10.5 Polar Coordinates

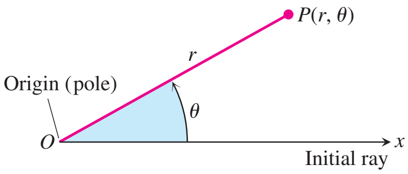


FIGURE 10.35 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

In this section, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. This has interesting consequences for graphing, as we will see in the next section.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O (Figure 10.35). Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.

> **Polar Coordinates**
>
> $P(r, \theta)$
>
> r: Directed distance from O to P
>
> θ: Directed angle from initial ray to OP

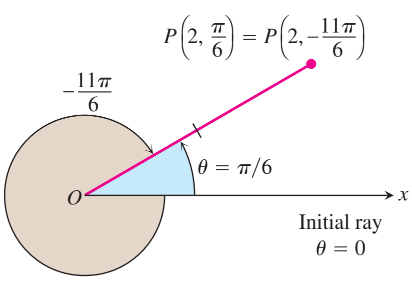


FIGURE 10.36 Polar coordinates are not unique.

As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique. For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2$, $\theta = \pi/6$. It also has coordinates $r = 2, \theta = -11\pi/6$ (Figure 10.36). There are occasions when we wish to allow r to be negative. That is why we use directed distance in defining $P(r, \theta)$. The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians counterclockwise from the initial ray and going forward 2 units (Figure 10.37). It can also be reached by turning $\pi/6$ radians counterclockwise from the initial ray and going *backward* 2 units. So the point also has polar coordinates $r = -2, \theta = \pi/6$.

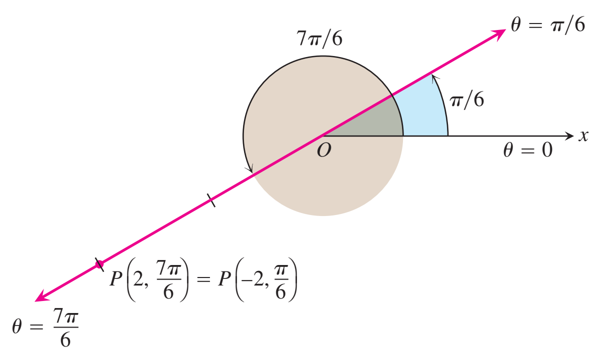


FIGURE 10.37 Polar coordinates can have negative r-values.

EXAMPLE 1 Finding Polar Coordinates

Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (Figure 10.38). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

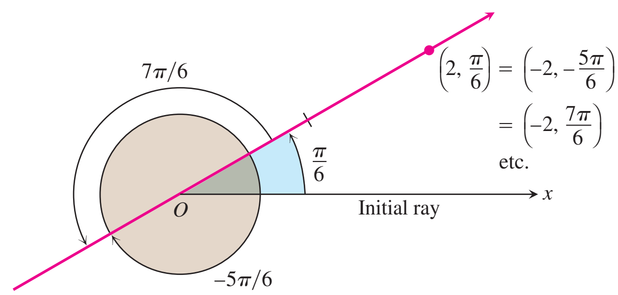


FIGURE 10.38 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \quad \ldots.$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \quad \ldots.$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■

Polar Equations and Graphs

If we hold r fixed at a constant value $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O (Figure 10.39).

If we hold θ fixed at a constant value $\theta = \theta_0$ and let r vary between $-\infty$ and ∞, the point $P(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray.

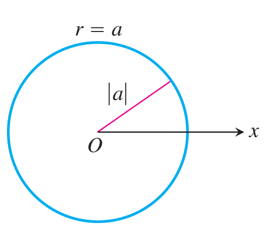


FIGURE 10.39 The polar equation for a circle is $r = a$.

Equation	Graph
$r = a$	Circle radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray

EXAMPLE 2 Finding Polar Equations for Graphs

(a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

(b) $\theta = \pi/6$, $\theta = 7\pi/6$, and $\theta = -5\pi/6$ are equations for the line in Figure 10.38.

Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments, and rays.

EXAMPLE 3 Identifying Graphs

Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \le r \le 2$ and $0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2$ and $\theta = \dfrac{\pi}{4}$

(c) $r \le 0$ and $\theta = \dfrac{\pi}{4}$

(d) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)

Solution The graphs are shown in Figure 10.40.

