10.7 Areas and Lengths in Polar Coordinates

This section shows how to calculate areas of plane regions, lengths of curves, and areas of surfaces of revolution in polar coordinates.

Area in the Plane

The region *OTS* in Figure 10.48 is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle *TOS*. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k, or

$$A_k = \frac{1}{2} r_k^2 \, \Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k .$$

The area of region *OTS* is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k .$$

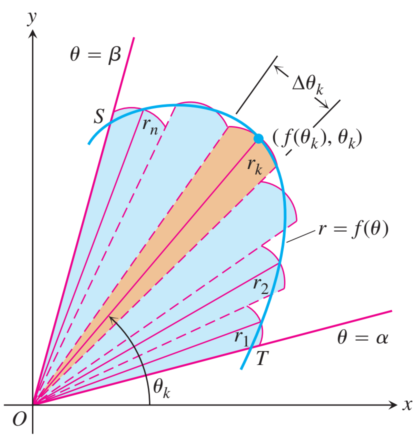


FIGURE 10.48 To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

If f is continuous, we expect the approximations to improve as the norm of the partition $\|P\| \to 0$, and we are led to the following formula for the region's area:

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$
$$= \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2\, d\theta.$$

This is the integral of the **area differential** (Figure 10.49)

$$dA = \frac{1}{2} r^2\, d\theta = \frac{1}{2}(f(\theta))^2\, d\theta.$$

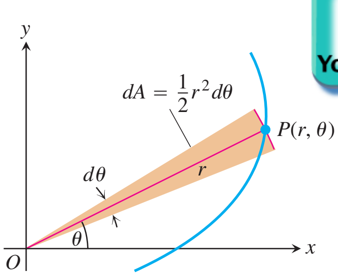


FIGURE 10.49 The area differential dA for the curve $n = f(\theta)$.

EXAMPLE 1 Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution We graph the cardioid (Figure 10.50) and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2\, d\theta$$
$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta)\, d\theta$$
$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta$$
$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta$$
$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.$$

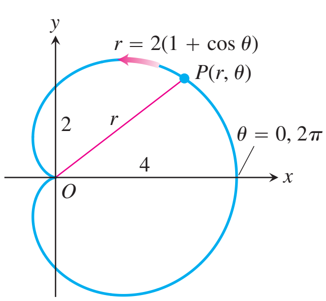


FIGURE 10.50 The cardioid in Example 1.

EXAMPLE 2 Finding Area

Find the area inside the smaller loop of the limaçon

$$r = 2\cos\theta + 1.$$

Solution After sketching the curve (Figure 10.51), we see that the smaller loop is traced out by the point (r, θ) as θ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. Since the curve is symmetric about the x-axis (the equation is unaltered when we replace θ by $-\theta$), we may calculate the area of the shaded half of the inner loop by integrating from $\theta = 2\pi/3$ to $\theta = \pi$. The area we seek will be twice the resulting integral:

$$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2\, d\theta = \int_{2\pi/3}^{\pi} r^2\, d\theta.$$

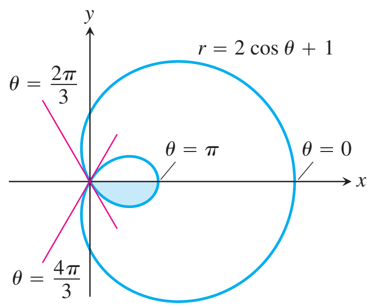


FIGURE 10.51 The limaçon in Example 2. Limaçon (pronounced LEE-ma-sahn) is an old French word for *snail*.

Since

$$\begin{aligned} r^2 &= (2\cos\theta + 1)^2 = 4\cos^2\theta + 4\cos\theta + 1 \\ &= 4\cdot\frac{1+\cos 2\theta}{2} + 4\cos\theta + 1 \\ &= 2 + 2\cos 2\theta + 4\cos\theta + 1 \\ &= 3 + 2\cos 2\theta + 4\cos\theta, \end{aligned}$$

we have

$$\begin{aligned} A &= \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta \\ &= \Big[3\theta + \sin 2\theta + 4\sin\theta\Big]_{2\pi/3}^{\pi} \\ &= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4\cdot\frac{\sqrt{3}}{2}\right) \\ &= \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$

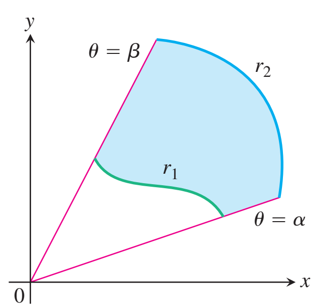


FIGURE 10.52 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

You Try It

To find the area of a region like the one in Figure 10.52, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2)r_1^2\, d\theta$ from the integral of $(1/2)r_2^2\, d\theta$. This leads to the following formula.

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\quad \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2\, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2\, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$

EXAMPLE 3 Finding Area Between Polar Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution We sketch the region to determine its boundaries and find the limits of integration (Figure 10.53). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \\ &= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry} \\ &= \int_{0}^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \\ &= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_{0}^{\pi/2} \left(2\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta \\ &= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{0}^{\pi/2} = 2 - \frac{\pi}{4}. \end{aligned}$$

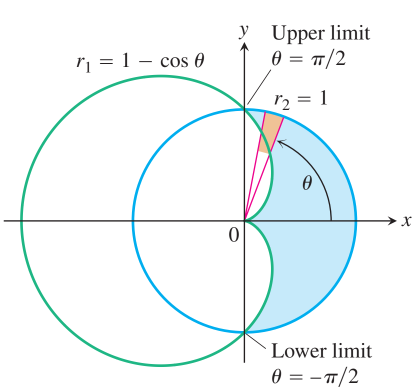


FIGURE 10.53 The region and limits of integration in Example 3.

Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta. \tag{2}$$

The parametric length formula, Equation (1) from Section 6.3, then gives the length as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This equation becomes

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

when Equations (2) are substituted for x and y (Exercise 33).

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3}$$

You Try It

EXAMPLE 4 Finding the Length of a Cardioid

Find the length of the cardioid $r = 1 - \cos\theta$.

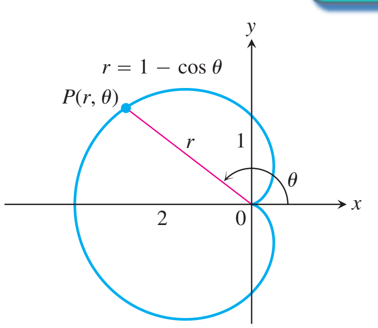


FIGURE 10.54 Calculating the length of a cardioid (Example 4).

Solution We sketch the cardioid to determine the limits of integration (Figure 10.54). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.

With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1 - \cos\theta)^2 + (\sin\theta)^2 \\ &= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta \end{aligned}$$

and

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \end{aligned}$$

$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \geq 0 \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8.$$

Area of a Surface of Revolution

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, with Equations (2) and apply the surface area equations for parametrized curves in Section 6.5.

