformer is the base line-to-line kv of the system and the base three-phase kva of the system is 3 × rated kva of each transformer, the per-unit impedance of the transformer on the base of the three-phase system is

$$Z_t = \frac{3Z_T(\text{kva})}{(\text{kv})^2 \times 1{,}000} = 3Z_{t'} \tag{14.33}$$

where (kv) and (kva) have the same values as in Eq. (14.32). In the equivalent circuit shown in Fig. 14.12, a per-unit series impedance of Z_t is shown, in agreement with the above discussion.

As shown by Fig. 14.12, the equivalent circuit of the open-Δ transformer bank is an example of Case III of Table 14.1. Since zero-sequence current cannot flow, the connection of the sequence networks is that shown in Fig. 14.10 with $Z_S = 2Z_t/3$, $Z_M = -Z_t/3$, and $Z_S - Z_M = Z_t$. From Eq. (14.29), the negative-sequence current in the transformer bank in terms of the positive-sequence current is

$$\begin{aligned} I_{a2} &= -I_{a1}\frac{-Z_t/3}{2Z_t/3 + Z_2} \\ I_{a2} &= I_{a1}\frac{Z_t}{2Z_t + 3Z_2} \end{aligned} \tag{14.34}$$

In Eq. (14.34) the impedances may be in per unit referred to a base as discussed above, or the impedances may be in ohms where the leakage impedance of the transformer is referred to the side of the transformer to which the motor is connected.

Example 14.3

A 500-hp, 6.9-kv, 0.8-power factor synchronous motor operates from an infinite bus through an open-Δ transformer bank. Each unit of the transformer bank is rated 13.8–6.9 kv, 200 kva with a leakage reactance of 8%. Find the negative-sequence current in the motor in per cent of the positive-sequence current.

Solution

Since the kva rating of the motor is not specified, it will be assumed to be 1.1 times the rated horsepower in accord with the rule given in Sec. 8.4. Thus,

$$\text{kva} = 1.1 \times 500 = 550$$

A typical value of 20% will be assumed for negative-sequence reactance, and negative-sequence resistance will be neglected.

Select a three-phase base of 600 kva, which is three times the rated kva of one transformer, and a line-to-line base voltage of 6.9 kv. Then, the per-unit leakage reactance of the transformer is

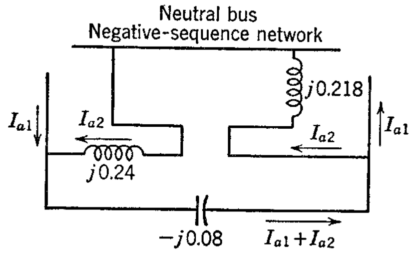


FIG. 14.13 Negative-sequence network and mutual impedance common to the positive- and negative-sequence networks for Example 14.3.

$$Z_{t'} = jX_{t'} = j0.08 \text{ per unit}$$

and on the three-phase system base

$$Z_t = 3Z_{t'} = 3 \times j0.08 = j0.24 \text{ per unit}$$

The negative-sequence reactance of the motor on the selected base is

$$Z_2 = jX_2 = j0.2 \times {}^{600}\!/_{550} = j0.218 \text{ per unit}$$

For the open-Δ transformer bank,

$$Z_a = 0 \qquad Z_b = Z_c = Z_t = j0.24$$
$$Z_S = \tfrac{1}{3}(0 + j0.24 + j0.24) = j0.16$$
$$Z_M = \tfrac{1}{3}(0 + aj0.24 + a^2j0.24) = -j0.08$$
$$Z_S - Z_M = j0.24$$

Since only the ratio I_{a2}/I_{a1} is desired, the positive-sequence network is not needed. The significant portion of the network connection is shown in Fig. 14.13. From Fig. 14.13,

$$I_{a2} = -I_{a1}\frac{-j0.08}{j0.24 - j0.08 + j0.218} = +I_{a1}\frac{j0.08}{j0.378} = 0.212I_{a1}$$

or from Eq. (14.34),

$$I_{a2} = I_{a1}\frac{j0.24}{j0.48 + j0.654} = I_{a1}\frac{j0.24}{j1.134} = 0.212I_{a1}$$

and

$$\frac{I_{a2}}{I_{a1}} = 21.2\%$$

If the supply is not an infinite bus, the impedance of the negative-sequence network between a and a' is larger than its value in the above problem, and I_{a2} is a smaller percentage of I_{a1}.

PROBLEMS

14.1 Balanced three-phase voltages of 100 volts line-to-line are applied to a Y-connected load consisting of three resistors. The neutral of the load is not grounded. The resistance in phase a is 10 ohms, in phase b 20 ohms, and in phase c 30 ohms. Find the current in phase a by symmetrical components.

14.2 If the load of Prob. 14.1 is supplied by a generator rated 500 volt-amps, 100 volts, find the current in phase a. Assume that the field of the generator is adjusted to give a terminal voltage of 100 volts line-to-line at no load. The per-unit reactances of the generator are $X_1 = 1.0$, $X_2 = 0.10$, and $X_0 = 0.05$.

14.3 A 1,000-hp, 2,300-volt, synchronous motor has per-unit reactances of $X_1 = 1.0$, $X'' = 0.15$, $X_2 = 0.10$, and $X_0 = 0.05$. It is delivering full load while operating at 2,300 volts, unity power factor. The efficiency of the motor is 90%. The motor is directly connected to a 2,000-kva, 2,300-volt generator having per-unit reactances of $X_1 = 1.0$, $X_2 = 0.15$, and $X_0 = 0.10$. Sometime later a single-phase lighting load rated 1,000 kw, 2,300 volts is connected between two of the lines. Find the ratio of the negative-sequence current in the generator to the rated current of the generator. Find the ratio of the negative-sequence current in the motor to the rated current of the motor. Find all terminal voltages.

14.4 The high-tension side of a transformer is connected to an infinite bus, and the low-tension side is connected through series reactors to a bus supplying a group of motors. The three-phase transformer is rated 4,500 kva, 2.4Δ–115Y kv with leakage reactance of 10%. The reactance of one of the current-limiting reactors connecting the transformer to the motor bus is 0.3 ohm. The reactance of each of the other two reactors is 0.2 ohm. The group of motors consists of induction motors having a total rating of 2,000 hp and synchronous motors having a total rating of 1,000 hp. All the motors are rated 2,300 volts. The induction motors are rated at 70.7% power-factor with 85% efficiency and have a negative-sequence reactance of 10%. The synchronous motors are rated at unity powerfactor with 90% efficiency and have a negative-sequence reactance of 20%. Determine the negative-sequence current in the series reactors in per cent of the positive-sequence current.

14.5 A 15-kva, 1,150-volt alternator has per-unit reactances of $X_1 = 1.0$, $X_2 = 0.20$, and $X_0 = 0.10$. The alternator supplies a balanced three-phase load of resistors rated 15 kva, 115 volts. The transformer connecting the alternator and load is a Δ-Δ bank, each unit of which is rated 5 kva, 1,150–115 volts with a leakage reactance of 10%. The voltage at the load is 115 volts when *two* 1.5-kva, 115-volt loads of pure resistance are connected across two pairs of lines on the low-tension side of the transformer. Find the ratio of negative-sequence to positive-sequence current in the generator.

14.6 A 10,000-kva, 11.5-kv generator supplies a 10,000-kva, 2,200-volt motor through an open-Δ transformer bank. Each transformer is rated 7,500 kva, 11.5–2.3 kv and has a leakage reactance of 10%. The motor draws rated kva at rated voltage. Based on their respective ratings the generator and motor have the same per-unit reactances of $X_1 = 1.0$, $X'' = X_2 = 0.10$, and $X_0 = 0.05$. Determine the ratio of negative- to positive-sequence current in the generator.

14.7 Two transformers rated 6.9–2.3 kv, 75 kva with 8% reactance are connected in open delta to the 6.9-kv lines of a very large system. The transformers supply a synchronous motor rated 2,300 volts, 100 hp (110 kva), at 0.8 power factor. The negative-sequence reactance of the motor is 25%. Find the ratio of negative-sequence current in the motor to positive-sequence current in the motor.

CHAPTER 15

POWER SYSTEM STABILITY

15.1 The Stability Problem. When a-c generators were driven by reciprocating steam engines, one of the major problems in the operation of machinery was hunting. The periodic variations in the torque applied to the generators caused periodic variations in speed. The resulting periodic variations in voltage and frequency were transmitted to the motors connected to the system. Oscillations of the motors caused by the variations in voltage and frequency sometimes caused the motors to lose synchronism entirely if their natural frequency of oscillation coincided with the frequency of oscillation caused by the engines driving the generators. Damper windings were first used to minimize hunting by the damping action of the losses resulting from the currents induced in the damper windings by any relative motion between the rotor and the rotating field set up by the armature current. The use of turbines has reduced the problem of hunting, although it is still present where the prime mover is a diesel engine. Maintaining synchronism between the various parts of a power system becomes increasingly difficult, however, as the systems and interconnections between systems continue to grow. The tendency of a power system or its component parts to develop forces to maintain synchronism and equilibrium is known as *stability*. Much study has been devoted to stability since about 1920.

Let us consider a synchronous motor connected through a transformer to a large power source. We shall see later that the power delivered to the motor is determined by the voltage of the source, the internal voltage of the motor, and the phase angle between these two voltages. The phase angle depends upon the position of the rotor of the motor. The power delivered to the motor when it is running at constant [illegible] course, equal to the power output of the motor plus lo[illegible] the motor. If the mechanical load on the motor is increased, the motor cannot supply the entire load until its power input increases. [illegible]refore, the motor slows down. The phase angle between the internal voltage of the motor and the voltage of the system increases until [illegible] electric power input to the motor is equal to the power output plus losses. While the angle is increasing the excess of the power required by the motor over the electric power input is supplied by the stored energy in the rotating system. As the motor speed decreases, the stored energy supplies part of the load. If the motor oscillates around the new point of equilibrium and eventually comes to rest, the application of the load has not caused the motor to lose stability. If the increase in load is too large or is applied too suddenly, the motor may lose synchronism, in which case the *stability limit* is said to have been exceeded. If the reactance of the transformer or of a transmission line between the motor and the power source is increased, the likelihood of maintaining stability is decreased, as we shall see later.

Lines having increased impedance and lower cost became practical with the advent of voltage regulators, but the increased reactance presented power system engineers with a more acute problem of stability. The rapid development of power systems after World War I was interrupted only temporarily by the depression of the early thirties, and as the individual power systems grew so did the interconnections between them. The interchange of power between different power companies and the long distance transmission of power now possible are a tribute to the ability of engineers to solve the stability problem in spite of the high reactance which is inherent in long lines between sources of power and the load. In the fall of 1941, when preparations for World War II were putting an increased demand on many power systems, an acute water shortage threatened to reduce the power available from hydro installations and curtail important war industries in Alabama, Georgia, and Tennessee. Because of the progress that had been made in the design of systems for stable operation, power was sent into the area of critical shortage from Texas, Florida, Virginia, the Carolinas, Pittsburgh, Pennsylvania and Chicago, Illinois. The continued development of the hydraulic resources of the United States with the resulting increase in distance from points of generation to load centers will increase the importance of stability studies in the years ahead.

Stability and the stability limit are defined in "American Standard Definitions of Electrical Terms," published by the American Institute of Electrical Engineers,[1] as follows:

"Stability, when used with reference to a power system, is that attribute of the system, or part of the system, which enables it to develop restoring forces between the elements thereof, equal to or greater than the disturbing forces so as to restore a state of equilibrium between the elements.

"A stability limit is the maximum power flow possible through some

[1] "American Standard Definitions of Electrical Terms," 35.20.200 and 35.20.203, American Institute of Electrical Engineers, New York, 1942.

particular point in the system when the entire system or the part of the system to which the stability limit refers is operating with stability."

The terms stability and stability limit are applied to both steady-state and transient conditions. *Steady-state stability limit* refers to the maximum flow of power possible through a particular point without the loss of stability when the power is increased very gradually. *Transient stability limit* refers to the maximum flow of power possible through a point without the loss of stability when a sudden disturbance occurs. The transient disturbance may be a sudden increment of load which could be carried with stability if it were applied gradually but which causes the loss of stability because of the rapidity of its application. More often the disturbance for which it is desired to know the transient stability limit is caused by a fault, or by switching one of several parallel lines out of the circuit, or by a combination of a fault and its subsequent isolation by disconnecting part of the system. The transient stability limit differs from the steady-state stability limit because the former depends on the nature and severity of the disturbance and is always below the steady-state stability limit. Because every system is subject to transient disturbances and because the transient stability limit is always lower than the steady-state stability limit, transient stability is the more important of the two and will be given more treatment in the discussion to follow.

A very simple power system consists of a generator or motor connected to an infinite bus. Almost as simple is a system containing only two synchronous machines. Since the machines located at any one point in a power system usually act together, it is customary in stability studies to consider all the machines at one point as one large machine. Often machines which are not actually connected to the same bus but which are not separated by lines of high reactance may be lumped together and considered as one machine. When studying the performance of one machine connected to a large system, the system may be considered to have constant voltage and constant frequency. Such a system is called an infinite bus. Thus, a multimachine system sometimes can be reduced to the equivalent of a two-machine system. The factors affecting the stability of a two-machine system, or the stability of one machine connected to an infinite bus, are the same as those which influence a multimachine system. The analysis of a two-machine system,[2] and of a system of one machine and an infinite bus, is so much less complicated than is the analysis of a multimachine system that stability problems are best

[2] A system consisting of two finite machines may be treated as an equivalent finite machine and an infinite bus if certain modifications are made. See E. W. Kimbark, "Power System Stability," Vol. I, "Elements of Stability Calculations," pp. [illegible], John Wiley & Sons, Inc., New York, 1948.

understood by studying one machine connected to an infinite bus. Multimachine stability problems are beyond the scope of this book.

15.2 Steady-state Stability. For balanced conditions the equivalent circuit of a two-machine system is a four-terminal network, as discussed in Chap. 6. The sending-end voltage for steady-state conditions is the voltage behind synchronous reactance of the generator, and the receiving-end voltage is the voltage behind synchronous reactance of the motor. Equations (7.6) and (7.8) were developed from the circle diagrams of a four-terminal network to give the power at the receiving end and sending end of the network. The same equations apply to the two-machine system and give power developed by the generator and the motor if the voltages behind synchronous reactance of the machines replace V_S and V_R and if the generalized circuit constants include the network formed by the synchronous impedances of the machines and the circuit connecting them. The equations become

$$\text{Motor:}\quad P_m = \frac{|E_g| \cdot |E_m|}{|B|}\cos(\beta - \delta) - \frac{|A| \cdot |E_m|^2}{|B|}\cos(\beta - \alpha) \tag{15.1}$$

$$\text{Generator:}\quad P_g = -\frac{|E_g| \cdot |E_m|}{|B|}\cos(\beta + \delta) + \frac{|D| \cdot |E_g|^2}{|B|}\cos(\beta - \Delta) \tag{15.2}$$

Similarly, from Eqs. (7.7) and (7.9) the maximum power developed by the motor and generator can be found from the equations below:

$$P_{m,\max} = \frac{|E_g| \cdot |E_m|}{|B|} - \frac{|A| \cdot |E_m|^2}{|B|}\cos(\beta - \alpha) \tag{15.3}$$

$$P_{g,\max} = \frac{|E_g| \cdot |E_m|}{|B|} + \frac{|D| \cdot |E_g|^2}{|B|}\cos(\beta - \Delta) \tag{15.4}$$

The power given by the preceding equations is power per phase if the voltages are line-to-neutral voltages. If the voltages in the equations are three-phase line-to-line voltages, the power is total three-phase power. As explained in Sec. 7.4, if the circuit contains any resistance, the maximum power output of the generator cannot be realized, for the maximum power input to the motor will be exceeded before the maximum output of the generator is reached.