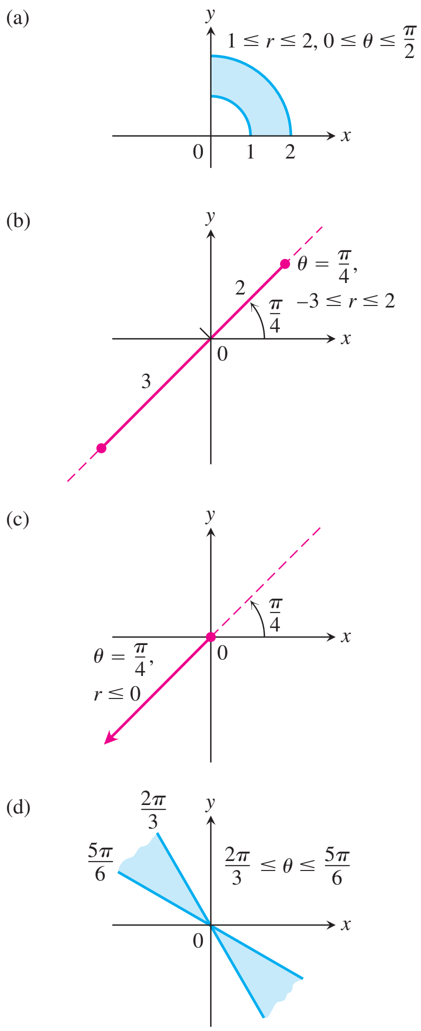


FIGURE 10.40 The graphs of typical inequalities in r and θ (Example 3).

Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The ray $\theta = \pi/2$, $r > 0$, becomes the positive y-axis (Figure 10.41). The two coordinate systems are then related by the following equations.

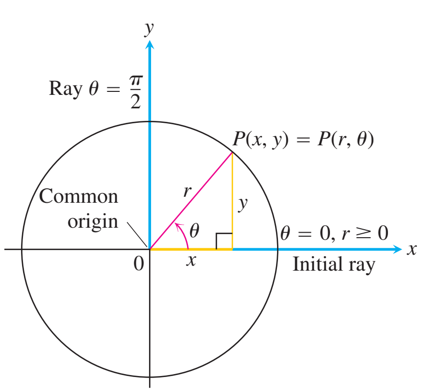


FIGURE 10.41 The usual way to relate polar and Cartesian coordinates.

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2$$

The first two of these equations uniquely determine the Cartesian coordinates x and y given the polar coordinates r and θ. On the other hand, if x and y are given, the third equation gives two possible choices for r (a positive and a negative value). For each selection, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point (x, y). The other polar coordinate representations for the point can be determined from these two, as in Example 1.

EXAMPLE 4 Equivalent Equations

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

With some curves, we are better off with polar coordinates; with others, we aren't. ■

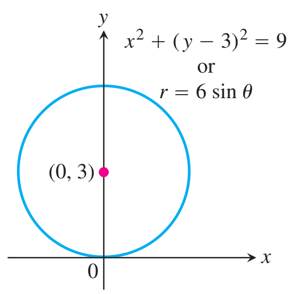


FIGURE 10.42 The circle in Example 5.

EXAMPLE 5 Converting Cartesian to Polar

Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$ (Figure 10.42).

Solution

$$
\begin{aligned}
x^2 + y^2 - 6y + 9 &= 9 && \text{Expand } (y-3)^2. \\
x^2 + y^2 - 6y &= 0 && \text{The 9's cancel.} \\
r^2 - 6r\sin\theta &= 0 && x^2 + y^2 = r^2 \\
r = 0 \quad \text{or} \quad r - 6\sin\theta &= 0 \\
r &= 6\sin\theta && \text{Includes both possibilities}
\end{aligned}
$$

We will say more about polar equations of conic sections in Section 10.8. ■

EXAMPLE 6 Converting Polar to Cartesian

Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

(a) $r\cos\theta = -4$

(b) $r^2 = 4r\cos\theta$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation:

$$
\begin{aligned}
r\cos\theta &= -4 \\
x &= -4
\end{aligned}
$$

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r\cos\theta$

The Cartesian equation:

$$
\begin{aligned}
r^2 &= 4r\cos\theta \\
x^2 + y^2 &= 4x \\
x^2 - 4x + y^2 &= 0 \\
x^2 - 4x + 4 + y^2 &= 4 && \text{Completing the square} \\
(x - 2)^2 + y^2 &= 4
\end{aligned}
$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation:

$$\begin{aligned} r(2\cos\theta - \sin\theta) &= 4 \\ 2r\cos\theta - r\sin\theta &= 4 \\ 2x - y &= 4 \\ y &= 2x - 4 \end{aligned}$$