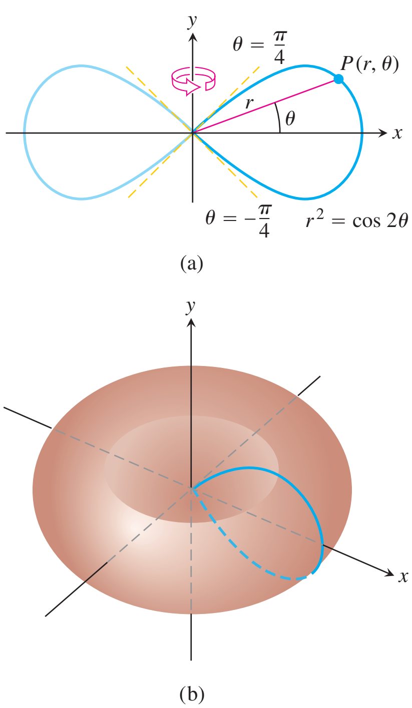


FIGURE 10.55 The right-hand half of a lemniscate (a) is revolved about the y-axis to generate a surface (b), whose area is calculated in Example 5.

Area of a Surface of Revolution of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the x- and y-axes are given by the following formulas:

1. Revolution about the x-axis $(y \geq 0)$:

$$S = \int_\alpha^\beta 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{4}$$

2. Revolution about the y-axis $(x \geq 0)$:

$$S = \int_\alpha^\beta 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{5}$$

EXAMPLE 5 Finding Surface Area

Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

Solution We sketch the loop to determine the limits of integration (Figure 10.55a). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from $-\pi/4$ to $\pi/4$, so these are the values we take for α and β.

We evaluate the area integrand in Equation (5) in stages. First,

$$2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\pi \cos\theta \sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}. \tag{6}$$

Next, $r^2 = \cos 2\theta$, so

$$2r\frac{dr}{d\theta} = -2\sin 2\theta$$

$$r\frac{dr}{d\theta} = -\sin 2\theta$$

$$\left(r\frac{dr}{d\theta}\right)^2 = \sin^2 2\theta.$$

Finally, $r^4 = (r^2)^2 = \cos^2 2\theta$, so the square root on the right-hand side of Equation (6) simplifies to

$$\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1.$$

All together, we have

$$\begin{aligned} S &= \int_\alpha^\beta 2\pi r\cos\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \qquad \text{Equation (5)} \\ &= \int_{-\pi/4}^{\pi/4} 2\pi\cos\theta\cdot(1)\,d\theta \\ &= 2\pi\Big[\sin\theta\Big]_{-\pi/4}^{\pi/4} \\ &= 2\pi\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 2\pi\sqrt{2}. \end{aligned}$$

■

EXERCISES 10.7

Areas Inside Polar Curves

Find the areas of the regions in Exercises 1–6.

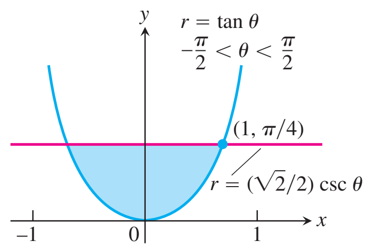
1. Inside the oval limaçon $r = 4 + 2\cos\theta$
2. Inside the cardioid $r = a(1 + \cos\theta), \quad a > 0$
3. Inside one leaf of the four-leaved rose $r = \cos 2\theta$
4. Inside the lemniscate $r^2 = 2a^2\cos 2\theta, \quad a > 0$
5. Inside one loop of the lemniscate $r^2 = 4\sin 2\theta$
6. Inside the six-leaved rose $r^2 = 2\sin 3\theta$

Areas Shared by Polar Regions

Find the areas of the regions in Exercises 7–16.

7. Shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$
8. Shared by the circles $r = 1$ and $r = 2\sin\theta$
9. Shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$
10. Shared by the cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$
11. Inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$
12. Inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta), a > 0$
13. Inside the circle $r = -2\cos\theta$ and outside the circle $r = 1$
14. **a.** Inside the outer loop of the limaçon $r = 2\cos\theta + 1$ (See Figure 10.51.)

    b. Inside the outer loop and outside the inner loop of the limaçon $r = 2\cos\theta + 1$
15. Inside the circle $r = 6$ above the line $r = 3\csc\theta$
16. Inside the lemniscate $r^2 = 6\cos 2\theta$ to the right of the line $r = (3/2)\sec\theta$
17. **a.** Find the area of the shaded region in the accompanying figure.



    b. It looks as if the graph of $r = \tan\theta, -\pi/2 < \theta < \pi/2$, could be asymptotic to the lines $x = 1$ and $x = -1$. Is it? Give reasons for your answer.
18. The area of the region that lies inside the cardioid curve $r = \cos\theta + 1$ and outside the circle $r = \cos\theta$ is not

$$\frac{1}{2}\int_0^{2\pi} [(\cos\theta + 1)^2 - \cos^2\theta]\, d\theta = \pi.$$

Why not? What *is* the area? Give reasons for your answers.

Lengths of Polar Curves

Find the lengths of the curves in Exercises 19–27.

19. The spiral $r = \theta^2$, $0 \le \theta \le \sqrt{5}$

20. The spiral $r = e^{\theta}/\sqrt{2}$, $0 \le \theta \le \pi$

21. The cardioid $r = 1 + \cos\theta$

22. The curve $r = a\sin^2(\theta/2)$, $0 \le \theta \le \pi$, $a > 0$

23. The parabolic segment $r = 6/(1 + \cos\theta)$, $0 \le \theta \le \pi/2$

24. The parabolic segment $r = 2/(1 - \cos\theta)$, $\pi/2 \le \theta \le \pi$

25. The curve $r = \cos^3(\theta/3)$, $0 \le \theta \le \pi/4$

26. The curve $r = \sqrt{1 + \sin 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$

27. The curve $r = \sqrt{1 + \cos 2\theta}$, $0 \le \theta \le \pi\sqrt{2}$

28. Circumferences of circles As usual, when faced with a new formula, it is a good idea to try it on familiar objects to be sure it gives results consistent with past experience. Use the length formula in Equation (3) to calculate the circumferences of the following circles $(a > 0)$:

a. $r = a$ **b.** $r = a\cos\theta$ **c.** $r = a\sin\theta$

Surface Area

Find the areas of the surfaces generated by revolving the curves in Exercises 29–32 about the indicated axes.