The circle diagrams of the power developed by the generator and motor of a two-machine system are shown in Fig. 15.1. The circles are drawn for equal values of E_g and E_m and are similar to the receiving-end and sending-end circles discussed in Chap. 7. The point marked $P_{m,\max}$ is the maximum power that can be developed by the motor. If the torque angle δ is less than β, any additional load placed on the shaft will cause δ to increase. The load may be increased until $\delta = \beta$, for which condition the motor is developing maximum power. If the load on the shaft requires a power greater than that developed when $\delta = \beta$, δ will continue to increase since the motor cannot maintain synchronous speed if its developed power is less than the power output to the shaft. The excess of power required over power developed must come from the stored energy of the rotating system as it slows down. The resulting increase in δ above the value of β causes a smaller developed power, and the motor slows down still further, causing larger values of δ and still lower developed power. The motor will lose synchronism completely.

The point marked $P_{g,\max}$ in Fig. 15.1 is the theoretical maximum power that can be developed by the generator, but it need not be considered in the two-machine system since the motor loses synchronism when $\delta = \beta$ and before the generator is developing its maximum power. The difference between motor and generator developed power at any torque angle is the power loss in the connecting network.

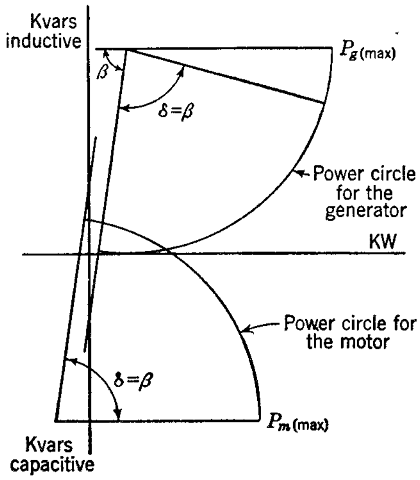


FIG. 15.1 Power circle diagrams for the generator and motor of a two-machine system.

If resistance is neglected, the positive-sequence impedance diagram for a two-machine system is that shown in Fig. 15.2, where X includes the per-unit synchronous reactances of the generator and motor and the reactances of the connecting circuit. Since resistance and shunt admittance are neglected, the generalized circuit constants of the network are

$$A = 1\underline{/0^\circ} \qquad B = |X|\underline{/90^\circ}$$
$$C = 0 \qquad D = 1\underline{/0^\circ}$$

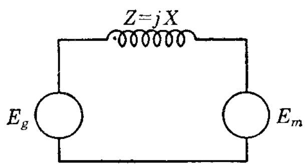


FIG. 15.2 Positive-sequence impedance diagram of a two-machine system.

When the above generalized circuit constants are substituted in Eqs. (15.1) and (15.2), power transferred between the machines is given by

$$P = \frac{|E_g| \cdot |E_m|}{|X|} \sin \delta \qquad (15.5)$$

Similarly, from Eqs. (15.3) and (15.4) the maximum possible power transferred is

$$P_{\max} = \frac{|E_g| \cdot |E_m|}{|X|} \qquad (15.6)$$

Since resistance is neglected, there are no I^2R losses, and all the electric power developed by the generator is received by the motor.

Neglecting resistance and shunt capacitance results in a higher calculated value for the steady-state stability limit for the motor than actually exists, as may be seen by comparing Eqs. (15.3) and (15.6). The constant B in Eq. (15.3) is the series impedance of the circuit, and, if resistance is included, B is slightly larger than the term X in Eq. (15.6). The angle β, which is the impedance angle, is less than 90° if resistance is considered. Both these factors make the power calculated by including resistance smaller than that calculated when resistance is neglected; that is, omission of R gives an optimistic result. When shunt capacitance is included, the transmission line between the sending and receiving ends may be represented by a nominal or an equivalent π. For a symmetrical π, Eqs. (6.20) give for the generalized circuit constants

$$A = 1 + \frac{ZY}{2} \qquad \text{and} \qquad B = Z$$

When Y is zero, the constant A is $1.0\underline{/0^\circ}$, but when both resistance and shunt capacitance are included, A is less than 1.0, and α is a small positive angle. Decreasing A and increasing α have opposite effects on the maximum power. Usually, neglecting shunt capacitance gives a pessimistic result for the stability limit. In stability calculations the same judgment must be used with regard to the inclusion of resistance and shunt admittance in the analysis as is used when making any other calculations. Often the degree of accuracy obtained by making a more exact calculation does not justify the additional complications involved. In the case of transient stability, resistance is important in damping oscillations, and its neglect gives a pessimistic result.

The methods of increasing the steady-state stability limits of a system are suggested by Eq. (15.6). An increase in the excitation of the generator or motor, or both, increases the maximum power that can be transferred between the machines. If the internal voltages are increased without an increase in the power transferred, the torque angle δ decreases, as may be seen from Eq. (15.5). Any reduction in the reactance of the network increases the stability limit. If the transmission lines contribute a significant amount to the total reactance of the system, an [illegible] the stability limit can be obtained by using two pa[illegible] lines. The installation of parallel lines will also increase the dependability of the system since one line will still carry power while a fault is on the other line. Series capacitors have been used in lines to improve voltage regulation, and more such installations are constantly being built. By decreasing the line reactance they raise the stability limit.

15.3 Transient Stability—Review of Mechanics. The analysis of any power system to determine its transient stability involves some mechanical properties of the machines of the system, for after every disturbance the machines must adjust the relative angles of their rotors to meet the conditions of power transfer imposed. The problem is mechanical as well as electrical, and certain mechanical principles must be kept clearly in mind when considering the problem. Table 15.1 gives the quantities which appear in problems concerned with the mechanics of linear motion, or translation. The corresponding quantities for the mechanics of rotation are also given in the table. The relations for rotational systems apply to the solutions of transient-stability problems and may be visualized by comparison with the more familiar relations for translational systems.

TABLE 15.1 COMPARISON OF QUANTITIES USED IN THE MECHANICS OF TRANSLATION AND ROTATION

Translation				Rotation			
Quantity	Symbol	Equation	Mks unit	Quantity	Symbol	Equation	Mks unit
Length	s	—	meter	Angular displacement	θ	$\theta = \frac{s}{r}$	radian
Mass	m	—	kilogram	Moment of inertia	I	$I = \int r^2\,dm$	kilogram-meter²
Time	t	—	second	Time	t	—	second
Velocity	v	$v = \frac{ds}{dt}$	meter/sec	Angular velocity	ω	$\omega = \frac{d\theta}{dt}$	rad/sec
Acceleration	a	$a = \frac{dv}{dt}$	meter/sec²	Angular acceleration	α	$\alpha = \frac{d\omega}{dt}$	rad/sec²
Force	F	$F = ma$	newton	Torque	T	$T = Fr = I\alpha$	newton-meter or joule/rad
Momentum	M'	$M' = mv$	newton-sec	Angular momentum	M	$M = I\omega$	joule-sec per radian
Work	W	$W = \int F\,ds$	joule	Work	W	$W = \int T\,d\theta$	joule
Power	P	$P = \frac{dW}{dt}$	watt	Power	P	$P = \frac{dW}{dt} = T\omega$	watt

The kinetic energy of a rotating body is

$$\text{K.E.} = \tfrac{1}{2} I\omega^2 \qquad \text{joules} \tag{15.7}$$

which is analogous to the kinetic energy of translation, $\frac{1}{2}mv^2$. Since ω is in radians per second, Eq. (15.7) shows that moment of inertia may be expressed in units of joule-seconds squared per radian squared, from which the unit of joule-second per radian is derived for angular momentum. The stored energy of an electric machine is more conveniently expressed in megajoules, and in engineering work angles are more often measured in degrees. Accordingly, angular momentum M is often measured in megajoule-seconds per electrical degree. When M is computed from $I\omega$ with ω determined by the synchronous speed of the machine, M is called the inertia constant. This practice leads to some confusion, since another term denoted by the letter H is also called the inertia constant. The inertia constant H is defined as the megajoules of stored energy of a machine at synchronous speed per megavolt-ampere of the machine

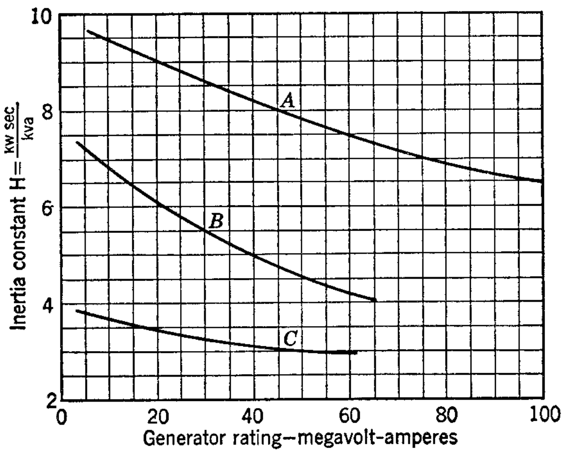


Fig. 15.3 Inertia constants of large steam turbogenerators, including the turbine. A, 1,800 rpm condensing; B, 3,600 rpm condensing; C, 3,600 rpm noncondensing. (Republished from *Elec. Eng.*, vol. 56, p. 268, February, 1937.)

rating. When so defined, the relation between M and H is derived as follows:

Let

$$H = \frac{\text{stored energy in megajoules}}{\text{machine rating in megavolt-amperes}}$$

and

$$G = \text{rating of machine in megavolt-amperes}$$

Then

$$GH = \text{megajoules of stored energy}$$

From Eq. (15.7),

$$\text{Stored energy} = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}M\omega \qquad (\text{since } M = I\omega) \qquad (15.8)$$

If M is in megajoule-seconds per electrical degree and ω is in electrical degrees per second, the stored energy will be given by Eq. (15.8) in megajoules. In electrical degrees per second, $\omega = 360f$, where f is frequency in cycles per second. Thus, Eq. (15.8) becomes

$$GH = \tfrac{1}{2}M(360f)$$

and

$$M = \frac{GH}{180f} \quad \text{megajoule-sec/electrical degree} \qquad (15.9)$$

As we shall see later, M must be determined in order to study transient stability, but M depends on the size of a machine as well as on its type, whereas H does not vary widely with size.

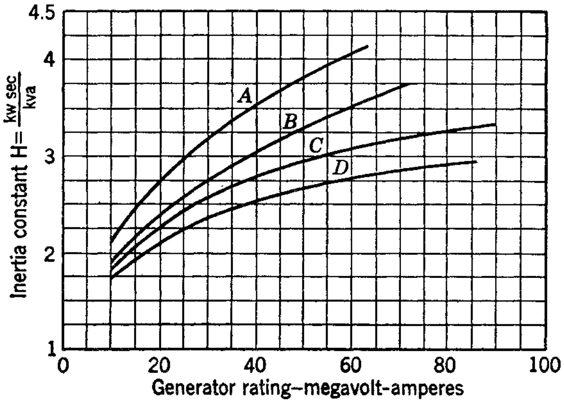


Fig. 15.4 Inertia constants of large vertical-type water-wheel generators, including allowance of 15% for water wheels. *A*, 450–514 rpm; *B*, 200–400 rpm; *C*, 138–180 rpm; *D*, 80–120 rpm. (Republished from *Elec. Eng.*, vol. 56, p. 268, February, 1937.)

The quantity H has a relatively narrow range of values for each class of machine regardless of the kva and speed rating of the machine. In 1937 an AIEE Subcommittee on Interconnection and Stability Factors[3] presented data on average values of H, which are given in the table below and in Figs. 15.3 and 15.4.

Table 15.2. Inertia Constants of Synchronous Machines*

Type	Inertia constant H, megajoules/megavolt-amp
Turbogenerators	See Fig. 15.3
Waterwheel generators	See Fig. 15.4
Synchronous condensers:†	
Large	1.25
Small	1.00
Synchronous motors	2.00

* Republished from *Elec. Eng.*, p. 266, February, 1937.
† Hydrogen cooled, 25% less.

If the WR^2 of the machine is known,[4] including the p... for a

[3] AIEE Subcommittee on Interconnection and Stability F... Report of Power System Stability," *Elec. Eng.*, vol. 56, pp. 261–282, F... 1937.

[4] The term WR^2 is equal to the weight of the rotating parts ... chine (including the prime mover or load) multiplied by the square of t... in feet. WR^2 is moment of inertia in pound-feet squared. $WR^2/$... in slug-feet squared.

generator and the connected load for a motor, H may be determined as follows:

From Eq. (15.7), using English units, we obtain

$$\text{K.E.} = \frac{1}{2}\frac{WR^2}{32.2}\left[\frac{2\pi(\text{rpm})}{60}\right]^2 \quad \text{ft-lb} \tag{15.10}$$

Converting foot-pounds to megajoules and dividing by the machine rating in megavolt-amperes, we obtain

$$H = \frac{\dfrac{746}{550} \times 10^{-6}\dfrac{1}{2}\dfrac{WR^2}{32.2}\left[\dfrac{2\pi(\text{rpm})}{60}\right]^2}{\text{mva rating}} \tag{15.11}$$

$$H = \frac{2.31 \times 10^{-10}WR^2(\text{rpm})^2}{\text{mva rating}} \tag{15.12}$$

When several machines at one location are considered as one machine, the single equivalent machine has a rating equal to the sum of the ratings of the several machines considered to act together during the transient period. The inertia constant M of the equivalent machine is the sum of the inertia constants M of the individual machines.

15.4 The Swing Equation. If the torque caused by friction, windage, and core loss in a machine is disregarded, any difference between the shaft torque and the electromagnetic torque developed must cause acceleration or deceleration of the machine. If T_s represents shaft torque and T_e is electromagnetic torque and if these values of torque are considered positive for a generator (that is, with mechanical input on the shaft and electric output torque developed), the torque causing acceleration is

$$T_a = T_s - T_e \tag{15.13}$$

and T_a is positive denoting acceleration when T_s is greater than T_e. When the same equation is used for a motor with T_s and T_e both negative to denote electric input and mechanical output, T_a is positive and indicates acceleration when T_e is greater than T_s. A similar equation holds for accelerating power, namely,

$$P_a = P_s - P_e \tag{15.14}$$

where P_s is the shaft power and P_e is the electric power developed for a generator. For a motor, P_e is the negative of the difference between the electric power input and the electric losses of the motor, that is, P_e is the negative of the electric power developed. If rotational losses (friction, windage, and core losses including eddy-current losses in the damper winding) are considered, P_s is the negative of the shaft power output plus rotational losses for a motor, and P_s is the shaft power input minus rotational losses for a generator.