The graph: Line, slope $m = 2$, y-intercept $b = -4$ ■

EXERCISES 10.5

Polar Coordinate Pairs

1. Which polar coordinate pairs label the same point?
   a. $(3, 0)$ **b.** $(-3, 0)$ **c.** $(2, 2\pi/3)$
   d. $(2, 7\pi/3)$ **e.** $(-3, \pi)$ **f.** $(2, \pi/3)$
   g. $(-3, 2\pi)$ **h.** $(-2, -\pi/3)$
2. Which polar coordinate pairs label the same point?
   a. $(-2, \pi/3)$ **b.** $(2, -\pi/3)$ **c.** (r, θ)
   d. $(r, \theta + \pi)$ **e.** $(-r, \theta)$ **f.** $(2, -2\pi/3)$
   g. $(-r, \theta + \pi)$ **h.** $(-2, 2\pi/3)$
3. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.
   a. $(2, \pi/2)$ **b.** $(2, 0)$
   c. $(-2, \pi/2)$ **d.** $(-2, 0)$
4. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.
   a. $(3, \pi/4)$ **b.** $(-3, \pi/4)$
   c. $(3, -\pi/4)$ **d.** $(-3, -\pi/4)$

Polar to Cartesian Coordinates

5. Find the Cartesian coordinates of the points in Exercise 1.
6. Find the Cartesian coordinates of the following points (given in polar coordinates).
   a. $(\sqrt{2}, \pi/4)$ **b.** $(1, 0)$
   c. $(0, \pi/2)$ **d.** $(-\sqrt{2}, \pi/4)$
   e. $(-3, 5\pi/6)$ **f.** $(5, \tan^{-1}(4/3))$
   g. $(-1, 7\pi)$ **h.** $(2\sqrt{3}, 2\pi/3)$

Graphing Polar Equations and Inequalities

Graph the sets of points whose polar coordinates satisfy the equations and inequalities in Exercises 7–22.

7. $r = 2$
8. $0 \le r \le 2$
9. $r \ge 1$
10. $1 \le r \le 2$
11. $0 \le \theta \le \pi/6, \quad r \ge 0$
12. $\theta = 2\pi/3, \quad r \le -2$
13. $\theta = \pi/3, \quad -1 \le r \le 3$
14. $\theta = 11\pi/4, \quad r \ge -1$
15. $\theta = \pi/2, \quad r \ge 0$
16. $\theta = \pi/2, \quad r \le 0$
17. $0 \le \theta \le \pi, \quad r = 1$
18. $0 \le \theta \le \pi, \quad r = -1$
19. $\pi/4 \le \theta \le 3\pi/4, \quad 0 \le r \le 1$
20. $-\pi/4 \le \theta \le \pi/4, \quad -1 \le r \le 1$
21. $-\pi/2 \le \theta \le \pi/2, \quad 1 \le r \le 2$
22. $0 \le \theta \le \pi/2, \quad 1 \le |r| \le 2$

Polar to Cartesian Equations

Replace the polar equations in Exercises 23–48 by equivalent Cartesian equations. Then describe or identify the graph.

23. $r\cos\theta = 2$
24. $r\sin\theta = -1$
25. $r\sin\theta = 0$
26. $r\cos\theta = 0$
27. $r = 4\csc\theta$
28. $r = -3\sec\theta$
29. $r\cos\theta + r\sin\theta = 1$
30. $r\sin\theta = r\cos\theta$
31. $r^2 = 1$
32. $r^2 = 4r\sin\theta$
33. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$
34. $r^2\sin 2\theta = 2$
35. $r = \cot\theta\csc\theta$
36. $r = 4\tan\theta\sec\theta$
37. $r = \csc\theta\, e^{r\cos\theta}$
38. $r\sin\theta = \ln r + \ln\cos\theta$
39. $r^2 + 2r^2\cos\theta\sin\theta = 1$
40. $\cos^2\theta = \sin^2\theta$
41. $r^2 = -4r\cos\theta$
42. $r^2 = -6r\sin\theta$
43. $r = 8\sin\theta$
44. $r = 3\cos\theta$
45. $r = 2\cos\theta + 2\sin\theta$
46. $r = 2\cos\theta - \sin\theta$
47. $r\sin\left(\theta + \dfrac{\pi}{6}\right) = 2$
48. $r\sin\left(\dfrac{2\pi}{3} - \theta\right) = 5$