29. $r = \sqrt{\cos 2\theta}$, $0 \le \theta \le \pi/4$, y-axis

30. $r = \sqrt{2}e^{\theta/2}$, $0 \le \theta \le \pi/2$, x-axis

31. $r^2 = \cos 2\theta$, x-axis

32. $r = 2a\cos\theta$, $a > 0$, y-axis

Theory and Examples

33. The length of the curve $r = f(\theta), \alpha \le \theta \le \beta$ Assuming that the necessary derivatives are continuous, show how the substitutions

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

(Equations 2 in the text) transform

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

into

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

34. Average value If f is continuous, the average value of the polar coordinate r over the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, with respect to θ is given by the formula

$$r_{\text{av}} = \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} f(\theta)\, d\theta.$$

Use this formula to find the average value of r with respect to θ over the following curves $(a > 0)$.

a. The cardioid $r = a(1 - \cos\theta)$

b. The circle $r = a$

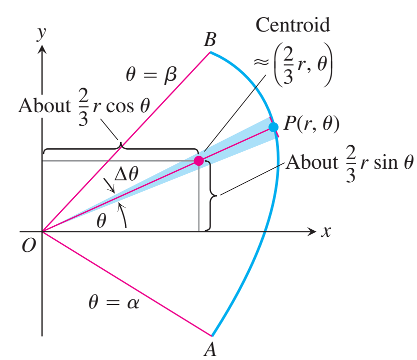
c. The circle $r = a\cos\theta$, $-\pi/2 \le \theta \le \pi/2$

35. $r = f(\theta)$ vs. $r = 2f(\theta)$ Can anything be said about the relative lengths of the curves $r = f(\theta)$, $\alpha \le \theta \le \beta$, and $r = 2f(\theta)$, $\alpha \le \theta \le \beta$? Give reasons for your answer.

36. $r = f(\theta)$ vs. $r = 2f(\theta)$ The curves $r = f(\theta)$, $\alpha \le \theta \le \beta$, and $r = 2f(\theta)$, $\alpha \le \theta \le \beta$, are revolved about the x-axis to generate surfaces. Can anything be said about the relative areas of these surfaces? Give reasons for your answer.

Centroids of Fan-Shaped Regions

Since the centroid of a triangle is located on each median, two-thirds of the way from the vertex to the opposite base, the lever arm for the moment about the x-axis of the thin triangular region in the accompanying figure is about $(2/3)r\sin\theta$. Similarly, the lever arm for the moment of the triangular region about the y-axis is about $(2/3)r\cos\theta$. These approximations improve as $\Delta\theta \to 0$ and lead to the following formulas for the coordinates of the centroid of region AOB:



$$\bar{x} = \frac{\displaystyle\int \frac{2}{3}r\cos\theta\cdot\frac{1}{2}r^2\, d\theta}{\displaystyle\int \frac{1}{2}r^2\, d\theta} = \frac{\displaystyle\frac{2}{3}\int r^3\cos\theta\, d\theta}{\displaystyle\int r^2\, d\theta},$$

$$\bar{y} = \frac{\displaystyle\int \frac{2}{3}r\sin\theta\cdot\frac{1}{2}r^2\, d\theta}{\displaystyle\int \frac{1}{2}r^2\, d\theta} = \frac{\displaystyle\frac{2}{3}\int r^3\sin\theta\, d\theta}{\displaystyle\int r^2\, d\theta},$$

with limits $\theta = \alpha$ to $\theta = \beta$ on all integrals.

37. Find the centroid of the region enclosed by the cardioid $r = a(1 + \cos\theta)$.

38. Find the centroid of the semicircular region $0 \le r \le a$, $0 \le \theta \le \pi$.

10.8 Conic Sections in Polar Coordinates

Polar coordinates are important in astronomy and astronautical engineering because the ellipses, parabolas, and hyperbolas along which satellites, moons, planets, and comets approximately move can all be described with a single relatively simple coordinate equation. We develop that equation here.

Lines

Suppose the perpendicular from the origin to line L meets L at the point $P_0(r_0, \theta_0)$, with $r_0 \geq 0$ (Figure 10.56). Then, if $P(r, \theta)$ is any other point on L, the points P, P_0, and O are the vertices of a right triangle, from which we can read the relation

$$r_0 = r\cos(\theta - \theta_0).$$

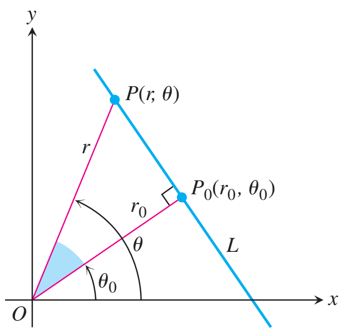


FIGURE 10.56 We can obtain a polar equation for line L by reading the relation $r_0 = r\cos(\theta - \theta_0)$ from the right triangle OP_0P.

> **The Standard Polar Equation for Lines**
> If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line L, and $r_0 \geq 0$, then an equation for L is
> $$r\cos(\theta - \theta_0) = r_0. \qquad (1)$$

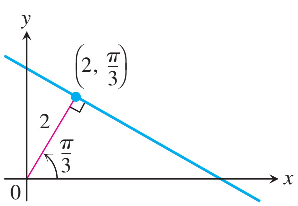


FIGURE 10.57 The standard polar equation of this line converts to the Cartesian equation $x + \sqrt{3}\,y = 4$ (Example 1).

EXAMPLE 1 Converting a Line's Polar Equation to Cartesian Form

Use the identity $\cos(A - B) = \cos A\cos B + \sin A\sin B$ to find a Cartesian equation for the line in Figure 10.57.

Solution

$$\begin{aligned}
r\cos\left(\theta - \frac{\pi}{3}\right) &= 2 \\
r\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) &= 2 \\
\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta &= 2 \\
\frac{1}{2}x + \frac{\sqrt{3}}{2}y &= 2 \\
x + \sqrt{3}\,y &= 4
\end{aligned}$$

Circles

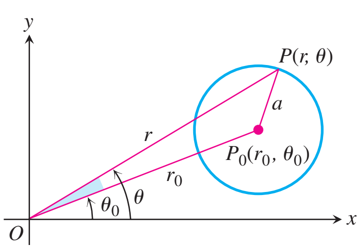


FIGURE 10.58 We can get a polar equation for this circle by applying the Law of Cosines to triangle OP_0P.

To find a polar equation for the circle of radius a centered at $P_0(r_0, \theta_0)$, we let $P(r, \theta)$ be a point on the circle and apply the Law of Cosines to triangle OP_0P (Figure 10.58). This gives

$$a^2 = r_0^2 + r^2 - 2r_0r\cos(\theta - \theta_0).$$

If the circle passes through the origin, then $r_0 = a$ and this equation simplifies to

$$\begin{aligned} a^2 &= a^2 + r^2 - 2ar\cos(\theta - \theta_0) \\ r^2 &= 2ar\cos(\theta - \theta_0) \\ r &= 2a\cos(\theta - \theta_0). \end{aligned}$$

If the circle's center lies on the positive x-axis, $\theta_0 = 0$ and we get the further simplification

$$r = 2a\cos\theta$$

(see Figure 10.59a).