Since power is equal to torque times angular velocity,

$$P_a = T_a\omega = I\alpha\omega = M\alpha \tag{15.15}$$

The accelerating power P_a is in megawatts if M is in megajoule-seconds per electrical degree and the angular acceleration α is in electrical degrees per second squared. The acceleration α expressed in terms of the angular position θ of the rotor is

$$\alpha = \frac{d^2\theta}{dt^2} \tag{15.16}$$

Since θ is continually changing with time, it is more convenient to measure angular position with respect to a reference axis which is rotating at synchronous speed. If δ is the angular displacement in electrical degrees from the synchronously rotating reference axis and ω_s is the synchronous speed in electrical degrees per second,

$$\theta = \omega_s t + \delta \tag{15.17}$$

Taking the derivative with respect to t, we obtain

$$\frac{d\theta}{dt} = \omega_s + \frac{d\delta}{dt} \tag{15.18}$$

and taking the derivative again

$$\frac{d^2\theta}{dt^2} = \frac{d^2\delta}{dt^2} \tag{15.19}$$

From Eqs. (15.15), (15.16), and (15.19), we obtain

$$M\frac{d^2\delta}{dt^2} = P_a = P_s - P_e \tag{15.20}$$

Equation (15.20) is called the *swing equation.* The angle δ for a machine connected to an infinite bus is the torque angle as used in Eqs. (15.1), (15.2), and (15.5) since this angle is the difference between the internal angle of the machine and the angle of the synchronously rotating reference frame, which in this case is the infinite bus. For a two-machine system two swing equations are necessary, one for each machine. The torque angle between the two machines depends on the angles between each machine and the synchronously rotating reference frame.

The angular momentum M of a machine is not constant since the angular velocity is changing, but M may be treated as a constant since the speed of the machine does not differ much from the synchronous speed unless the stability limit is exceeded. The inertia constant (also designated as M) is truly constant because it is defined as angular momentum at synchronous speed. Shaft power P_s is considered constant in the solution of the equation. For a generator this assumption is justified, even though the input from the prime mover may be controlled by governor action, since governors do not act until the speed changes by at least 1% and response to their action is not instantaneous in any case. For a motor the load remains constant since the speed does not change appreciably unless stability is lost. Electric power P_e is given by Eqs. (15.1), (15.2), or (15.5). Transient reactance is used to determine the generalized circuit constants in Eqs. (15.1) and (15.2) and to determine X for Eq. (15.5), where resistance is neglected. Transient reactance is the best value to use because the rotor of the machine is constantly changing position with respect to the mmf of the armature current so that flux changes over the rotor face in a manner similar to that of the changing flux when transient reactance is evaluated. E_g and E_m are voltages behind the transient reactance of the generator and of the motor. From Eq. (15.5), the swing equation becomes

$$M \frac{d^2\delta}{dt^2} = P_s - \frac{|E_g| \cdot |E_m|}{|X|} \sin \delta \tag{15.21}$$

or from Eq. (15.6),

$$M \frac{d^2\delta}{dt^2} = P_s - P_{\max} \sin \delta \tag{15.22}$$

For a multimachine system involving several swing equations, no formal solution of the equation is attempted, and a point-by-point solution, usually with the help of a calculating board, must be made. Even for the simple case of one machine and an infinite bus with resistance neglected, which involves the solution of Eq. (15.22) alone, the formal solution requires the use of elliptic integrals. The solution gives values of δ for different times, and a graph of δ versus t is usually plotted. Such a graph is called the *swing curve*. If the swing curve indicates that the angle δ starts to decrease after reaching a maximum value, it is usually assumed that the system will not lose stability and that the oscillations of δ around the equilibrium point will become successively smaller and eventually be damped out.

15.5 Equal-area Criterion of Stability. In a system where one machine is swinging with respect to an infinite bus, it is not necessary to plot and inspect the swing curves to determine whether the torque angle of the machine increases indefinitely or oscillates around an equilibrium position. Solution of the swing equation, with the usual assumptions of constant P_s, a purely reactive network, and constant voltage behind transient reactance, shows that δ oscillates around the equilibrium point with constant amplitude if the transient stability limit is not exceeded. The principle by which stability under transient conditions is determined without solving the swing equation is called the *equal-area criterion* of stability. It may be applied to any two-machine system, but it is especially simple for one machine and an infinite bus, for which the derivation follows.

The swing equation for the machine connected to the bus is

$$M \frac{d^2\delta}{dt^2} = P_s - P_e \tag{15.23}$$

Multiplying both sides of the equation by $\frac{d\delta}{dt}$, we have

$$M \frac{d^2\delta}{dt^2}\frac{d\delta}{dt} = (P_s - P_e)\frac{d\delta}{dt} \tag{15.24}$$

The left side of Eq. (15.24) may be rewritten to give[5]

$$\frac{1}{2} M \frac{d(d\delta/dt)^2}{dt} = (P_s - P_e)\frac{d\delta}{dt} \tag{15.25}$$

Rearranging, multiplying by dt, and integrating, we obtain

$$\left(\frac{d\delta}{dt}\right)^2 = \int_{\delta_0}^{\delta} \frac{2(P_s - P_e)}{M}\, d\delta \tag{15.26}$$

or

$$\frac{d\delta}{dt} = \sqrt{\int_{\delta_0}^{\delta} \frac{2(P_s - P_e)}{M}\, d\delta} \tag{15.27}$$

where δ_0 is the torque angle when the machine is operated synchronously before the disturbance occurs, at which time $d\delta/dt = 0$. The angle δ will cease to change, and the machine will again be operating at synchronous speed after a disturbance, when $d\delta/dt = 0$ or when

$$\int_{\delta_0}^{\delta} \frac{2(P_s - P_e)}{M}\, d\delta = 0 \tag{15.28}$$

As we shall see later, the machine will not remain at rest with respect to the infinite bus the first time $d\delta/dt$ becomes zero, but the fact that δ has momentarily stopped changing may be taken to indicate stability, which corresponds to the interpretation that the swing curve indicates stability when the angle δ reaches a maximum and starts to decrease.

Some of the conditions caused by the sudden increase in the mechanical load on a synchronous motor connected to an infinite bus can be predicted by analyzing Fig. 15.5. The sinusoidal curve P_e is a plot of the electric power input to the motor with all resistance neglected. The curve P_e is plotted from Eqs. (15.5) and (15.6), where |[illegible] the voltage

[5] The rearrangement of the left side of Eq. (15.24) to obtain Eq. (15.25) may be verified by substituting dx/dt for x in the formula

$$\frac{dx^2}{dt} = 2x\frac{dx}{dt}$$

of the infinite bus, $|E_m|$ is the voltage behind transient reactance of the motor, and X is determined from the transient reactance of the motor plus the reactance of the transformer and line, if any, between the motor and the infinite bus.

Originally the motor is operating at synchronous speed with a torque angle of δ_0, and the mechanical power output P_0 is equal to the electric power input P_e corresponding to δ_0. When the mechanical load is suddenly increased so that the power output is P_s, which is greater than the electric power input at δ_0, the difference in power must come from the kinetic energy stored in the rotating system. This can be accomplished only by a decrease in speed, which results in an increase in the torque angle δ. As δ increases, the electric power received from the bus increases until $P_e = P_s$ at point b on the curve. At this point there is an equilibrium of input and output torque so that acceleration is zero, but the motor is running at less than synchronous speed so that δ is increasing. The angle δ continues to increase, but after passing through point b the electric power input P_e is greater than P_s, and the difference must be stored in the system through an increase in kinetic energy accompanying an increase in speed. Thus, between points b and c as δ increases, the speed is increasing, until synchronous speed is again reached at point c, where the torque angle is δ_m. At point c, P_e is still greater than P_s, and speed continues to increase, but δ starts to decrease as soon as the speed of the motor exceeds synchronous speed. The maximum value of δ is δ_m, at point c. As δ decreases, point b is reached again with the speed above synchronous speed so that δ continues to decrease until point a is reached. The motor is again operating at synchronous speed, and the cycle is repeated.

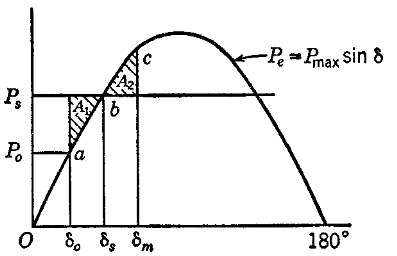


FIG. 15.5 Electric power input to a motor as a function of torque angle δ. The load is suddenly increased from P_0 to P_s, and the motor oscillates around δ_s between δ_0 and δ_m.

The motor oscillates around the equilibrium torque angle δ_s between angles δ_0 and δ_m. If damping is present, the oscillations decrease, and stable operation results at δ_s. Table 15.3 shows the changes in speed, angle, electric power input, mechanical power output, st[illegible] and acceleration or deceleration as the machine oscill[illegible] thorough study of this table will lead to a better understan[illegible]g of transient disturbances.

The changing position of the synchronous motor [illegible]ing with respect to an infinite bus may be visualized by an analo[illegible] Consider a pendulum swinging from a pivot on a stationary frame, [illegible] shown in Fig. 15.6a.

Points a and c are the maximum points of the oscillation of the pendulum about the equilibrium point b. Damping will eventually bring the pendulum to rest at b. Now imagine a disk rotating in a clockwise direction about the pivot of the pendulum, as shown in Fig. 15.6b, and superimpose the motion of the pendulum on the motion of the disk.

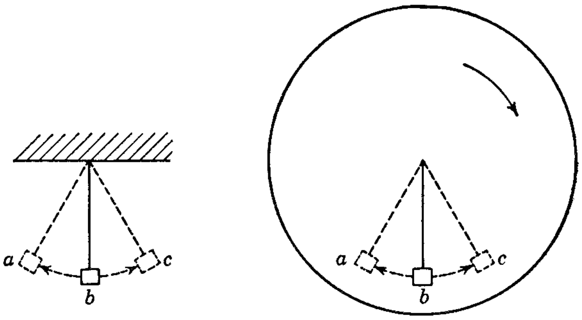
TABLE 15.3 CHANGING CONDITIONS IN A SYNCHRONOUS MOTOR SWINGING WITH RESPECT TO AN INFINITE BUS BECAUSE OF A SUDDEN INCREASE IN LOAD

Position in cycle (see Fig. 15.5)	Motor speed, ω	Torque angle, δ	Electric power, P_e	Stored energy, $\frac{1}{2}I\omega^2 = W$	Rotating system is undergoing—
At point a	$\omega = \omega_s$, decreasing	$\delta = \delta_0$, minimum	$P_e < P_s$, minimum	$W = W_s$,* decreasing	Deceleration
From a toward b	$\omega < \omega_s$, decreasing	increasing	$P_e < P_s$, increasing	$W < W_s$, decreasing	Deceleration
At point b	$\omega < \omega_s$, minimum	$\delta = \delta_s$, increasing	$P_e = P_s$, increasing	$W < W_s$, minimum	
From b toward c	$\omega < \omega_s$, increasing	increasing	$P_e > P_s$, increasing	$W < W_s$, increasing	Acceleration
At point c	$\omega = \omega_s$, increasing	$\delta = \delta_m$, maximum	$P_e > P_s$, maximum	$W = W_s$, increasing	Acceleration
From c toward b	$\omega > \omega_s$, increasing	decreasing	$P_e > P_s$, decreasing	$W > W_s$, increasing	Acceleration
At point b	$\omega > \omega_s$, maximum	$\delta = \delta_s$, decreasing	$P_e = P_s$, decreasing	$W > W_s$, maximum	
From b toward a	$\omega > \omega_s$, decreasing	decreasing	$P_e < P_s$, minimum	$W > W_s$, decreasing	Deceleration
At point a	Cycle starts to repeat as above				

* W_s is the stored energy at synchronous speed; that is, $W_s = \frac{1}{2}I\omega_s^2$.

When the pendulum is moving from a to c, the combined angular velocity is slower than that of the disk. When the pendulum is moving from c to a, the combined angular velocity is faster than that of the disk. At points a and c, the angular velocity of the pendulum alone is zero, and the combined angular velocity equals that of the disk. If the angular velocity of the disk corresponds to the synchronous speed of the motor and if the motion of the pendulum alone represents the swinging of the motor with respect to an infinite bus, the superimposed motion of the pendulum on the motion of the disk represents the actual oscillation of the motor.

The maximum swing of the motor, to a torque angle δ_m, is found by a graphic interpretation of Eq. (15.28). When this equation is satisfied, the maximum value of δ is reached and $d\delta/dt = 0$. The shaded area A_1 in



(a) Pendulum (b) Pendulum on disc

FIG. 15.6 Pendulum and rotating disk to illustrate a motor swinging with respect to an infinite bus.

Fig. 15.5 is

$$A_1 = \int_{\delta_0}^{\delta_s} (P_s - P_e)\, d\delta \tag{15.29}$$

Similarly, the shaded area A_2 is

$$A_2 = \int_{\delta_s}^{\delta_m} (P_e - P_s)\, d\delta \tag{15.30}$$

and

$$A_1 - A_2 = \int_{\delta_0}^{\delta_s} (P_s - P_e)\, d\delta - \int_{\delta_s}^{\delta_m} (P_e - P_s)\, d\delta \tag{15.31}$$

$$A_1 - A_2 = \int_{\delta_0}^{\delta_m} (P_s - P_e)\, d\delta \tag{15.32}$$

Equation (15.28) is satisfied and $d\delta/dt = 0$ when $A_1 = A_2$. The maximum torque angle δ_m is located graphically so as to make A_2 equal to A_1. A study of Table 15.3 shows that the energy lost as the motor decelerates and δ increases to δ_s is regained by the time δ_m is reached.

Figure 15.7 shows a suddenly applied load which is larger than that shown in Fig. 15.5. The area A_2 above P_s under the curve P_e is less than A_1, and $d\delta/dt$ is not zero at $\delta = \delta_x$. Therefore δ continues to increase after $\delta = \delta_x$. P_e again becomes less than P_s. The torque a[illegible]tinues to increase beyond δ_x, and restoring forces are [illegible] encountered. The system is stable only if an area A_2 can be located above P_s equal to A_1. The test of equal areas is called the equal-area criterion. The maximum allowable increase in the power suddenly taken from the motor originally supplying the power P_0 is shown in Fig. 15.8. A suddenly applied load greater than that shown in Fig. 15[illegible] would not permit the torque angle of the motor to stop decreasing before the input power became less than the power required, since the area above P_s would be less than A_1.