Cartesian to Polar Equations

Replace the Cartesian equations in Exercises 49–62 by equivalent polar equations.

49. $x = 7$
50. $y = 1$
51. $x = y$
52. $x - y = 3$
53. $x^2 + y^2 = 4$
54. $x^2 - y^2 = 1$
55. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$
56. $xy = 2$

57. $y^2 = 4x$

58. $x^2 + xy + y^2 = 1$

59. $x^2 + (y - 2)^2 = 4$

60. $(x - 5)^2 + y^2 = 25$

61. $(x - 3)^2 + (y + 1)^2 = 4$

62. $(x + 2)^2 + (y - 5)^2 = 16$

Theory and Examples

63. Find all polar coordinates of the origin.

64. Vertical and horizontal lines

a. Show that every vertical line in the xy-plane has a polar equation of the form $r = a \sec \theta$.

b. Find the analogous polar equation for horizontal lines in the xy-plane.

10.6 Graphing in Polar Coordinates

This section describes techniques for graphing equations in polar coordinates.

Symmetry

Figure 10.43 illustrates the standard polar coordinate tests for symmetry.

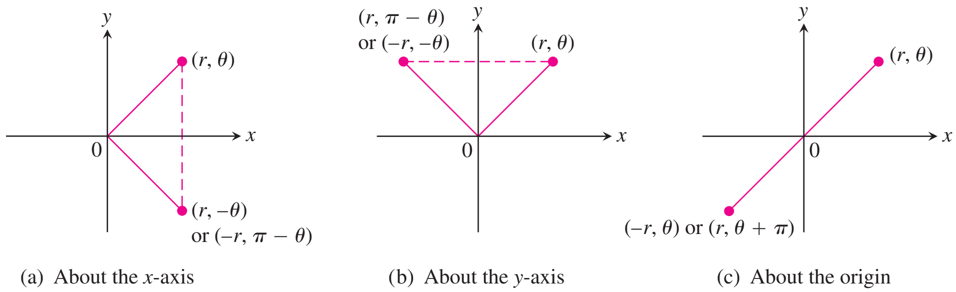


FIGURE 10.43 Three tests for symmetry in polar coordinates.

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 10.43a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 10.43b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 10.43c).

Slope

The slope of a polar curve $r = f(\theta)$ is given by dy/dx, not by $r' = df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

If f is a differentiable function of θ, then so are x and y and, when $dx/d\theta \neq 0$, we can calculate dy/dx from the parametric formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \qquad \text{Section 3.5, Equation (2) with } t = \theta$$

$$= \frac{\dfrac{d}{d\theta}(f(\theta) \cdot \sin\theta)}{\dfrac{d}{d\theta}(f(\theta) \cdot \cos\theta)}$$

$$= \frac{\dfrac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\dfrac{df}{d\theta}\cos\theta - f(\theta)\sin\theta} \qquad \text{Product Rule for derivatives}$$

Slope of the Curve $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta},$$

provided $dx/d\theta \neq 0$ at (r, θ).

If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$, and the slope equation gives

$$\left.\frac{dy}{dx}\right|_{(0,\,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0.$$

If the graph of $r = f(\theta)$ passes through the origin at the value $\theta = \theta_0$, the slope of the curve there is $\tan\theta_0$. The reason we say "slope at $(0, \theta_0)$" and not just "slope at the origin" is that a polar curve may pass through the origin (or any point) more than once, with different slopes at different θ-values. This is not the case in our first example, however.

EXAMPLE 1 A Cardioid

Graph the curve $r = 1 - \cos\theta$.