If the center lies on the positive y-axis, $\theta = \pi/2$, $\cos(\theta - \pi/2) = \sin\theta$, and the equation $r = 2a\cos(\theta - \theta_0)$ becomes

$$r = 2a\sin\theta$$

(see Figure 10.59b).

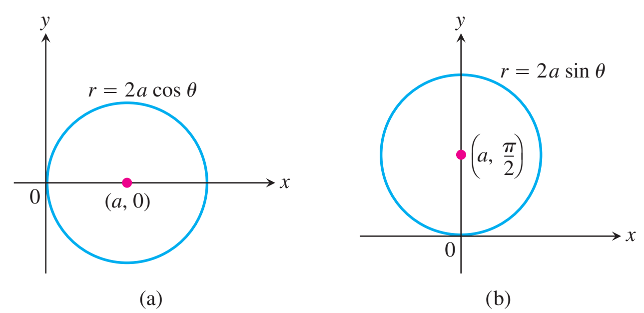


FIGURE 10.59 Polar equation of a circle of radius a through the origin with center on (a) the positive x-axis, and (b) the positive y-axis.

Equations for circles through the origin centered on the negative x- and y-axes can be obtained by replacing r with $-r$ in the above equations (Figure 10.60).

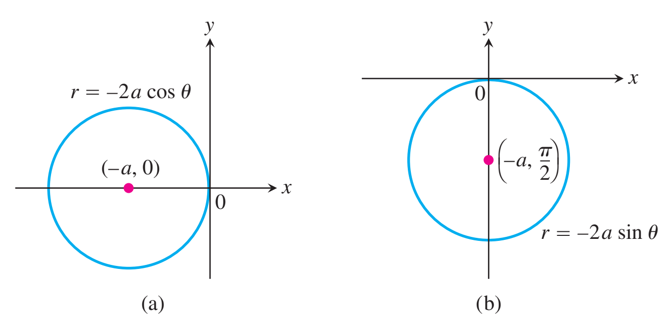


FIGURE 10.60 Polar equation of a circle of radius a through the origin with center on (a) the negative x-axis, and (b) the negative y-axis.

EXAMPLE 2 Circles Through the Origin

Radius	Center (polar coordinates)	Polar equation
3	$(3, 0)$	$r = 6\cos\theta$
2	$(2, \pi/2)$	$r = 4\sin\theta$
1/2	$(-1/2, 0)$	$r = -\cos\theta$
1	$(-1, \pi/2)$	$r = -2\sin\theta$

Ellipses, Parabolas, and Hyperbolas

To find polar equations for ellipses, parabolas, and hyperbolas, we place one focus at the origin and the corresponding directrix to the right of the origin along the vertical line $x = k$ (Figure 10.61). This makes

$$PF = r$$

and

$$PD = k - FB = k - r\cos\theta.$$

The conic's focus–directrix equation $PF = e \cdot PD$ then becomes

$$r = e(k - r\cos\theta),$$

which can be solved for r to obtain

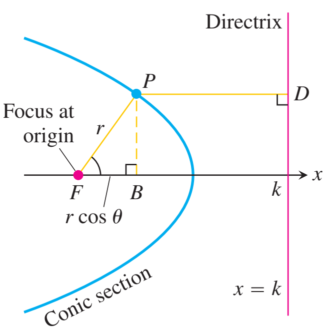


FIGURE 10.61 If a conic section is put in the position with its focus placed at the origin and a directrix perpendicular to the initial ray and right of the origin, we can find its polar equation from the conic's focus–directrix equation.

Polar Equation for a Conic with Eccentricity e

$$r = \frac{ke}{1 + e\cos\theta}, \qquad (2)$$

where $x = k > 0$ is the vertical directrix.

This equation represents an ellipse if $0 < e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$. That is, ellipses, parabolas, and hyperbolas all have the same basic equation expressed in terms of eccentricity and location of the directrix.

EXAMPLE 3 Polar Equations of Some Conics

$e = \frac{1}{2}$: ellipse $\quad r = \dfrac{k}{2 + \cos\theta}$

$e = 1$: parabola $\quad r = \dfrac{k}{1 + \cos\theta}$

$e = 2$: hyperbola $\quad r = \dfrac{2k}{1 + 2\cos\theta}$

You may see variations of Equation (2) from time to time, depending on the location of the directrix. If the directrix is the line $x = -k$ to the left of the origin (the origin is still a focus), we replace Equation (2) by

$$r = \frac{ke}{1 - e\cos\theta}.$$

The denominator now has a $(-)$ instead of a $(+)$. If the directrix is either of the lines $y = k$ or $y = -k$, the equations have sines in them instead of cosines, as shown in Figure 10.62.

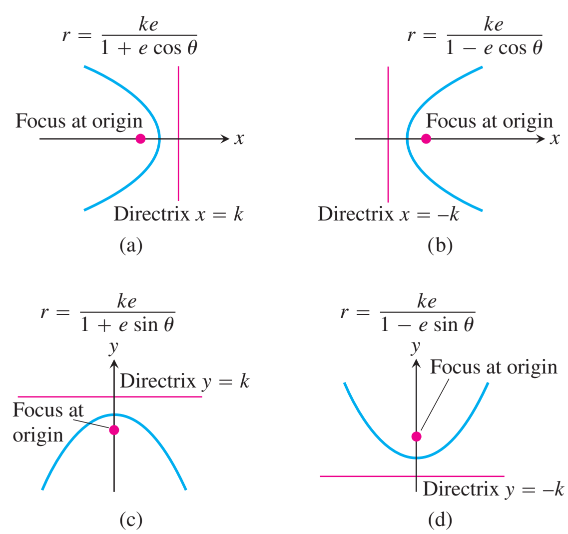


FIGURE 10.62 Equations for conic sections with eccentricity $e > 0$, but different locations of the directrix. The graphs here show a parabola, so $e = 1$.

EXAMPLE 4 Polar Equation of a Hyperbola

Find an equation for the hyperbola with eccentricity $3/2$ and directrix $x = 2$.

Solution We use Equation (2) with $k = 2$ and $e = 3/2$:

$$r = \frac{2(3/2)}{1 + (3/2)\cos\theta} \quad \text{or} \quad r = \frac{6}{2 + 3\cos\theta}.$$ ■

EXAMPLE 5 Finding a Directrix

Find the directrix of the parabola

$$r = \frac{25}{10 + 10\cos\theta}.$$

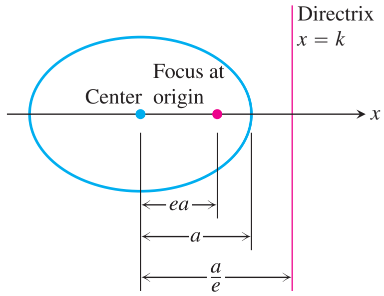


FIGURE 10.63 In an ellipse with semimajor axis a, the focus–directrix distance is $k = (a/e) - ea$, so $ke = a(1 - e^2)$.