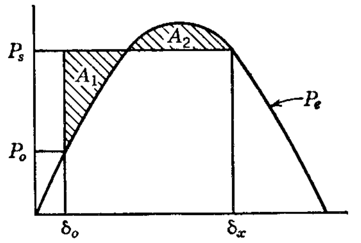


FIG. 15.7 Electric power input to a motor as a function of torque angle for a suddenly increased load such that $A_2 < A_1$.

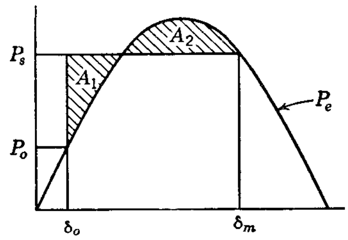


FIG. 15.8 Electric power input to a motor as a function of torque angle for the maximum sudden increase of load without loss of stability.

15.6 Further Applications of the Equal-area Criterion. The equal-area criterion of transient stability can be applied to disturbances other than the sudden increase in load on a motor. We shall discuss some of the more important disturbances.

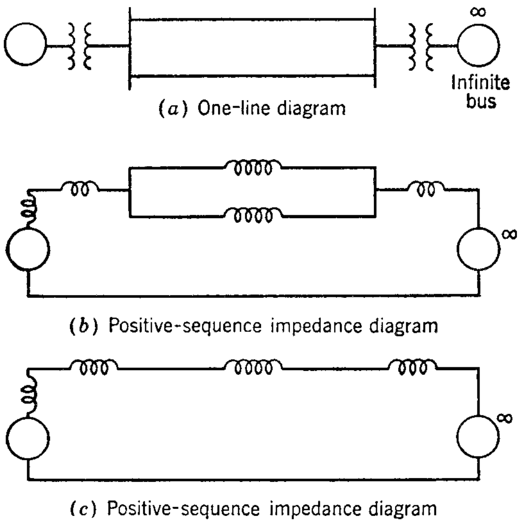


FIG. 15.9 One-line diagram and positive-sequence impedance diagrams of a generator supplying power to an infinite bus through two transformers at opposite ends of two parallel transmission lines.

When a generator is supplying power to an infinite bus over two parallel transmission lines, the opening of one of the lines may cause the generator to lose synchronism even though the load could be supplied over the remaining line under steady-state conditions. The one-line diagram of such a system is shown in Fig. 15.9a. The positive-sequence impedance diagram is shown in Fig. 15.9*b*. When one line is opened, the positive-sequence impedance diagram becomes that of Fig. 15.9*c*. The opening of one line increases the reactance between the generator and the bus. Increased reactance means that the torque angle must increase in order to transfer the same power over the system as was carried before opening the line. The generator is accelerated because the reduced power output resulting from the increased reactance is less than the power input. Acceleration increases the torque angle. Power output curves plotted against torque angle are shown in Fig. 15.10 for the double-circuit and single-circuit line. P_s is the mechanical power input to the generator. When one line is opened, power output drops from the point determined by the δ_0 ordinate and the upper power curve to the power determined by the δ_0 ordinate and the lower power curve. The excess of power input over power output causes the acceleration, which in turn results in an increased torque angle. The torque angle oscillates between δ_0 and δ_m around the equilibrium value δ_s, as determined by the equal-area criterion. As the line P_s is raised, a value of P_s will be found where equal areas A_1 and A_2 are determined when δ_m is at the intersection of P_s and the lower electric power output curve. This value of P_s is the transient stability limit for the switching operation described.

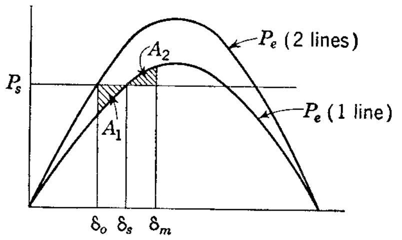


Fig. 15.10 Equal-area criterion applied to the opening of one transmission line in the system of Fig. 15.9.

Short-circuit faults often cause loss of stability, even though they may be removed by isolating the fault from the rest of the system in a relatively short time. A three-phase fault at one end of a double-circuit line is represented in Fig. 15.11. All the current from the generator flows through pure reactances to the fault. Only reactive power flows, and the real power output of the generator is zero. If the fault is sustained and constant input is assumed, δ will increase indefinitely because all the input power must go into acceleration.

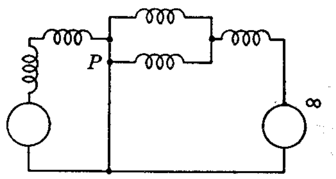


Fig. 15.11 Reactance diagram for a three-phase fault at the end of a double-circuit line connecting a generator to an infinite bus.

When a three-phase fault at one end of a double-circuit line is removed by opening breakers at both ends of the faulted circuit, power is again transmitted from the generator to the infinite bus over the remaining line. The equal-area criterion is applied as shown in Fig. 15.12. The upper curve shows the variation of transmitted power plotted against torque angle before the fault occurs. During the fault no power is transmitted. When the fault is cleared at $\delta = \delta_c$ the power transmitted over the remaining circuit is shown by the lower curve. The maximum angle of swing is fixed by the condition that A_1 and A_2 must be equal. If clearing of the fault takes place later than for the case illustrated in Fig. 15.12 (that is, for a larger δ_c), it may be impossible to make the area A_2 above P_s equal to A_1, and the system will lose stability. Thus, for any P_s there is a *critical clearing angle*, and, unless the fault is cleared before the torque angle equals the critical clearing angle, the machine will lose synchronism. It is evident from Fig. 15.12 that larger values of P_s require quicker clearing of the fault to maintain stable operation.

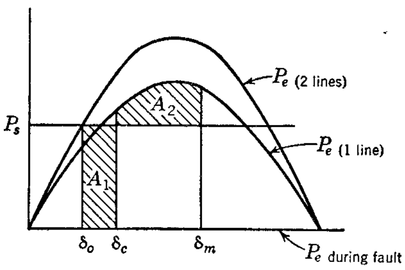
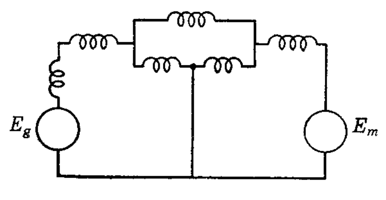
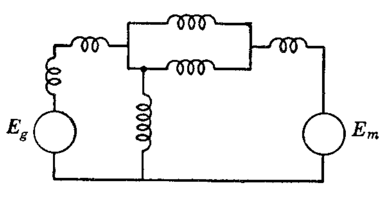


FIG. 15.12 Equal-area criterion applied to a three-phase fault removed from a system by opening one of two parallel transmission lines. No power is transmitted during the fault.

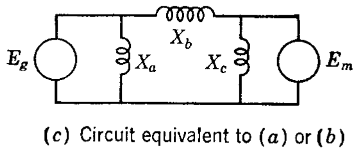
When a three-phase fault occurs at some point on a double-circuit line other than on the paralleling busses or at the extreme ends of the line, there is some impedance between the paralleling busses and the fault. Therefore, some power is transmitted while the fault is still on the system. Regardless of their location, short-circuit faults not involving all three



(*a*) Circuit for three-phase fault at middle of one of two parallel lines



(*b*) Circuit for fault other than a three-phase fault at end of one line



(*c*) Circuit equivalent to (*a*) or (*b*)

FIG. 15.13 Network reduction preceding stability deter[...]n.

phases allow the transmission of some power, because [...] are represented by connecting some impedance between the fault poi[...] and the reference bus in the positive-sequence impedance diagram [...]er than [...] a short circuit as shown for a three-phase fault in Fig. [...]11. The larger the impedance shunted across the positive-sequence network to represent the fault, the larger the power transmitted during the fault.

The power transmitted during a fault may be calculated after reducing the network which represents the fault condition to a Δ-connected circuit between the internal voltage of the generator and the infinite bus. Two circuits before reduction are shown in Figs. 15.13*a* and 15.13*b*. Either may be reduced to the Δ network of Fig. 15.13*c*. The current in the reactance X_a is 90° out of phase with the generator voltage, and the power in this leg is reactive. Real power transmitted is $\frac{|E_g| \cdot |E_m|}{|X_b|} \sin \delta$, where E_g is the voltage behind transient reactance of the generator and E_m is the voltage of the infinite bus.

If power is transmitted during the fault, the equal-area criterion is applied as shown in Fig. 15.14, where $P_{\max} \sin \delta$ is the power transmitted before the fault, $r_1 P_{\max} \sin \delta$ is the power transmitted during the fault, and $r_2 P_{\max} \sin \delta$ is the power transmitted after the fault is cleared by switching at the instant when $\delta = \delta_c$. The terms r_1 and r_2 are ratios of the maximum power that can be transmitted during and after the fault, respectively, to the maximum power that can be transmitted before the fault. For the case illustrated in Fig. 15.14, δ_c is evidently the critical clearing angle, since A_2 becomes equal to A_1 for the torque angle where P_s intersects $r_2 P_{\max} \sin \delta$. The power transmitted during the fault helps to reduce the value of A_1 for any given clearing angle. Thus, smaller values of r_1 result in greater disturbances to the system, for low r_1 means low power transmitted during the fault.

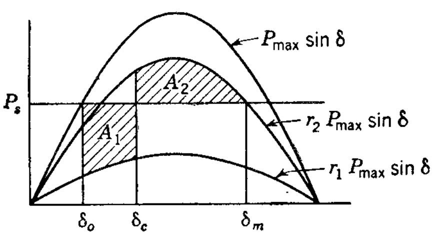


FIG. 15.14 Equal-area criterion applied to fault clearing when power is transmitted during the fault.

In order of increasing severity (decreasing $r_1 P_{\max}$) the various faults are:

1. Single line-to-ground fault
2. Line-to-line fault
3. Double line-to-ground fault
4. Three-phase fault

The single line-to-ground fault occurs most frequently, and the three-phase fault is least frequent. For complete reliability a system should be designed for transient stability for three-phase faults at the worst locations. If this is impracticable from the economic standpoint, reliability may be sacrificed to the extent of designing for transient stability for double line-to-ground faults.

An analysis can be made of Fig. 15.14 to obtain a formula for the critical clearing angle. The shaded areas are

$$A_1 = P_s(\delta_c - \delta_0) - \int_{\delta_0}^{\delta_c} r_1 P_{\max} \sin \delta \, d\delta \tag{15.33}$$

$$A_1 = P_s(\delta_c - \delta_0) + r_1 P_{\max}(\cos \delta_c - \cos \delta_0) \tag{15.34}$$

and

$$A_2 = \int_{\delta_c}^{\delta_m} r_2 P_{\max} \sin \delta \, d\delta - P_s(\delta_m - \delta_c) \tag{15.35}$$

$$A_2 = r_2 P_{\max}(\cos \delta_c - \cos \delta_m) - P_s(\delta_m - \delta_c) \tag{15.36}$$

For stability, $A_1 = A_2$, or

$$P_s\delta_c - P_s\delta_0 + r_1 P_{\max} \cos \delta_c - r_1 P_{\max} \cos \delta_0 = r_2 P_{\max} \cos \delta_c - r_2 P_{\max} \cos \delta_m - P_s\delta_m + P_s\delta_c \tag{15.37}$$

$$(r_1 - r_2) P_{\max} \cos \delta_c = P_s(\delta_0 - \delta_m) + r_1 P_{\max} \cos \delta_0 - r_2 P_{\max} \cos \delta_m \tag{15.38}$$

Solving for δ_c, we obtain

$$\delta_c = \cos^{-1}\left[\frac{(P_s/P_{\max})(\delta_m - \delta_0) + r_2 \cos \delta_m - r_1 \cos \delta_0}{r_2 - r_1}\right] \tag{15.39}$$

To evaluate δ_c, we note that

$$P_s = P_{\max} \sin \delta_0 \tag{15.40}$$

$$P_s = r_2 P_{\max} \sin \delta_m \tag{15.41}$$

$$\delta_0 = \sin^{-1} \frac{P_s}{P_{\max}} \qquad \text{where } \delta_0 < 90° \tag{15.42}$$

$$\delta_m = \sin^{-1} \frac{P_s}{r_2 P_{\max}} \qquad \text{where } \delta_m > 90° \tag{15.43}$$

The reader may derive special formulas for the critical clearing angle with zero transmitted power during the fault, or for the maximum value of P_s with a sustained fault.

Example 15.1

The single-line diagram of Fig. 15.15 shows a generator connected through parallel high-voltage transmission lines to a large metropolitan system considered as an infinite bus. Numbers on the diagram indicate the values of the reactances in per unit. The transient reactance of the

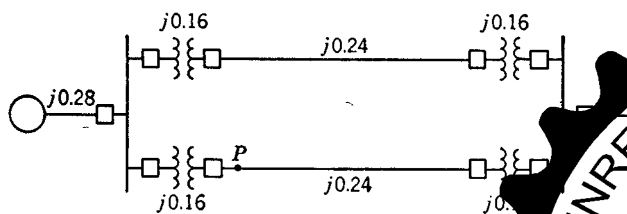


Fig. 15.15 One-line diagram for Example 15.1.

generator is included in the values marked. Breakers adjacent to a fault on both sides are arranged to clear simultaneously. Specify in electrical degrees the critical clearing angle for the generator for a three-phase

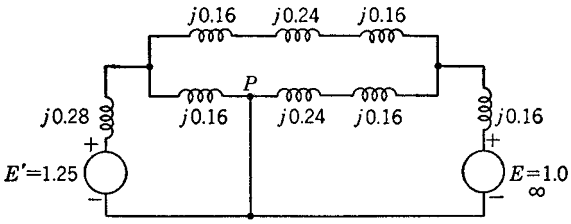


FIG. 15.16 Positive-sequence impedance diagram for the system of Fig. 15.15.

fault at the point P when the generator is delivering 1.0 per-unit power. Assume that the voltage behind transient reactance is 1.25 per unit for the generator and that the voltage at the infinite bus is 1.0 per unit.