Solution We divide the numerator and denominator by 10 to put the equation in standard form:

$$r = \frac{5/2}{1 + \cos\theta}.$$

This is the equation

$$r = \frac{ke}{1 + e\cos\theta}$$

with $k = 5/2$ and $e = 1$. The equation of the directrix is $x = 5/2$. ■

From the ellipse diagram in Figure 10.63, we see that k is related to the eccentricity e and the semimajor axis a by the equation

$$k = \frac{a}{e} - ea.$$

From this, we find that $ke = a(1 - e^2)$. Replacing ke in Equation (2) by $a(1 - e^2)$ gives the standard polar equation for an ellipse.

Polar Equation for the Ellipse with Eccentricity *e* and Semimajor Axis *a*

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{3}$$

Notice that when $e = 0$, Equation (3) becomes $r = a$, which represents a circle.

Equation (3) is the starting point for calculating planetary orbits.

EXAMPLE 6 The Planet Pluto's Orbit

Find a polar equation for an ellipse with semimajor axis 39.44 AU (astronomical units) and eccentricity 0.25. This is the approximate size of Pluto's orbit around the sun.

Solution We use Equation (3) with $a = 39.44$ and $e = 0.25$ to find

$$r = \frac{39.44(1 - (0.25)^2)}{1 + 0.25\cos\theta} = \frac{147.9}{4 + \cos\theta}.$$

At its point of closest approach (perihelion) where $\theta = 0$, Pluto is

$$r = \frac{147.9}{4 + 1} = 29.58 \text{ AU}$$

from the sun. At its most distant point (aphelion) where $\theta = \pi$, Pluto is

$$r = \frac{147.9}{4 - 1} = 49.3 \text{ AU}$$

from the sun (Figure 10.64). ■

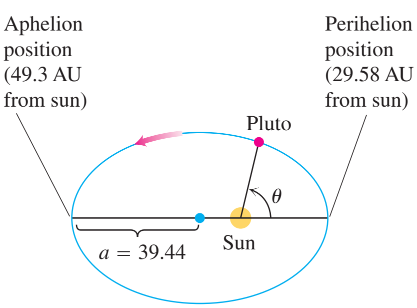


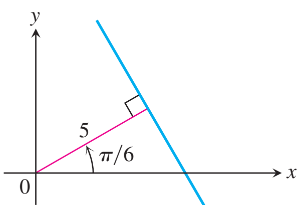
FIGURE 10.64 The orbit of Pluto (Example 6).

EXERCISES 10.8

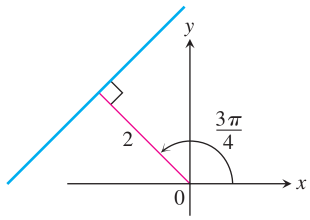
Lines

Find polar and Cartesian equations for the lines in Exercises 1–4.

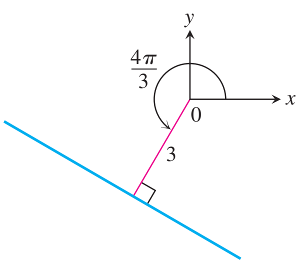
1.



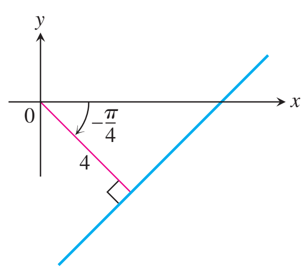
2.



3.



4.



Sketch the lines in Exercises 5–8 and find Cartesian equations for them.

5. $r\cos\left(\theta - \dfrac{\pi}{4}\right) = \sqrt{2}$ **6.** $r\cos\left(\theta + \dfrac{3\pi}{4}\right) = 1$

7. $r\cos\left(\theta - \dfrac{2\pi}{3}\right) = 3$ **8.** $r\cos\left(\theta + \dfrac{\pi}{3}\right) = 2$

Find a polar equation in the form $r\cos(\theta - \theta_0) = r_0$ for each of the lines in Exercises 9–12.

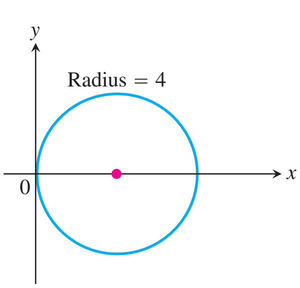
9. $\sqrt{2}x + \sqrt{2}y = 6$ **10.** $\sqrt{3}x - y = 1$

11. $y = -5$ **12.** $x = -4$

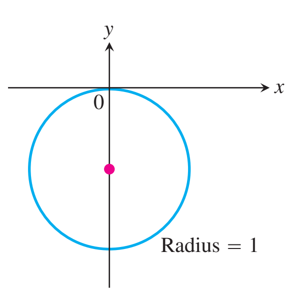
Circles

Find polar equations for the circles in Exercises 13–16.

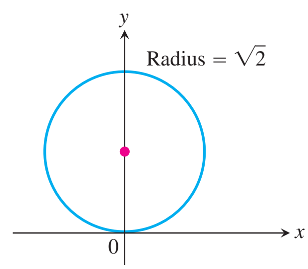
13.



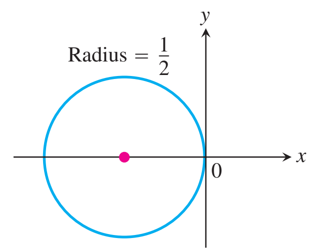
14.



15.



16.



Sketch the circles in Exercises 17–20. Give polar coordinates for their centers and identify their radii.

17. $r = 4\cos\theta$ **18.** $r = 6\sin\theta$

19. $r = -2\cos\theta$ **20.** $r = -8\sin\theta$

Find polar equations for the circles in Exercises 21–28. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

21. $(x - 6)^2 + y^2 = 36$ **22.** $(x + 2)^2 + y^2 = 4$

23. $x^2 + (y - 5)^2 = 25$ **24.** $x^2 + (y + 7)^2 = 49$

25. $x^2 + 2x + y^2 = 0$ **26.** $x^2 - 16x + y^2 = 0$

27. $x^2 + y^2 + y = 0$ **28.** $x^2 + y^2 - \dfrac{4}{3}y = 0$

Conic Sections from Eccentricities and Directrices

Exercises 29–36 give the eccentricities of conic sections with one focus at the origin, along with the directrix corresponding to that focus. Find a polar equation for each conic section.

29. $e = 1, \quad x = 2$ **30.** $e = 1, \quad y = 2$

31. $e = 5, \quad y = -6$ **32.** $e = 2, \quad x = 4$

33. $e = 1/2, \quad x = 1$ **34.** $e = 1/4, \quad x = -2$

35. $e = 1/5, \quad y = -10$ **36.** $e = 1/3, \quad y = 6$

Parabolas and Ellipses

Sketch the parabolas and ellipses in Exercises 37–44. Include the directrix that corresponds to the focus at the origin. Label the vertices with appropriate polar coordinates. Label the centers of the ellipses as well.