Solution

The positive-sequence impedance diagram is shown in Fig. 15.16. Before the fault the impedance between the generator and the infinite bus is

$$X = 0.28 + 0.16 + \frac{0.16 + 0.24 + 0.16}{2} = 0.72$$

After the fault is cleared by opening the circuit breakers at both ends of the faulted line, the impedance between the generator and the infinite bus is

$$X = 0.28 + 0.16 + 0.16 + 0.24 + 0.16 = 1.00$$

During the fault the circuit is represented by the network shown in Fig. 15.17*a*. Reduction of this circuit to a Δ is accomplished by two Y-Δ transformations as shown in Figs. 15.17*b* and 15.17*c*. The calculations are as follows:

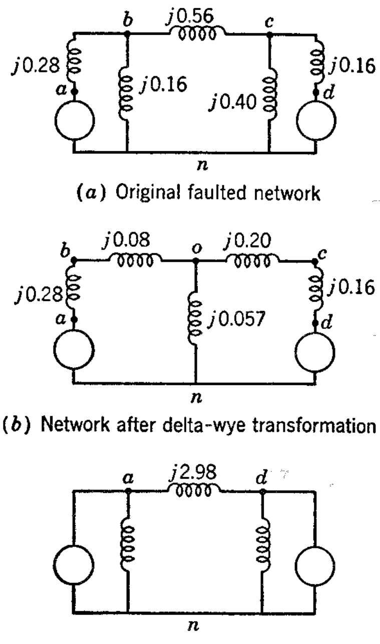


FIG. 15.17 Steps in circuit reduction for Example 15.1.

$$X_{bo} = \frac{0.56 \times 0.16}{0.56 + 0.16 + 0.40} = \frac{0.0895}{1.12} = 0.080$$

$$X_{co} = \frac{0.40 \times 0.56}{1.12} = \frac{0.224}{1.12} = 0.200$$

$$X_{no} = \frac{0.16 \times 0.40}{1.12} = \frac{0.064}{1.12} = 0.057$$
$$X_{ao} = 0.28 + 0.08 = 0.36$$
$$X_{do} = 0.20 + 0.16 = 0.36$$
$$X_{ad} = \frac{0.36 \times 0.057 + 0.36 \times 0.36 + 0.36 \times 0.057}{0.057} = 2.98$$

It is not necessary to calculate X_{an} and X_{dn} since these purely reactive shunts across the voltages of the generator and the infinite bus can absorb no real power.

Equations for the power output of the generator are

Before the fault: $P_{\max} \sin \delta = \dfrac{1.0 \times 1.25}{0.72} \sin \delta = 1.735 \sin \delta$

During the fault: $r_1 P_{\max} \sin \delta = \dfrac{1.0 \times 1.25}{2.98} \sin \delta = 0.42 \sin \delta$

After the fault: $r_2 P_{\max} \sin \delta = \dfrac{1.0 \times 1.25}{1.00} \sin \delta = 1.25 \sin \delta$

Hence,

$$r_1 = \frac{0.42}{1.735} = 0.242$$
$$r_2 = \frac{1.25}{1.735} = 0.72$$

From Eqs. (15.42) and (15.43), we obtain

$$\delta_0 = \sin^{-1} \frac{1.0}{1.735} = 35.2°, \text{ or } 0.615 \text{ radian}$$
$$\delta_m = \sin^{-1} \frac{1.0}{1.25} = 126.9°, \text{ or } 2.22 \text{ radians}$$

Substituting in Eq. (15.39) gives

$$\delta_c = \cos^{-1} \left[\frac{(1.0/1.735)(2.22 - 0.615) + 0.72 \cos 126.9° - 0.242 \cos 35.2°}{0.72 - 0.242} \right]$$
$$= \cos^{-1} \frac{0.925 - 0.432 - 0.197}{0.478} = \cos^{-1} 0.62 = 51.6°$$

15.7 Point-by-point Solution of the Swing Curve. The equal-area criterion of stability is useful in determining whether or not a machine will remain stable for a sustained fault and in determining the angle through which a machine may be permitted to swing before a fault is cleared. It does not determine directly the length of time permitted before clearing a fault if stability is to be maintained. Standard operating times for circuit breakers and their associated relays are 8, 5, or 3 cycles after a fault occurs. Constant progress in the design of high-speed breakers is enabling them to operate in circuits of lower and lower voltage. In order to specify a breaker of the proper speed the engineer must know the *critical clearing time*, which is the time for a machine to swing from its original position to its critical clearing angle. The critical clearing time should be calculated for a fault in the position which will allow the least transfer of power from the machine and for the most severe type of fault for which protection against loss of stability is justified. If the critical clearing angle has been determined by the equal-area criterion, a method must be found to relate the angular position of the rotor to time. As was previously pointed out, the solution of the swing equation yields values from which a curve of angular position versus time, called the swing curve, may be plotted. From the swing curve the time from the beginning of a disturbance until the rotor reaches any angular position may be read. A swing curve for the data of Example 15.1 could be used to find the critical clearing time, corresponding to the critical clearing angle determined by the equal-area criterion.

Generally, the only feasible method of solving the swing curve is by making a point-by-point solution. In a point-by-point solution the change in the angular position of the rotor during a short interval of time is computed by assuming the variables in the swing equation to be constant or to vary linearly over the interval. New values of the variables are calculated at the end of each interval. The solution progresses through enough intervals to obtain points for plotting the swing curve. The accuracy of the resulting curve depends on the nature of the assumptions and the length of the intervals. Greater accuracy is obtained when small intervals are used. An interval of 0.05 sec is usually satisfactory.

Several methods of point-by-point solution have been proposed.[6] The most widely used method is based on the following assumptions:

1. The accelerating power P_a computed at the beginning of an interval is constant from the middle of the preceding interval to the middle of the interval considered.

2. The angular velocity is constant throughout any interval at the value computed for the middle of the interval. Of course, neither of the assumptions is true, since δ is changing continuously and both P_a and ω are functions of δ. As the time interval is decreased the computed swing curve approaches the true curve.

Figure 15.18 will help in visualizing the assumptions. The accelerating power is computed for the points enclosed in circles at the ends of the $n - 2$, $n - 1$, and n intervals, which are the beginnings of the $n - 1$, n, and $n + 1$ intervals. The step curve of P_a in Fig. 15.18 results from the assumption that P_a is constant between mid-points of the intervals.

[6] O. G. C. Dahl, "Electric Power Circuits," Vol. II, "Power System Stability," pp. 391–401, McGraw-Hill Book Company, Inc., New York, 1938.

Similarly, ω', the *excess* of the angular velocity ω over the synchronous angular velocity ω_s, is shown as a step curve which is constant throughout the interval at the value computed for the mid-point. Between the ordinates $n - 3/2$ and $n - 1/2$ there is a change of speed caused by the constant accelerating power. The change in speed is the product of the acceleration and the time interval, so

$$\omega'_{n-1/2} - \omega'_{n-3/2} = \frac{d^2\delta}{dt^2}\,\Delta t = \frac{P_{a(n-1)}}{M}\,\Delta t \tag{15.44}$$

The change in δ over any interval is the product of ω' for the interval and the time of the interval. Thus, the change in δ during the $n - 1$ interval is

$$\Delta\delta_{n-1} = \delta_{n-1} - \delta_{n-2} = \Delta t\,\omega'_{n-3/2} \tag{15.45}$$

and during the nth interval

$$\Delta\delta_n = \delta_n - \delta_{n-1} = \Delta t\,\omega'_{n-1/2} \tag{15.46}$$

Subtracting Eq. (15.45) from Eq. (15.46) and substituting Eq. (15.44) in the resulting equation to eliminate all values of ω' yields

$$\Delta\delta_n = \Delta\delta_{n-1} + \frac{P_{a(n-1)}}{M}\,(\Delta t)^2 \tag{15.47}$$

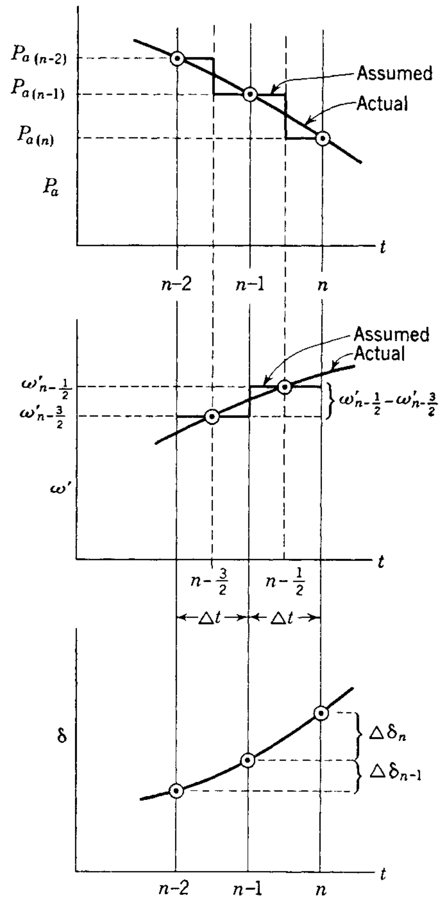


FIG. 15.18 Actual and assumed values of P_a, ω', and δ as functions of time.

Equation (15.47) is the important one for the point-by-point solution of the swing equation, for it shows how to calculate the change in δ during an interval if the change in δ for the previous interval and the accelerating power for the interval in question are known. Equation (15.47) shows that, subject to the stated assumptions, the change in torque angle during a given interval is equal to the change in torque angle during the preceding interval plus the accelerating power at the beginning of the interval times $(\Delta t)^2/M$.

The occurrence of a fault causes a discontinuity in the accelerating power P_a which is zero before the fault and a definite amount immediately following the fault. The discontinuity occurs at the beginning of the interval, when $t = 0$. Reference to Fig. 15.18 shows that our method of calculation assumes the accelerating power computed at the beginning of an interval to be constant from the middle of the preceding interval to the middle of the interval considered. When the fault occurs, we have two values of P_a at the beginning of an interval, and we must take the average of these two values as our constant accelerating power. The procedure is illustrated in the following example.

Example 15.2

Specify in cycles the critical clearing time for the breakers at the ends of the high-tension lines of Example 15.1 for the fault condition described. Also plot swing curves for clearing the fault at two values of t less than the critical clearing time. For the generator, assume $H = 3.0$, and carry out the calculations in per unit.

Solution

The inertia constant is

$$M = \frac{GH}{180f} = \frac{1.0 \times 3.0}{180 \times 60} = 2.78 \times 10^{-4} \text{ per unit}$$

For the time interval $\Delta t = 0.05$ sec,

$$\frac{(\Delta t)^2}{M} = \frac{25 \times 10^{-4}}{2.78 \times 10^{-4}} = 9.0$$

From Example 15.1,

When the fault occurs: $\delta = 35.2°$
During the fault: $P_e = 0.42 \sin \delta$

Therefore

$$P_a = P_s - P_e = 1.0 - 0.42 \sin \delta$$

At the beginning of the first interval there is a discontinuity in the accelerating power. Just before the fault occurs, $P_a = 0$, and just after the fault occurs

$$P_a = 1.0 - 0.42 \sin 35.2° = 1.0 - 0.242 = 0.75\ldots \text{ unit}$$

The average value of P_a is $\frac{1}{2} \times 0.758 = 0.379$ pe[illegible]

$$\frac{(\Delta t)^2}{M} P_a = 9 \times 0.379 = 3\ldots$$

$$\Delta\delta_n = 0 + 3.41 = 3.4\ldots$$

when $t = 0.05$ sec,

$$\delta_n = 35.2° + 3.41° = 38.61°$$

$$P_a = 1.0 - 0.42 \sin 38.61° = 1.0 - 0.262 = 0.738$$

$$\frac{(\Delta t)^2}{M} P_a = 9 \times 0.738 = 6.64$$

$$\Delta\delta_n = 3.41° + 6.64° = 10.05°, \text{ or } 10.1°$$

When $t = 0.10$ sec,

$$\delta_n = 38.6° + 10.1° = 48.7°$$

The computations are shown in Table 15.4.

TABLE 15.4 COMPUTATIONS OF SWING CURVE FOR SUSTAINED FAULT

t, sec	P_e	P_a	$\frac{(\Delta t)^2}{M} P_a$	$\Delta\delta_n$, degrees	δ_n, degrees
0−	1.0	0.00	....	.....	35.2
0+	0.242	0.758	....	.....	35.2
0 av	.....	0.379	3.41	3.41	35.2
0.05	0.262	0.738	6.64	10.05	38.6
0.10	0.315	0.685	6.17	16.22	48.7
0.15	0.380	0.620	5.58	21.80	64.9
0.20	0.419	0.581	5.23	27.03	86.7
0.25	.....	.....	....	.....	113.7

In Example 15.1 the critical clearing angle was found to be 51.6°. The critical clearing time may be estimated from the values shown in Table 15.4. The swing curve is plotted in Fig. 15.19, and the critical clearing time, corresponding to the critical clearing angle, may be read from the curve. The critical clearing time is 0.11 sec, or 6.6 cycles. A 5-cycle breaker would be satisfactory for the application. An 8-cycle breaker would not isolate the fault quickly enough, and the machine would lose synchronism. The swing curves for clearing the fault in 3 cycles and 4.5 cycles are also shown in Fig. 15.19. The data for the latter curves are shown in Tables 15.5 and 15.6.

At the instant the fault is cleared a discontinuity occurs in the accelerating power P_a. When clearing is at 3 cycles, the discontinuity occurs at 0.05 sec, which is at the beginning of an interval. Two values of accelerating power result from the two expressions (one during the fault and one after clearing) for the power output of the generator at the beginning of the interval. Since the discontinuity occurs at the beginning of an interval, the average of the two values is assumed as the constant value of P_a from 0.025 to 0.075 sec. The procedure is the same as that followed upon the occurrence of the fault.

When clearing is at 4.5 cycles, the discontinuity is at 0.075 sec, which is at the middle of an interval. No special procedure is required, because we assume a discontinuity in accelerating power at the middle of an interval. The assumed constant value of P_a at the beginning of the interval during which the fault is cleared is determined by the electric power input during the fault for the value of δ at the beginning of the

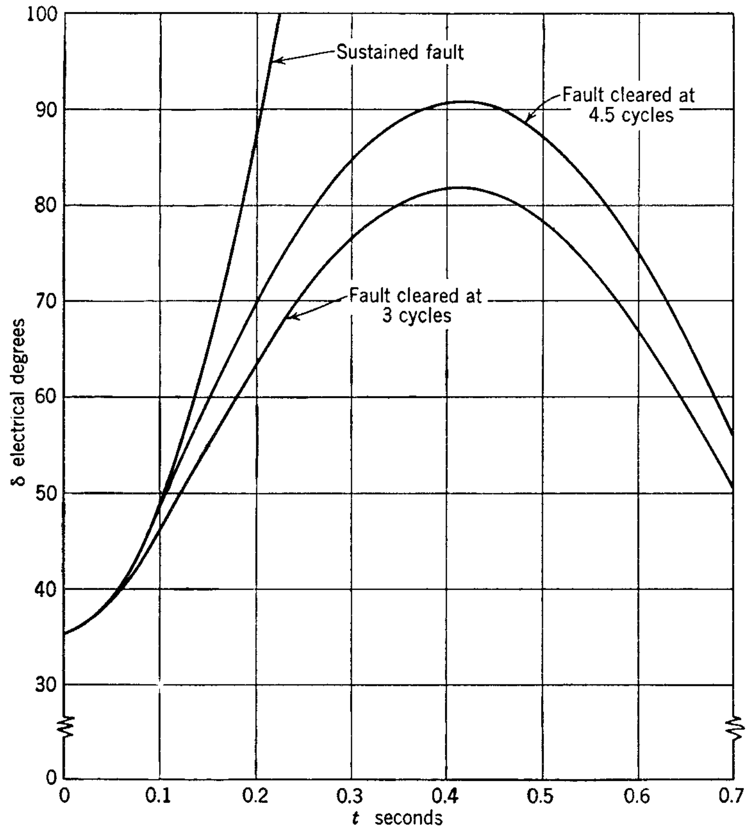


Fig. 15.19 Swing curves for Example 15.2 for a sustained fault and for clearing in 3.0 and 4.5 cycles.

interval. At the beginning of the interval following clearing, the assumed constant value of P_a is that computed from the electric power input after clearing for the value of δ at the beginning of the interval following clearing. Careful study of Table 15.6 will clarify the procedure.