37. $r = \dfrac{1}{1 + \cos\theta}$ **38.** $r = \dfrac{6}{2 + \cos\theta}$

39. $r = \dfrac{25}{10 - 5\cos\theta}$ **40.** $r = \dfrac{4}{2 - 2\cos\theta}$

41. $r = \dfrac{400}{16 + 8\sin\theta}$ **42.** $r = \dfrac{12}{3 + 3\sin\theta}$

43. $r = \dfrac{8}{2 - 2\sin\theta}$ **44.** $r = \dfrac{4}{2 - \sin\theta}$

Graphing Inequalities

Sketch the regions defined by the inequalities in Exercises 45 and 46.

45. $0 \le r \le 2\cos\theta$ **46.** $-3\cos\theta \le r \le 0$

Grapher Explorations

Graph the lines and conic sections in Exercises 47–56.

47. $r = 3\sec(\theta - \pi/3)$ **48.** $r = 4\sec(\theta + \pi/6)$

49. $r = 4\sin\theta$ **50.** $r = -2\cos\theta$

51. $r = 8/(4 + \cos\theta)$ **52.** $r = 8/(4 + \sin\theta)$

53. $r = 1/(1 - \sin\theta)$ **54.** $r = 1/(1 + \cos\theta)$

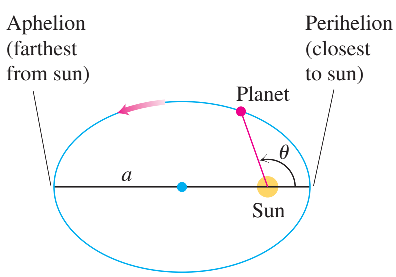
55. $r = 1/(1 + 2\sin\theta)$ **56.** $r = 1/(1 + 2\cos\theta)$

Theory and Examples

57. Perihelion and aphelion A planet travels about its sun in an ellipse whose semimajor axis has length a. (See accompanying figure.)

a. Show that $r = a(1 - e)$ when the planet is closest to the sun and that $r = a(1 + e)$ when the planet is farthest from the sun.

b. Use the data in the table in Exercise 58 to find how close each planet in our solar system comes to the sun and how far away each planet gets from the sun.



58. Planetary orbits In Example 6, we found a polar equation for the orbit of Pluto. Use the data in the table below to find polar equations for the orbits of the other planets.

Planet	Semimajor axis (astronomical units)	Eccentricity
Mercury	0.3871	0.2056
Venus	0.7233	0.0068
Earth	1.000	0.0167
Mars	1.524	0.0934
Jupiter	5.203	0.0484
Saturn	9.539	0.0543
Uranus	19.18	0.0460
Neptune	30.06	0.0082
Pluto	39.44	0.2481

59. a. Find Cartesian equations for the curves $r = 4\sin\theta$ and $r = \sqrt{3}\sec\theta$.

b. Sketch the curves together and label their points of intersection in both Cartesian and polar coordinates.

60. Repeat Exercise 59 for $r = 8\cos\theta$ and $r = 2\sec\theta$.

61. Find a polar equation for the parabola with focus (0, 0) and directrix $r\cos\theta = 4$.

62. Find a polar equation for the parabola with focus (0, 0) and directrix $r\cos(\theta - \pi/2) = 2$.

63. a. The space engineer's formula for eccentricity The space engineer's formula for the eccentricity of an elliptical orbit is

$$e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}},$$

where r is the distance from the space vehicle to the attracting focus of the ellipse along which it travels. Why does the formula work?

b. Drawing ellipses with string You have a string with a knot in each end that can be pinned to a drawing board. The string is 10 in. long from the center of one knot to the center of the other. How far apart should the pins be to use the method illustrated in Figure 10.5 (Section 10.1) to draw an ellipse of eccentricity 0.2? The resulting ellipse would resemble the orbit of Mercury.

64. Halley's comet (See Section 10.2, Example 1.)

a. Write an equation for the orbit of Halley's comet in a coordinate system in which the sun lies at the origin and the other focus lies on the negative x-axis, scaled in astronomical units.

b. How close does the comet come to the sun in astronomical units? In kilometers?

c. What is the farthest the comet gets from the sun in astronomical units? In kilometers?

In Exercises 65–68, find a polar equation for the given curve. In each case, sketch a typical curve.

65. $x^2 + y^2 - 2ay = 0$ **66.** $y^2 = 4ax + 4a^2$

67. $x\cos\alpha + y\sin\alpha = p$ (α, p constant)

68. $(x^2 + y^2)^2 + 2ax(x^2 + y^2) - a^2y^2 = 0$

COMPUTER EXPLORATIONS

69. Use a CAS to plot the polar equation

$$r = \frac{ke}{1 + e\cos\theta}$$

for various values of k and e, $-\pi \le \theta \le \pi$. Answer the following questions.

a. Take $k = -2$. Describe what happens to the plots as you take e to be $3/4$, 1, and $5/4$. Repeat for $k = 2$.

b. Take $k = -1$. Describe what happens to the plots as you take e to be $7/6$, $5/4$, $4/3$, $3/2$, 2, 3, 5, 10, and 20. Repeat for $e = 1/2$, $1/3$, $1/4$, $1/10$, and $1/20$.

c. Now keep $e > 0$ fixed and describe what happens as you take k to be -1, -2, -3, -4, and -5. Be sure to look at graphs for parabolas, ellipses, and hyperbolas.

70. Use a CAS to plot the polar ellipse

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

for various values of $a > 0$ and $0 < e < 1$, $-\pi \le \theta \le \pi$.

a. Take $e = 9/10$. Describe what happens to the plots as you let a equal 1, $3/2$, 2, 3, 5, and 10. Repeat with $e = 1/4$.

b. Take $a = 2$. Describe what happens as you take e to be $9/10$, $8/10$, $7/10$, $\ldots$, $1/10$, $1/20$, and $1/50$.