Rather than actually plotting the swing curve of a sustained fault on a system consisting of one machine and an infinite bus, we may use precalculated swing curves.[7] The data must be put in a modified,

[7] I. H. Summers and J. B. McClure, "Progress in the Study of System Stability," *Trans. AIEE*, vol. 49, pp. 132–158, January, 1930.

TABLE 15.5 COMPUTATIONS OF SWING CURVE FOR FAULT CLEARED AT t = 0.05 SEC

t, sec	P_e	P_a	$\frac{(\Delta t)^2}{M} P_a$	$\Delta\delta_n$, degrees	δ_n, degrees
0−	1.0	0.00	......	......	35.2
0+	0.242	0.758	......	......	35.2
0 av	.....	0.379	3.41	3.41	35.2
0.05−	0.262	0.738	......	......	38.6
0.05+	0.780	0.220	......	......	38.6
0.05 av	.....	0.479	4.31	7.72	38.6
0.10	0.905	0.095	0.86	8.58	46.3
0.15	1.02	−0.02	−0.20	8.38	54.9
0.20	1.12	−0.12	−1.08	7.30	63.3
0.25	1.18	−0.18	−1.62	5.68	70.6
0.30	1.22	−0.22	−1.98	3.70	76.3
0.35	1.23	−0.23	−2.07	1.63	80.0
0.40	1.24	−0.24	−2.16	−0.53	81.6
0.45	1.24	−0.24	−2.16	−2.69	81.1
0.50	1.23	−0.23	−2.07	−4.76	78.4
0.55	1.20	−0.20	−1.80	−6.56	73.6
0.60	1.15	−0.15	−1.35	−7.91	67.0
0.65	1.07	−0.07	−0.63	−8.54	59.1
0.70	.....	....	......	......	50.6

Note: During the fault, $P_e = 0.42 \sin \delta$.
After the fault, $P_e = 1.25 \sin \delta$.

TABLE 15.6 COMPUTATIONS OF SWING CURVE FOR FAULT CLEARED AT t = 0.075 SEC

t, sec	P_e	P_a	$\frac{(\Delta t)^2}{M} P_a$	$\Delta\delta_n$, degrees	δ_n, degrees
0−	1.0	0.00	......	....	35.2
0+	0.242	0.758	......	....	35.2
0 av	.....	0.379	3.41	3.41	35.2
0.05	0.262	0.738	6.64	10.05	38.6
0.10	0.094	0.06	0.54	10.59	48.7
0.15	1.08	−0.08	−0.72	9.87	59.3
0.20	1.17	−0.17	−1.53	8.34	69.2
0.25	1.22	−0.22	−1.98	6.36	78.2
0.30	1.24	−0.24	−2.16	4.20	84.5
0.35	1.25	−0.25	−2.25	1.95	88.7
0.40	1.25	−0.25	−2.25	− 0.30	90.7
0.45	1.25	−0.25	−2.25	− 2.55	90.4
0.50	1.25	−0.25	−2.25	− 4.80	[illegible]
0.55	1.24	−0.24	−2.16	− 6.96	[illegible]
0.60	1.21	−0.21	−1.89	[illegible]	75.3
0.65	1.15	−0.15	−1.35	[illegible]	[illegible]
0.70	.....	....	......	......	56.2

Note: During the fault, $P_e = 0.42 \sin \delta$.
After the fault, $P_e = 1.25 \sin \delta$.

dimensionless form. The appropriate swing curve is then selected from a group of curves found in the article referred to in footnote 7. A method is also available for determining the critical clearing time for one machine connected to an infinite bus.[8] Either of these methods may be used for a system consisting of two finite machines.

15.8 Use of the Calculating Board to Determine Swing Curves. The computation of swing curves for all the machines in a multimachine system is best accomplished with the aid of a calculating board. The same point-by-point method is still followed, but the calculating board greatly simplifies the work. The developed power (and hence the accelerating power) of each machine is dependent on the angular position of every machine in the system. If a calculating board is not used, a very great amount of network reduction and tedious calculations are necessary to determine the accelerating power of each machine at the beginning of each time interval.

When a calculating board is used the positive-sequence network is set up, and the fault is represented by the appropriate impedance. Each machine should be represented by its transient reactance and its voltage behind transient reactance adjusted to the proper phase angle and magnitude to give the power flow and power factor existing before the fault occurs. The fault is then applied by closing a switch, and the electric power P_e of each machine is read. The accelerating power is the difference between the electric power developed and the constant value assumed for P_s. The change in angular position with respect to the synchronous position is then calculated for each machine over the selected interval by Eq. (15.47). The generators representing each machine are then adjusted to agree with the rotor position determined. Electric power P_e for each machine is read again, and new angular positions are computed. The point-by-point analysis is continued in this manner until enough points on the curves have been found to indicate which machines, if any, are unstable.

15.9 Some Factors Affecting Transient Stability. Aside from the type of fault and its location, which are beyond the control of the system designer, there are certain other factors which affect transient stability and which may be altered in order to raise the transient stability limit of the system. Inspection of Eq. (15.47) indicates that an increase in the inertia constant M of a machine reduces the angle through which it swings during any time interval and thus allows a longer time for breaker operation to isolate the fault before the machine passes through its critical clearing angle. An increase in M offers a means of increasing stability, but it has not been used extensively for economic reasons.

[8] H. L. Byrd and S. R. Pritchard, Jr., "Solution of the Two-machine Stability Problem," *Gen. Elec. Rev.*, vol. 36, pp. 81–93, February 1933.

The methods frequently used to increase stability are:

1. Increase in the system voltage.
2. Reduction of series reactance by parallel lines.
3. Use of high-speed circuit breakers, including reclosing breakers.

As shown by Eq. (15.6), P_{max} is increased by an increase in the internal voltage of a machine or in the voltage of the infinite bus to which the machine is connected through a reactance. For a given shaft power the initial torque angle δ_0 is decreased by an increase in P_{max}, as shown by Eq. (15.42). Examination of Fig. 15.14 shows that all three power curves are raised when P_{max} is increased, and the results are a lower δ_0, an increased δ_m, and a greater difference between δ_0 and δ_c. Therefore, increasing P_{max} allows a machine to swing through a larger angle from its original position before it reaches the critical clearing angle. Thus, raising P_{max} increases the critical clearing time and the probability of maintaining stability.

Reducing the reactance of a transmission line has the same effect as raising P_{max}. Compensation for line reactance by series capacitors is economical for increasing the stability of lines more than 200 miles long. Increasing the number of parallel lines between two points is a common means of reducing reactance. When parallel transmission lines are used instead of a single line, some power is transferred over the remaining line even during a three-phase fault on one of the lines unless the fault occurs at a paralleling bus. For other types of faults on one line, more power is transferred during the fault if there are two lines in parallel than is transferred over a single faulted line. For more than two lines in parallel the power transferred during the fault is even greater. Power transferred is subtracted from power input to obtain accelerating power. Thus increased power transferred during a fault means lower accelerating power for the machine and increased chance of stability.

Obviously the quicker a fault is isolated from a system the less disturbance it causes. It has been pointed out that there is a critical clearing time before which circuit breakers must operate to clear the fault if stability is to be maintained. The use of high-speed circuit breakers on power systems has greatly improved their stability and at the same time has reduced the need for making other changes in design to effect stable operation. It is still important, however, for the system designer and operator to understand the reasons for the loss of stability and the means for its prevention.

PROBLEMS

15.1 The generalized circuit constants of a nominal-π circuit representing a three-phase transmission line are

$$A = D = 0.980\underline{/0.3°}$$
$$B = 82.5\underline{/76.0°} \text{ ohms}$$
$$C = 0.0005\underline{/90°} \text{ mho}$$

Find the steady-state stability limit of the line if $|V_S|$ and $|V_R|$ are held constant at 110 kv. What is the steady-state stability limit if the shunt admittance is assumed to be zero? What is the steady-state stability limit if the shunt admittance is assumed to be zero and the series resistance is neglected?

15.2 A 60-cycle, four-pole turbogenerator rated 20,000 kva, 13.2 kv has an inertia constant of $H = 9.0$ kw-sec/kva. Find the kinetic energy stored in the rotor at synchronous speed. Find the acceleration if the input less the rotational losses is 26,800 hp and the electric power developed is 16,000 kw.

15.3 If the acceleration computed for the generator described in Prob. 15.2 is constant for a period of 15 cycles, find the change in torque angle in that period and the rpm at the end of 15 cycles. Assume that the generator is synchronized with a large system and has no accelerating torque before the 15-cycle period begins.

15.4 The generator of Prob. 15.2 is delivering rated kva at 0.8 power factor lag when a fault reduces the electric power output by 50%. Determine the accelerating torque at the time the fault occurs. Neglect losses and assume constant power input to the shaft.

15.5 A motor is receiving 25% of the power which it is capable of receiving from an infinite bus. If the load on the motor is doubled, calculate the maximum value of δ during the swinging of the motor around its new equilibrium position.

15.6 A 60-cycle generator is delivering 50% of the power which it is capable of delivering through a transmission line to an infinite bus. A fault occurs which increases the reactance between the generator and the infinite bus to 400% of the value before the fault. When the fault is isolated the maximum power which can be delivered is 75% of the original maximum value. Determine the critical clearing angle for the condition described.

15.7 If the generator of Prob. 15.6 has an inertia constant of $H = 5.0$ megajoules/megavolt-amp, find the critical clearing time for the condition described. Use $\Delta t = 0.05$ sec.

15.8 A 60-cycle generator with an inertia constant of $H = 5.0$ megajoules/megavolt-amp is connected through a step-up transformer to a transmission line. At the other end of the line is a step-down transformer which connects the line to a large system which may be treated as an infinite bus. Reduced to a common base the per-unit reactances of the generator are $X' = 0.3$, $X_2 = 0.15$, and $X_0 = 0.05$, those of the transformers are $X_1 = X_2 = X_0 = 0.10$, and those of the transmission line are $X_1 = X_2 = 0.25$ and $X_0 = 0.70$. The transformers are connected in Δ on the low-tension side and in Y, with a solidly grounded neutral, on the high-tension side. A single line-to-ground fault occurs on the high-tension side of the transformer connected to the generator when the generator is delivering power of 1.0 per unit. The voltage behind transient reactance of the generator is 1.30 per unit, and the voltage of the infinite bus is 1.0 per unit. The fault is cleared by the simultaneous action of single-pole breakers on both sides of the fault. The breakers [illegible] the faulted phase open 0.15 sec after the fault occurs and rec[illegible] cycles after opening. The fault does not recur after reclosure. Plot the swing curve for the generator.

APPENDIX

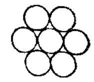


TABLE A.1 CHARACTERISTICS OF COPPER CONDUCTORS, HARD-DRAWN, 97.3% CONDUCTIVITY*

Size of conductor: Circular mils	Size of conductor: A.W.G. or B. & S.	Number of strands	Diameter of individual strands, in.	Outside diameter, in.	Breaking strength, lb	Weight, lb/mile	Approx. current-carrying capacity,† amp	Geometric mean radius at 60 cycles, ft	Resistance, ohms/conductor/mile: 25°C (77°F) d-c	25°C (77°F) 25 cycles	25°C (77°F) 50 cycles	25°C (77°F) 60 cycles	50°C (122°F) d-c	50°C (122°F) 25 cycles	50°C (122°F) 50 cycles	50°C (122°F) 60 cycles	Inductive reactance, ohms/conductor/mile at 1-ft spacing: 25 cycles	50 cycles	60 cycles	Shunt capacitive reactance, megohms/conductor/mile at 1-ft spacing: 25 cycles	50 cycles	60 cycles
1,000,000	...	37	0.1644	1.151	43,830	16,300	1,300	0.0368	0.0585	0.0594	0.0620	0 0634	0.0640	0 0648	0 0672	0.0685	0.1666	0.333	0.400	0.216	0.1081	0.0901
900,000	...	37	.1560	1.092	39,510	14,670	1,220	.0349	.0650	.0658	.0682	.0695	.0711	.0718	.0740	.0752	.1693	.339	.406	.220	.1100	.0916
800,000	...	37	.1470	1.029	35,120	13,040	1,130	.0329	.0731	.0739	.0760	.0772	.0800	.0806	.0826	.0837	.1722	.344	.413	.224	.1121	.0934
750,000	...	37	.1424	0.997	33,400	12,230	1,090	.0319	.0780	.0787	.0807	.0818	.0853	.0859	.0878	.0888	.1739	.348	.417	.226	.1132	.0943
700,000	...	37	.1375	.963	31,170	11,410	1,040	.0308	.0836	.0842	.0861	.0871	.0914	.0920	.0937	.0947	.1759	.352	.422	.229	.1145	.0954
600,000	...	37	.1273	.891	27,020	9,781	940	.0285	.0975	.0981	.0997	.1006	.1066	.1071	.1086	.1095	.1799	.360	.432	.235	.1173	.0977
500,000	...	37	.1162	.814	22,510	8,151	840	.0260	.1170	.1175	.1188	.1196	.1280	.1283	.1296	.1303	.1845	.369	.443	.241	.1205	.1004
500,000	...	19	.1622	.811	21,590	8,151	840	.0256	.1170	.1175	.1188	.1196	.1280	.1283	.1296	.1303	.1853	.371	.445	.241	.1206	.1005
450,000	...	19	.1539	.770	19,750	7,336	780	.0243	.1300	.1304	.1316	.1323	.1422	.1426	.1437	.1443	.1879	.376	.451	.245	.1224	.1020
400,000	...	19	.1451	.726	17,560	6,521	730	.0229	.1462	.1466	.1477	.1484	.1600	.1603	.1613	.1619	.1909	.382	.458	.249	.1245	.1038
350.000	...	19	.1357	.679	15,590	5,706	670	.0214	.1671	.1675	.1684	.1690	.1828	.1831	.1840	.1845	.1943	.389	.466	.254	.1269	.1058
350,000	...	12	.1708	.710	15,140	5,706	670	.0225	.1671	.1675	.1684	.1690	.1828	.1831	.1840	.1845	.1918	.384	.460	.251	.1253	.1044
300,000	...	19	.1257	.629	13,510	4,891	610	.01987	.1950	.1953	.1961	.1966	.213	.214	.214	.215	.1982	.396	.476	.259	.1296	.1080
300,000	...	12	.1581	.657	13,170	4,891	610	.0208	.1950	.1953	.1961	.1966	.213	.214	.214	.215	.1957	.392	.470	.256	.1281	.1068
250,000	...	19	.1147	.574	11,360	4,076	540	.01813	.234	.234	.235	.235	.256	.256	.257	.257	.203	.406	.487	.266	.1329	.1108
250,000	...	12	.1443	.600	11,130	4,076	540	.01902	.234	.234	.235	.235	.256	.256	.257	.257	.200	.401	.481	.263	.1313	.1094
211,600	4/0	19	.1055	.528	9,617	3,450	480	.01668	.276	.277	.277	.278	.302	.303	.303	.303	.207	.414	.497	.272	.1359	.1132
211,600	4/0	12	.1328	.552	9,483	3,450	490	.01750	.276	.277	.277	.278	.302	.303	.303	.303	.205	.409	.491	.269	.1343	.1119
211,600	4/0	7	.1739	.522	9,154	3,450	480	.01579	.276	.277	.277	.278	.302	.303	.303	.303	.210	.420	.503	.273	.1363	.1136
167,800	3/0	12	.1183	.492	7,556	2,736	420	.01559	.349	.349	.349	.350	.381	.381	.382	.382	.210	.421	.505	.277	.1384	.1153

Table A.1 continued on page 354

TABLE A.1 CHARACTERISTICS OF COPPER CONDUCTORS, HARD-DRAWN, 97.3% CONDUCTIVITY.* (*Continued*)

Size of conductor		Number of strands	Diameter of individual strands, in.	Outside diameter, in.	Breaking strength, lb	Weight, lb/mile	Approx. current-carrying capacity,† amp	Geometric mean radius at 60 cycles, ft	Resistance, ohms/conductor/mile								Inductive reactance, ohms/conductor/mile at 1-ft spacing			Shunt capacitive reactance, megohms/conductor/mile at 1-ft spacing		
									25°C (77°F)				50°C (122°F)									
Circular mils	A.W.G. or B. & S.								d-c	25 cycles	50 cycles	60 cycles	d-c	25 cycles	50 cycles	60 cycles	25 cycles	50 cycles	60 cycles	25 cycles	50 cycles	60 cycles
167,800	3/0	7	.1548	.464	7,366	2,736	420	.01404	.349	.349	.349	.350	.381	.381	.382	.382	.216	.431	.518	.281	.1405	.1171
133,100	2/0	7	.1379	.414	5,926	2,170	360	.01252	.440	.440	.440	.440	.481	.481	.481	.481	.222	.443	.532	.289	.1445	.1205
105,500	1/0	7	.1228	.368	4,752	1,720	310	.01113	.555	.555	.555	.555	.606	.607	.607	.607	.227	.455	.546	.298	.1488	.1240
83,690	1	7	.1093	.328	3,804	1,364	270	.00992	.699	.699	.699	.699	.765				.233	.467	.560	.306	.1528	.1274
83,690	1	3	.1670	.360	3,620	1,351	270	.01016	.692	692	.692	.692	.757				.232	.464	.557	.299	.1495	.1246
66,370	2	7	.0974	.292	3,045	1,082	230	.00883	.881	.882	.882	.882	.964				.239	.478	.574	.314	.1570	.1308
66,370	2	3	.1487	.320	2,913	1,071	240	.00903	.873				.955				238	.476	.571	.307	.1537	.1281
66,370	2	1	......	.258	3,003	1,061	220	.00836	.864				.945				.242	.484	.581	.323	.1614	.1345
52,630	3	7	.0867	.260	2,433	858	200	.00787	1 112				1.216				.245	.490	.588	.322	.1611	.1343
52,630	3	3	.1325	.285	2,359	850	200	.00805	1 101				1.204				.244	.488	.585	.316	.1578	.1315
52,630	3	1	.....	.229	2,439	841	190	.00745	1.090		Same as d-c		1.192		Same as d-c		.248	.496	.595	.331	.1656	.1380
41,740	4	3	.1180	.254	1,879	674	180	.00717	1.388				1.518				.250	.499	.599	.324	.1619	.1349
41,740	4	1	.....	.204	1,970	667	170	.00663	1.374				1.503				.254	.507	.609	.339	.1697	.1415
33,100	5	3	.1050	.226	1,505	534	150	.00638	1.750				1.914				.256	.511	.613	.332	.1661	.1384
33,100	5	1	.....	.1819	1.591	529	140	.00590	1.733				1.895				.260	.519	.623	.348	.1738	.1449
26,250	6	3	.0935	.201	1,205	424	130	.00568	2.21				2.41				.262	.523	.628	.341	.1703	.1419
26,250	6	1	.....	.1620	1,280	420	120	.00526	2.18				2.39				.265	.531	.637	.356	.1779	.1483
20,820	7	1	.....	.1443	1,030	333	110	.00468	2.75				3.01				.271	.542	.651	.364	.1821	.1517
16,510	8	1	.....	.1285	.826	264	90	.00417	3.47				3.80				.277	.554	.665	.372	.1862	.1552

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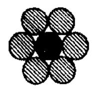
† For conductor at 75°C, air at 25°C, wind 1.4 miles/hour (2 ft/sec), frequency = 60 cycles.

Table A.2 Characteristics of Aluminum Cable, Steel-reinforced*

(Aluminum Company of America)



Multilayer Conductors

Circular mils or A.W.G. aluminum	Aluminum: Strands	Aluminum: Layers	Aluminum: Strand diameter, in.	Steel: Strands	Steel: Strand diameter, in.	Outside diameter, in.	Copper equivalent,† circular mils or A.W.G.	Ultimate strength, lb	Weight, lb/mile	Geometric mean radius at 60 cycles, ft	Approx. current-carrying capacity,§ amp	Resistance, ohms/conductor/mile, 25°C (77°F) Small currents: d-c	25 cycles	50 cycles	60 cycles	Resistance, 50°C (122°F) Current approx. 75% capacity‡: d-c	25 cycles	50 cycles	60 cycles	Inductive reactance, ohms/conductor/mile at 1-ft spacing all currents: 25 cycles	50 cycles	60 cycles	Shunt capacitive reactance, megohms/conductor/mile at 1-ft spacing: 25 cycles	50 cycles	60 cycles
1,590,000	54	3	0.1716	19	0.1030	1.545	1,000,000	56,000	10,777	0.0520	1,380	0.0587	0.0588	0.0590	0.0591	0.0646	0.0656	0.0675	0.0684	0.1495	0.299	0.359	0.1953	0.0977	0.0814
1,510,500	54	3	.1673	19	.1004	1.506	950,000	53,200	10,237	.0507	1,340	.0618	.0619	.0621	.0622	.0680	.0690	.0710	.0720	.1508	.302	.362	.1971	.0986	.0821
1,431,000	54	3	.1628	19	.0977	1 465	900,000	50,400	9,699	.0493	1,300	.0652	.0653	.0655	.0656	.0718	.0729	.0749	.0760	.1522	.304	.365	.1991	.0996	.0830
1,351,000	54	3	.1582	19	.0949	1.424	850,000	47,600	9,160	.0479	1,250	.0691	.0692	.0694	.0695	.0761	.0771	.0792	.0803	.1536	.307	.369	.201	.1006	.0838
1,272,000	54	3	.1535	19	.0921	1.382	800,000	44,800	8,621	.0465	1,200	.0734	.0735	.0737	.0738	.0808	.0819	.0840	.0851	.1551	.310	.372	.203	.1016	.0847
1,192,500	54	3	.1486	19	.0892	1.338	750,000	43,100	8,082	.0450	1,160	.0783	.0784	.0786	.0788	.0862	.0872	.0894	.0906	.1568	.314	.376	.206	.1028	.0857
1,113,000	54	3	.1436	19	.0862	1.293	700,000	40,200	7,544	0435	1,110	.0839	.0840	.0842	.0844	.0924	.0935	.0957	.0969	.1585	.317	.380	.208	.1040	.0867
1,033,500	54	3	.1384	7	.1384	1.246	650,000	37,100	7,019	0420	1,060	.0903	.0905	.0907	.0909	.0994	.1005	.1025	.1035	.1603	.321	.385	.211	.1053	.0878
954,000	54	3	.1329	7	.1329	1 196	600,000	34,200	6,479	.0403	1,010	.0979	.0980	.0981	.0982	.1078	.1088	.1118	.1128	.1624	.325	.390	.214	.1068	.0890
900,000	54	3	.1291	7	.1291	1.162	566,000	32,300	6,112	.0391	970	.104	.104	.104	.104	.1145	.1155	.1175	.1185	.1639	.328	.393	.216	.1078	.0898
874,500	54	3	.1273	7	.1273	1.146	550,000	31,400	5,940	.0386	950	.107	.107	.107	.108	.1178	.1188	.1218	.1228	.1646	.329	.395	.217	.1083	.0903
795,000	54	3	.1214	7	.1214	1.093	500,000	28,500	5,399	.0368	900	.117	.118	.118	.119	.1288	.1308	.1358	.1378	.1670	.334	.401	.220	.1100	.0917
795,000	26	2	.1749	7	.1360	1.108	500,000	31,200	5,770	.0375	900	.117	.117	.117	.117	.1288	.1288	.1288	.1288	.1660	.332	.399	.219	.1095	.0912
795,000	30	3	.1628	19	.0977	1.140	500,000	38,400	6,517	.0393	910	.117	.117	.117	.117	.1288	.1288	.1288	.1288	.1637	.327	.393	.217	.1085	.0904
715,500	54	3	.1151	7	.1151	1.036	450,000	26,300	4,859	0349	830	.131	.131	.131	.132	.1442	.1452	.1472	.1482	.1697	.339	.407	.224	.1119	.0932
715,500	26	2	.1659	7	.1290	1.051	450,000	28,100	5,193	.0355	840	.131	.131	.131	.131	.1442	.1442	.1442	.1442	.1687	.337	.405	.223	.1114	.0928
715,500	30	2	.1544	19	.0926	1.081	450,000	34,600	5,865	.0372	840	.131	.131	.131	.131	.1442	.1442	.1442	.1442	.1664	.333	.399	.221	.1104	.0920
666,600	54	3	.1111	7	.1111	1.000	419,000	24,500	4,527	.0337	800	.140	.140	.141	.141	.1541	.1571	.1591	.1601	.1715	.343	.412	.226	.1132	.0943
636,000	54	3	.1085	7	.1085	0.977	400,000	23,600	4,319	.0329	770	.147	.147	.148	.148	.1618	.1638	.1678	.1688	.1726	.345	.414	.228	.1140	.0950
636,000	26	2	.1564	7	.1216	.990	400,000	25,000	4,616	.0335	780	.147	.147	.147	.147	.1618	.1618	.1618	.1618	.1718	.344	.412	.227	.1135	.0946
636,000	30	2	.1456	19	.0874	1.019	400,000	31,500	5,213	.0351	780	.147	.147	.147	.147	.1618	.1618	.1618	.1618	.1693	.339	.406	.225	.1125	.0937
605,000	54	3	.1059	7	.1059	0.953	380,500	22,500	4,109	.0321	750	.154	.155	.155	.155	.1695	.1715	.1755	.1775	.1739	.348	.417	.230	.1149	.0957
605,000	26	2	.1525	7	.1186	.966	380,500	24,100	4,391	.0327	760	.154	.154	.154	.154	.1700	.1720	.1720	.1720	.1730	.346	.415	.229	.1144	.0953
556,500	26	2	.1463	7	.1138	.927	350,000	22,400	4,039	.0313	730	.168	.168	.168	.168	.1849	.1859	.1859	.1859	.1751	.350	.420	.232	.1159	.0965

Table A.2 continued on page 356

TABLE A.2 CHARACTERISTICS OF ALUMINUM CABLE, STEEL-REINFORCED.* (*Continued*)

Circular mils or A.W.G. aluminum	Aluminum: Strands	Aluminum: Layers	Aluminum: Strand diameter, in.	Steel: Strands	Steel: Strand diameter, in.	Outside diameter, in.	Copper equivalent,† circular mils or A.W.G.	Ultimate strength, lb	Weight, lb/mile	Geometric mean radius at 60 cycles, ft	Approx. current-carrying capacity,§ amp	Resistance, ohms/conductor/mile: 25°C (77°F) Small currents: d-c	25°C: 25 cycles	25°C: 50 cycles	25°C: 60 cycles	50°C (122°F) Current approx. 75% capacity‡: d-c	50°C: 25 cycles	50°C: 50 cycles	50°C: 60 cycles	Inductive reactance, ohms/conductor/mile at 1-ft spacing all currents: 25 cycles	50 cycles	60 cycles	Shunt capacitive reactance, megohms/conductor/mile at 1-ft spacing: 25 cycles	50 cycles	60 cycles
556,500	30	2	.1362	7	.1362	.953	350,000	27,200	4,588	.0328	730	.168	.168	.168	.168	.1849	.1859	.1859	.1859	.1728	.346	.415	.230	.1149	.0957
500,000	30	2	.1291	7	.1291	.904	314,500	24,400	4,122	.0311	690	.187	.187	.187	.187	.206				.1754	.351	.421	.234	.1167	.0973
477,000	26	2	.1355	7	.1054	.858	300,000	19,430	3,462	.0290	670	.196	.196	.196	.196	.216				.1790	.358	.430	.237	.1186	.0988
477,000	30	2	.1261	7	.1261	.883	300,000	23,300	3,933	.0304	670	.196	.196	.196	.196	.216				.1766	.353	.424	.235	.1176	.0980
397,500	26	2	.1236	7	.0961	.783	250,000	16,190	2,885	.0265	590	.235				.259				.1836	.367	.441	.244	.1219	.1015
397,500	30	2	.1151	7	.1151	.806	250,000	19,980	3,277	.0278	600	.235		Same as d-c		.259		Same as d-c		.1812	.362	.435	.242	.1208	.1006
336,400	26	2	.1138	7	.0885	.721	4/0	14,050	2,442	.0244	530	.278				.306				.1872	.376	.451	.250	.1248	.1039
336,400	30	2	.1059	7	.1059	.741	4/0	17,040	2,774	.0255	530	.278				.306				.1855	.371	.445	.248	.1238	.1032
300,000	26	2	.1074	7	.0835	.680	188,700	12,650	2,178	.0230	490	.311				.342				.1908	.382	.458	.254	.1269	.1057
300,000	30	2	.1000	7	.1000	.700	188,700	15,430	2,473	.0241	500	311				.342				.1883	377	.452	.252	.1258	1049
266,800	26	2	.1013	7	.0788	.642	3/0	11,250	1,936	.0217	460	.350				.385				.1936	.387	.465	.258	.1289	.1074

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† Based on copper 97%, aluminum 61% conductivity.

‡ "Current approx. 75% capacity" is 75% of the "Approx. current-carrying capacity, amp" shown in Column 9 and is approximately the current which will produce 50°C conductor temp. (25°C rise) with 25°C air temp., wind 1.4 miles/hour.

§ For conductor at 75°C, air at 25°C, wind 1.4 miles/hour (2 ft/sec), frequency = 60 cycles.