Chapter 10 Questions to Guide Your Review

1. What is a parabola? What are the Cartesian equations for parabolas whose vertices lie at the origin and whose foci lie on the coordinate axes? How can you find the focus and directrix of such a parabola from its equation?
2. What is an ellipse? What are the Cartesian equations for ellipses centered at the origin with foci on one of the coordinate axes? How can you find the foci, vertices, and directrices of such an ellipse from its equation?
3. What is a hyperbola? What are the Cartesian equations for hyperbolas centered at the origin with foci on one of the coordinate axes? How can you find the foci, vertices, and directrices of such an ellipse from its equation?
4. What is the eccentricity of a conic section? How can you classify conic sections by eccentricity? How are an ellipse's shape and eccentricity related?
5. Explain the equation $PF = e \cdot PD$.
6. What is a quadratic curve in the xy-plane? Give examples of degenerate and nondegenerate quadratic curves.
7. How can you find a Cartesian coordinate system in which the new equation for a conic section in the plane has no xy-term? Give an example.
8. How can you tell what kind of graph to expect from a quadratic equation in x and y? Give examples.
9. What are some typical parametrizations for conic sections?
10. What is a cycloid? What are typical parametric equations for cycloids? What physical properties account for the importance of cycloids?
11. What are polar coordinates? What equations relate polar coordinates to Cartesian coordinates? Why might you want to change from one coordinate system to the other?
12. What consequence does the lack of uniqueness of polar coordinates have for graphing? Give an example.
13. How do you graph equations in polar coordinates? Include in your discussion symmetry, slope, behavior at the origin, and the use of Cartesian graphs. Give examples.
14. How do you find the area of a region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$, in the polar coordinate plane? Give examples.
15. Under what conditions can you find the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, in the polar coordinate plane? Give an example of a typical calculation.
16. Under what conditions can you find the area of the surface generated by revolving a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, about the x-axis? The y-axis? Give examples of typical calculations.
17. What are the standard equations for lines and conic sections in polar coordinates? Give examples.

Chapter 10 Practice Exercises

Graphing Conic Sections

Sketch the parabolas in Exercises 1–4. Include the focus and directrix in each sketch.

1. $x^2 = -4y$

2. $x^2 = 2y$

3. $y^2 = 3x$

4. $y^2 = -(8/3)x$

Find the eccentricities of the ellipses and hyperbolas in Exercises 5–8. Sketch each conic section. Include the foci, vertices, and asymptotes (as appropriate) in your sketch.

5. $16x^2 + 7y^2 = 112$

6. $x^2 + 2y^2 = 4$

7. $3x^2 - y^2 = 3$

8. $5y^2 - 4x^2 = 20$

Shifting Conic Sections

Exercises 9–14 give equations for conic sections and tell how many units up or down and to the right or left each curve is to be shifted. Find an equation for the new conic section and find the new foci, vertices, centers, and asymptotes, as appropriate. If the curve is a parabola, find the new directrix as well.

9. $x^2 = -12y$, right 2, up 3

10. $y^2 = 10x$, left 1/2, down 1

11. $\frac{x^2}{9} + \frac{y^2}{25} = 1$, left 3, down 5

12. $\frac{x^2}{169} + \frac{y^2}{144} = 1$, right 5, up 12

13. $\frac{y^2}{8} - \frac{x^2}{2} = 1$, right 2, up $2\sqrt{2}$

14. $\frac{x^2}{36} - \frac{y^2}{64} = 1$, left 10, down 3

Identifying Conic Sections

Identify the conic sections in Exercises 15–22 and find their foci, vertices, centers, and asymptotes (as appropriate). If the curve is a parabola, find its directrix as well.

15. $x^2 - 4x - 4y^2 = 0$

16. $4x^2 - y^2 + 4y = 8$

17. $y^2 - 2y + 16x = -49$

18. $x^2 - 2x + 8y = -17$

19. $9x^2 + 16y^2 + 54x - 64y = -1$

20. $25x^2 + 9y^2 - 100x + 54y = 44$

21. $x^2 + y^2 - 2x - 2y = 0$

22. $x^2 + y^2 + 4x + 2y = 1$

Using the Discriminant

What conic sections or degenerate cases do the equations in Exercises 23–28 represent? Give a reason for your answer in each case.

23. $x^2 + xy + y^2 + x + y + 1 = 0$

24. $x^2 + 4xy + 4y^2 + x + y + 1 = 0$

25. $x^2 + 3xy + 2y^2 + x + y + 1 = 0$

26. $x^2 + 2xy - 2y^2 + x + y + 1 = 0$

27. $x^2 - 2xy + y^2 = 0$

28. $x^2 - 3xy + 4y^2 = 0$

Rotating Conic Sections

Identify the conic sections in Exercises 29–32. Then rotate the coordinate axes to find a new equation for the conic section that has no cross product term. (The new equations will vary with the size and direction of the rotations used.)

29. $2x^2 + xy + 2y^2 - 15 = 0$

30. $3x^2 + 2xy + 3y^2 = 19$

31. $x^2 + 2\sqrt{3}\,xy - y^2 + 4 = 0$

32. $x^2 - 3xy + y^2 = 5$

Identifying Parametric Equations in the Plane

Exercises 33–36 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation and indicate the direction of motion and the portion traced by the particle.

33. $x = (1/2)\tan t, \quad y = (1/2)\sec t; \quad -\pi/2 < t < \pi/2$

34. $x = -2\cos t, \quad y = 2\sin t; \quad 0 \le t \le \pi$

35. $x = -\cos t, \quad y = \cos^2 t; \quad 0 \le t \le \pi$

36. $x = 4\cos t, \quad y = 9\sin t; \quad 0 \le t \le 2\pi$

Graphs in the Polar Plane

Sketch the regions defined by the polar coordinate inequalities in Exercises 37 and 38.

37. $0 \le r \le 6\cos\theta$

38. $-4\sin\theta \le r \le 0$

Match each graph in Exercises 39–46 with the appropriate equation (a)–(l). There are more equations than graphs, so some equations will not be matched.

a. $r = \cos 2\theta$

b. $r\cos\theta = 1$

c. $r = \frac{6}{1 - 2\cos\theta}$

d. $r = \sin 2\theta$

e. $r = \theta$

f. $r^2 = \cos 2\theta$

g. $r = 1 + \cos\theta$

h. $r = 1 - \sin\theta$

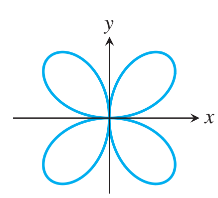
i. $r = \frac{2}{1 - \cos\theta}$

j. $r^2 = \sin 2\theta$

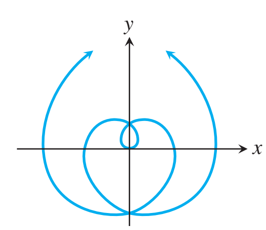
k. $r = -\sin\theta$

l. $r = 2\cos\theta + 1$

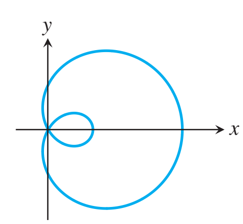
39. Four-leaved rose



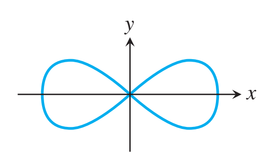
40. Spiral



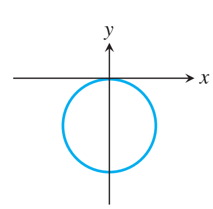
41. Limaçon



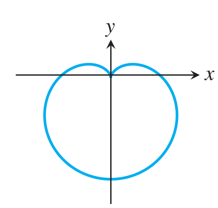
42. Lemniscate



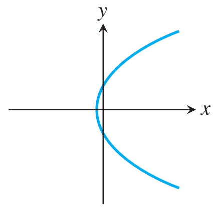
43. Circle



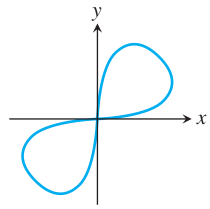
44. Cardioid



45. Parabola



46. Lemniscate



Intersections of Graphs in the Polar Plane

Find the points of intersection of the curves given by the polar coordinate equations in Exercises 47–54.