TABLE A.2 CHARACTERISTICS OF ALUMINUM CABLE, STEEL-REINFORCED.* (*Continued*)

Single-Layer Conductors

Circular mils or A.W.G. aluminum	Aluminum: Strands	Aluminum: Layers	Aluminum: Strand diameter, in.	Steel: Strands	Steel: Strand diameter, in.	Outside diameter, in.	Copper equivalent,† circular mils or A.W.G.	Ultimate strength, lb.	Weight, lb/mile	Geometric mean radius at 60 cycles, feet, for current approx. 75% capacity‡	Approx. current-carrying capacity,§ amp	Resistance ohms/conductor/mile, 25°C (77°F) Small currents: d-c	25 cycles	50 cycles	60 cycles	Resistance, 50°C (122°F) Current approx. 75% capacity‡: d-c	25 cycles	50 cycles	60 cycles	Inductive reactance, ohms/conductor/mile at 1-ft spacing, Small currents: 25 cycles	50 cycles	60 cycles	Inductive reactance, Current approx. 75% capacity‡: 25 cycles	50 cycles	60 cycles	Shunt capacitive reactance, megohms/conductor/mile at 1-ft spacing: 25 cycles	50 cycles	60 cycles
266,800	6	1	0.2109	7	0.0703	0.633	3/0	9,645	1,802	0.00684	460	0.351	0.351	0.351	0.352	0.386	0.430	0.510	0.552	0.194	0.388	0.466	0.252	0.504	0.605	0.259	0.1294	0.1079
4/0	6	1	.1878	1	.1878	.563	2/0	8,420	1,542	.00814	340	.441	.442	.444	.445	.485	.514	.567	.592	.218	.437	.524	.242	.484	.581	.267	.1336	.1113
3/0	6	1	.1672	1	.1672	.502	1/0	6,675	1,223	.00600	300	.556	.557	.559	.560	.612	.642	.697	.723	.225	.450	.540	.259	.517	.621	.275	.1377	.1147
2/0		1	.1490	1	.1490	.447	1	5,345	970	.00510	270	.702	.702	.704	.706	.773	.806	.866	.895	.231	.462	.554	.267	.534	.641	.284	.1418	.1182
1/0		1	.1327	1	.1327	.398	2	4,280	769	.00446	230	.885	.885	.887	.888	.974	1.01	1.08	1.12	.237	.473	.568	.273	.547	.656	.292	.1460	.1216
1		1	.1182	1	.1182	.355	3	3,480	610	.00418	200	1.12	1.12	1.12	1.12	1.23	1.27	1.34	1.38	.242	.483	.580	.277	.554	.665	.300	.1500	.1250
2	6	1	.1052	1	.1052	.316	4	2,790	484	.00418	180	1.41	1.41	1.41	1.41	1.55	1.59	1.66	1.69	.247	.493	.592	.277	.554	.665	.308	.1542	.1285
2	7	1	.0974	1	.1299	.325	4	3,525	566	.00504	180	1.41	1.41	1.41	1.41	1.55	1.59	1.62	1.65	.247	.493	.592	.267	.535	.642	.306	.1532	.1276
3	6	1	.0937	1	.0937	.281	5	2,250	384	.00430	160	1.78	1.78	1.78	1.78	1.95	1.98	2.04	2.07	.252	.503	.604	.275	.551	.661	.317	1583	.1320
4	6	1	.0834	1	.0834	.250	6	1,830	304	.00437	140	2.24	2 24	2.24	2.24	2.47	2.50	2.54	2.57	.257	.514	.611	.274	.549	.659	.325	1627	.1355
4	7	1	.0772	1	1029	.257	6	2,288	356	.00452	140	2.24	2.24	2.24	2 24	2.47	2 50	2.53	2.55	257	.515	.618	273	.545	.655	.323	1615	.1346
5	6	1	.0743	1	.0743	.223	7	1,460	241	.00416	120	2.82	2.82	2.82	2.82	3.10	3.12	3.16	3.18	.262	.525	.630	.279	.557	.665	.333	.1666	.1388
6	6	1	.0661	1	.0661	.198	8	1,170	191	.00394	100	3.56	3.56	3.56	3.56	3.92	3.94	3.97	3.98	.268	.536	.643	.281	.561	.673	.342	.1708	.1423

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† Based on copper 97%, aluminum 61% conductivity.

‡ "Current approx. 75% capacity" is 75% of the "Approx. current-carrying capacity, amp" shown in Column 9 and is approximately the current which will produce 50°C conductor temp. (25°C rise) with 25°C air temp., wind 1.4 miles/hour.

§ For conductor at 75°C, air at 25°C, wind 1.4 miles/hour (2 ft/sec), frequency = 60 cycles.

TABLE A.3 INDUCTIVE REACTANCE SPACING FACTOR AT 60 CYCLES*
(Ohms per conductor per mile)

Feet	Separation, Inches: 0	1	2	3	4	5	6	7	8	9	10	11
0	—	−.3015	−.2174	−.1682	−.1333	−.1062	−.0841	−.0654	−.0492	−.0349	−.0221	−.0106
1	0	.0097	.0187	.0271	.0349	.0423	.0492	.0558	.0620	.0679	.0735	.0789
2	.0841	.0891	.0938	.0984	.1028	.1071	.1112	.1152	.1190	.1227	.1264	.1299
3	.1333	.1366	.1399	.1430	.1461	.1491	.1520	.1549	.1577	.1604	.1631	.1657
4	.1682	.1707	.1732	.1756	.1779	.1802	.1825	.1847	.1869	.1891	.1912	.1933
5	.1953	.1973	.1993	.2012	.2031	.2050	.2069	.2087	.2105	.2123	.2140	.2157
6	.2174	.2191	.2207	.2224	.2240	.2256	.2271	.2287	.2302	.2317	.2332	.2347
7	.2361	.2376	.2390	.2404	.2418	.2431	.2445	.2458	.2472	.2485	.2498	.2511
8	.2523											
9	.2666											
10	.2794											
11	.2910											
12	.3015											
13	.3112											
14	.3202											
15	.3286											
16	.3364											
17	.3438											
18	.3507											
19	.3573											
20	.3635											
21	.3694											
22	.3751											
23	.3805											
24	.3856											
25	.3906											
26	.3953											
27	.3999											
28	.4043											
29	.4086											
30	.4127											
31	.4167											
32	.4205											
33	.4243											
34	.4279											
35	.4314											
36	.4348											
37	.4382											
38	.4414											
39	.4445											
40	.4476											
41	.4506											
42	.4535											
43	.4564											
44	.4592											
45	.4619											
46	.4646											
47	.4672											
48	.4697											
49	.4722											

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Table A.4 Shunt Capacitive Reactance Spacing Factor at 60 Cycles*
(Megohms per conductor per mile)

	Separation											
	Inches											
Feet	0	1	2	3	4	5	6	7	8	9	10	11
0	—	−.0737	−.0532	−.0411	−.0326	−.0260	−.0206	−.0160	−.0120	−.0085	−.0054	−.0026
1	0	.0024	.0046	.0066	.0085	.0103	.0120	.0136	.0152	.0166	.0180	.0193
2	.0206	.0218	.0229	.0241	.0251	.0262	.0272	.0282	.0291	.0300	.0309	.0318
3	.0326	.0334	.0342	.0350	.0357	.0365	.0372	.0379	.0385	.0392	.0399	.0405
4	.0411	.0417	.0423	.0429	.0435	.0441	.0446	.0452	.0457	.0462	.0467	.0473
5	.0478	.0482	.0487	.0492	.0497	.0501	.0506	.0510	.0515	.0519	0.523	0.527
6	.0532	.0536	.0540	.0544	.0548	.0552	.0555	.0559	.0563	.0567	.0570	.0574
7	.0577	.0581	.0584	.0588	.0591	.0594	.0598	.0601	.0604	.0608	.0611	.0614
8	.0617											
9	.0652											
10	.0683											
11	.0711											
12	.0737											
13	.0761											
14	.0783											
15	.0803											
16	.0823											
17	.0841											
18	.0858											
19	.0874											
20	.0889											
21	.0903											
22	.0917											
23	.0930											
24	.0943											
25	.0955											
26	.0967											
27	.0978											
28	.0989											
29	.0999											
30	.1009											
31	.1019											
32	.1028											
33	.1037											
34	.1046											
35	.1055											
36	.1063											
37	.1071											
38	.1079											
39	.1087											
40	.1094											
41	.1102											
42	.1109											
43	.1116											
44	.1123											
45	.1129											
46	.1136											
47	.1142											
48	.1149											
49	.1155											

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TABLE A.5 TYPICAL CONSTANTS OF THREE-PHASE SYNCHRONOUS MACHINES*
(Reactances are per unit. Values below the line give the normal range of values, while those above give an average value.)

	1	2	3	4	5	6
	X_d (unsat.)	X_q rated current	X'_d rated voltage	X''_d rated voltage	X_2 rated current	† X_0 rated current
Two-pole turbine generators	1.20 0.95–1.45	1.16 0.92–1.42	0.15 0.12–0.21	0.09 0.07–0.14	$= X''_d$	0.03 0.01–0.08
Four-pole turbine generators	1.20 1.00–1.45	1.16 0.92–1.42	0.23 0.20–0.28	0.14 0.12–0.17	$= X''_d$	0.08 0.015–0.14
Salient-pole generators and motors (with dampers)	1.25 0.60–1.50	0.70 0.40–0.80	0.30 0.20–0.50‡	0.20 0.13–0.32‡	0.20 0.13–0.32‡	0.18 0.03–0.23
Salient-pole generators (without dampers)	1.25 0.60–1.50	0.70 0.40–0.80	0.30 0.20–0.50‡	0.30 0.20–0.50‡	0.48 0.35–0.65	0.19 0.03–0.24
Condensers, air-cooled	1.85 1.25–2.20	1.15 0.95–1.30	0.40 0.30–0.50	0.27 0.19–0.30	0.26 0.18–0.40	0.12 0.025–0.15
Condensers, hydrogen-cooled at ½ psi kva rating	2.20 1.50–2.65	1.35 1.10–1.55	0.48 0.36–0.60	0.32 0.23–0.36	0.31 0.22–0.48	0.14 0.030–0.18

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† X_0 varies so critically with armature winding pitch that an average value can hardly be given. Variation is from 0.1 to 0.7 of X''_d. Low limit is for ⅔ pitch windings.

‡ High-speed units tend to have low reactance and low-speed units high reactance.

TABLE A.6 TRANSFORMER REACTANCES AND IMPEDANCES*
(Typical full-load values in per cent of full-load kva base)

Single-phase kva rating	2.5 Average reactance	2.5 Average impedance	15 Average reactance	15 Average impedance	25 Average reactance	25 Average impedance	69 Average reactance	69 Average impedance	138 Average reactance	138 Average impedance	161 Average reactance	161 Average impedance	230 Average reactance	230 Average impedance
3	1.1	2.2	0.8	2.8										
10	1.5	2.2	1.3	2.4	4.4	5.2								
25	2.0	2.5	1.7	2.3	4.8	5.2								
50	2.1	2.4	2.1	2.5	4.9	5.2	6.3	6.5						
100	3.1	3.3	2.9	3.2	5.0	5.2	6.3	6.5						
500	4.7	4.8	4.9	5.0	5.1	5.2	6.4	6.5						

Single-phase kva rating	2.5 Impedance range	15 Impedance range	25 Impedance range	69 Impedance range	138 Impedance range	161 Impedance range	230 Impedance range
1000	....	4.5–8.0	5.5–9.0	7.0–11.0	8.5–17.0		
5000	....	4.5–8.0	5.5–9.0	7.0–11.0	8.5–17.0	9.5–18.5	
10000	....	4.5–8.0	5.5–9.0	7.0–11.0	8.5–17.0	9.5–18.5	11.5–20.5
25000	....	....	5.5–9.0	7.0–11.0	8.5–17.0	9.5–18.5	11.5–20.5
50000	....	....	....	....	8.5–17.0	9.5–18.5	11.5–20.5

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Notes: (1) Above 500 kva, reactance and impedance values are nearly equal, and only the normal design impedance range is given.

(2) For three-phase transformers use ⅓ of the three-phase kva rating, and enter the table with rated line-to-line voltage.

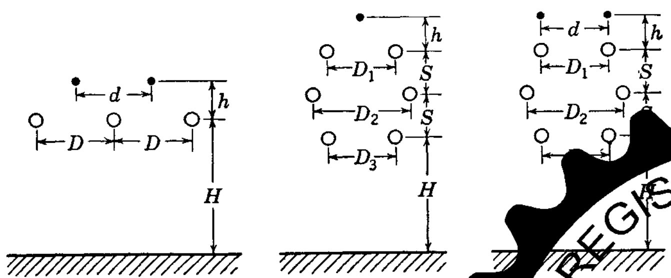
TABLE A.7 TYPICAL

Line No.	Company	Length, miles	Voltage, kv	Circuits	Conductor
1	Union Electric Company of Missouri	16.56	66	2	4/0 copper
2	Georgia Power Company	102.65	110	1	397,500 CM 26/7 ACSR
3	Carolina Power and Light Company	47.0	132	1	397,500 CM 26/7 ACSR
4	Union Electric Company of Missouri	119.5	132	2	250,000 CM 19-strand copper
5	Hydro-Electric Power Commission of Ontario	153.8	132	2	336,400 CM ACSR
6	Southern California Edison Company	233.4	220	1	605,000 CM 30/19 ACSR
7	Southern California Edison Company	7.0	220	1	1,033,500 CM 54/7 ACSR
8	American Gas and Electric Company	*	330	2	Expanded ACSR 1.6″ O.D.

* A single-circuit line of the first portion of the 330-kv system of the American Gas and Electric Company was completed between the Philip Sporn and Kanawa River plants, a distance of 63 miles, in 1952. A second circuit will be added later on the same towers.

POWER TRANSMISSION LINES

Ground wire	Location of line	Type of construction	Dimensions
3/0 ACSR	Venice-Alton	*B*	D_1 = 14′6″ S = 8′ D_2 = 18′6″ h = 8′4″ D_3 = 14′6″ H = 46′6″ av
3/8″ 7-strand galv. steel	South Macon–Vidalia	*A*	D = 14′ h = 12′6″ approx d = 14′ H = 28′8″ max = 19′7″ min
3/8″ S-M steel	Roxboro-Raleigh	*A*	D = 13′6″ h = 7′ d = 13′6″ H = 22′ av
7/16″ S-M steel	Osage-Rivermines	*C*	D_1 = 29′ d = 29′ D_2 = 35′ h = 20′8″ D_3 = 29′ H = 42′4″ av S = 14′
5/16″ steel		*B*	D_1 = 18′6″ h = 10′ approx D_2 = 28′6″ H = 55′ max D_3 = 18′6″ = 22′ min S = 11′
1/2″ 7-strand steel	Hoover Dam–Chino	*A*	D = 23′ h = 20′ d = 24′ H = 45′ av
1/2″ 7-strand steel	East Laguna Bell–Lighthipe	*A*	D = 23′ h = 20′ d = 24′ H = 45′ av
159,000 CM ACSR		*B*	D_1 = 35′6″ h = 15′ D_2 = 48′6″ H = 88′ max D_3 = 38′6″ = 35′ min S = 21′6″



Type *A* Type *B* Type *C*

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