47. $r = \sin\theta, \quad r = 1 + \sin\theta$

48. $r = \cos\theta, \quad r = 1 - \cos\theta$

49. $r = 1 + \cos\theta, \quad r = 1 - \cos\theta$

50. $r = 1 + \sin\theta, \quad r = 1 - \sin\theta$

51. $r = 1 + \sin\theta, \quad r = -1 + \sin\theta$

52. $r = 1 + \cos\theta, \quad r = -1 + \cos\theta$

53. $r = \sec\theta, \quad r = 2\sin\theta$

54. $r = -2\csc\theta, \quad r = -4\cos\theta$

Polar to Cartesian Equations

Sketch the lines in Exercises 55–60. Also, find a Cartesian equation for each line.

55. $r\cos\left(\theta + \dfrac{\pi}{3}\right) = 2\sqrt{3}$

56. $r\cos\left(\theta - \dfrac{3\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

57. $r = 2\sec\theta$

58. $r = -\sqrt{2}\sec\theta$

59. $r = -(3/2)\csc\theta$

60. $r = \left(3\sqrt{3}\right)\csc\theta$

Find Cartesian equations for the circles in Exercises 61–64. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

61. $r = -4\sin\theta$

62. $r = 3\sqrt{3}\sin\theta$

63. $r = 2\sqrt{2}\cos\theta$

64. $r = -6\cos\theta$

Cartesian to Polar Equations

Find polar equations for the circles in Exercises 65–68. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

65. $x^2 + y^2 + 5y = 0$

66. $x^2 + y^2 - 2y = 0$

67. $x^2 + y^2 - 3x = 0$

68. $x^2 + y^2 + 4x = 0$

Conic Sections in Polar Coordinates

Sketch the conic sections whose polar coordinate equations are given in Exercises 69–72. Give polar coordinates for the vertices and, in the case of ellipses, for the centers as well.

69. $r = \dfrac{2}{1 + \cos\theta}$

70. $r = \dfrac{8}{2 + \cos\theta}$

71. $r = \dfrac{6}{1 - 2\cos\theta}$

72. $r = \dfrac{12}{3 + \sin\theta}$

Exercises 73–76 give the eccentricities of conic sections with one focus at the origin of the polar coordinate plane, along with the directrix for that focus. Find a polar equation for each conic section.

73. $e = 2, \quad r\cos\theta = 2$

74. $e = 1, \quad r\cos\theta = -4$

75. $e = 1/2, \quad r\sin\theta = 2$

76. $e = 1/3, \quad r\sin\theta = -6$

Area, Length, and Surface Area in the Polar Plane

Find the areas of the regions in the polar coordinate plane described in Exercises 77–80.

77. Enclosed by the limaçon $r = 2 - \cos\theta$

78. Enclosed by one leaf of the three-leaved rose $r = \sin 3\theta$

79. Inside the "figure eight" $r = 1 + \cos 2\theta$ and outside the circle $r = 1$

80. Inside the cardioid $r = 2(1 + \sin\theta)$ and outside the circle $r = 2\sin\theta$

Find the lengths of the curves given by the polar coordinate equations in Exercises 81–84.

81. $r = -1 + \cos\theta$

82. $r = 2\sin\theta + 2\cos\theta, \quad 0 \le \theta \le \pi/2$

83. $r = 8\sin^3(\theta/3), \quad 0 \le \theta \le \pi/4$

84. $r = \sqrt{1 + \cos 2\theta}, \quad -\pi/2 \le \theta \le \pi/2$

Find the areas of the surfaces generated by revolving the polar coordinate curves in Exercises 85 and 86 about the indicated axes.

85. $r = \sqrt{\cos 2\theta}, \quad 0 \le \theta \le \pi/4, \quad x$-axis

86. $r^2 = \sin 2\theta, \quad y$-axis

Theory and Examples

87. Find the volume of the solid generated by revolving the region enclosed by the ellipse $9x^2 + 4y^2 = 36$ about **(a)** the x-axis, **(b)** the y-axis.

88. The "triangular" region in the first quadrant bounded by the x-axis, the line $x = 4$, and the hyperbola $9x^2 - 4y^2 = 36$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

89. A ripple tank is made by bending a strip of tin around the perimeter of an ellipse for the wall of the tank and soldering a flat bottom onto this. An inch or two of water is put in the tank and you drop a marble into it, right at one focus of the ellipse. Ripples radiate outward through the water, reflect from the strip around the edge of the tank, and a few seconds later a drop of water spurts up at the second focus. Why?

90. LORAN A radio signal was sent simultaneously from towers A and B, located several hundred miles apart on the northern California coast. A ship offshore received the signal from A 1400 microseconds before receiving the signal from B. Assuming that the signals traveled at the rate of 980 ft/microsecond, what can be said about the location of the ship relative to the two towers?

91. On a level plane, at the same instant, you hear the sound of a rifle and that of the bullet hitting the target. What can be said about your location relative to the rifle and target?

92. Archimedes spirals The graph of an equation of the form $r = a\theta$, where a is a nonzero constant, is called an *Archimedes spiral*. Is there anything special about the widths between the successive turns of such a spiral?

93. **a.** Show that the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the polar equation

$$r = \frac{k}{1 + e\cos\theta}$$

into the Cartesian equation

$$(1 - e^2)x^2 + y^2 + 2kex - k^2 = 0.$$

b. Then apply the criteria of Section 10.3 to show that

$$\begin{aligned} e = 0 &\Rightarrow \text{circle.} \\ 0 < e < 1 &\Rightarrow \text{ellipse.} \\ e = 1 &\Rightarrow \text{parabola.} \\ e > 1 &\Rightarrow \text{hyperbola.} \end{aligned}$$

94. A satellite orbit A satellite is in an orbit that passes over the North and South Poles of the earth. When it is over the South Pole it is at the highest point of its orbit, 1000 miles above the earth's surface. Above the North Pole it is at the lowest point of its orbit, 300 miles above the earth's surface.

a. Assuming that the orbit is an ellipse with one focus at the center of the earth, find its eccentricity. (Take the diameter of the earth to be 8000 miles.)

b. Using the north–south axis of the earth as the x-axis and the center of the earth as origin, find a polar equation for the orbit.