Solution The curve is symmetric about the x-axis because

$$(r, \theta) \text{ on the graph} \Rightarrow r = 1 - \cos\theta$$
$$\Rightarrow r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta)$$
$$\Rightarrow (r, -\theta) \text{ on the graph}.$$

As θ increases from 0 to π, $\cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ continues on from π to 2π, $\cos\theta$ increases from -1 back to 1 and r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π.

The curve leaves the origin with slope $\tan(0) = 0$ and returns to the origin with slope $\tan(2\pi) = 0$.

We make a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the x-axis to complete the graph (Figure 10.44). The curve is called a *cardioid* because of its heart shape. Cardioid shapes appear in the cams that direct the even layering of thread on bobbins and reels, and in the signal-strength pattern of certain radio antennas. ■

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)

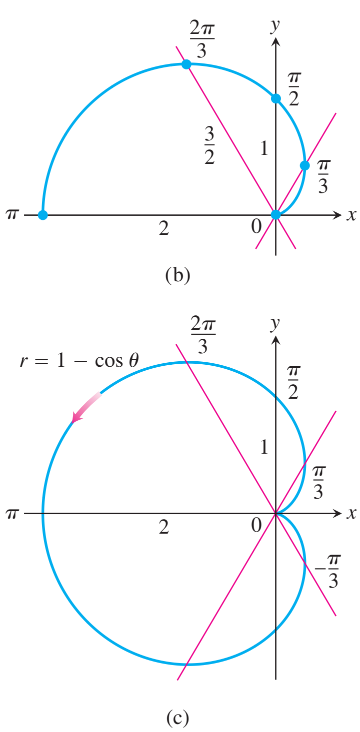


FIGURE 10.44 The steps in graphing the cardioid $r = 1 - \cos\theta$ (Example 1). The arrow shows the direction of increasing θ.

You Try It

EXAMPLE 2 Graph the Curve $r^2 = 4\cos\theta$.

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the x-axis because

$$\begin{aligned}(r, \theta) \text{ on the graph } &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.}\end{aligned}$$

The curve is also symmetric about the origin because

$$\begin{aligned}(r, \theta) \text{ on the graph } &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow (-r)^2 = 4\cos\theta \\ &\Rightarrow (-r, \theta) \text{ on the graph.}\end{aligned}$$

Together, these two symmetries imply symmetry about the y-axis.

The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because $\tan\theta$ is infinite.

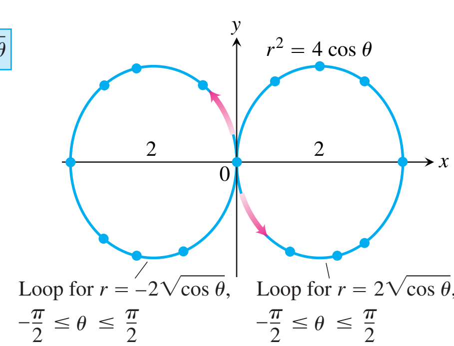
For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of r:

$$r = \pm 2\sqrt{\cos\theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve (Figure 10.45). ■

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

FIGURE 10.45 The graph of $r^2 = 4\cos\theta$. The arrows show the direction of increasing θ. The values of r in the table are rounded (Example 2).

A Technique for Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ)-values, plot the corresponding points, and connect them in order of increasing θ. This can work well if enough points have been plotted to reveal all the loops and dimples in the graph. Another method of graphing that is usually quicker and more reliable is to

1. first graph $r = f(\theta)$ in the *Cartesian* $r\theta$-plane,
2. then use the Cartesian graph as a "table" and guide to sketch the *polar* coordinate graph.

This method is better than simple point plotting because the first Cartesian graph, even when hastily drawn, shows at a glance where r is positive, negative, and nonexistent, as well as where r is increasing and decreasing. Here's an example.

EXAMPLE 3 A Lemniscate

Graph the curve

$$r^2 = \sin 2\theta .$$

Solution Here we begin by plotting r^2 (not r) as a function of θ in the Cartesian $r^2\theta$-plane. See Figure 10.46a. We pass from there to the graph of $r = \pm\sqrt{\sin 2\theta}$ in the $r\theta$-plane (Figure 10.46b), and then draw the polar graph (Figure 10.46c). The graph in Figure 10.46b "covers" the final polar graph in Figure 10.46c twice. We could have managed with either loop alone, with the two upper halves, or with the two lower halves. The double covering does no harm, however, and we actually learn a little more about the behavior of the function this way. ■

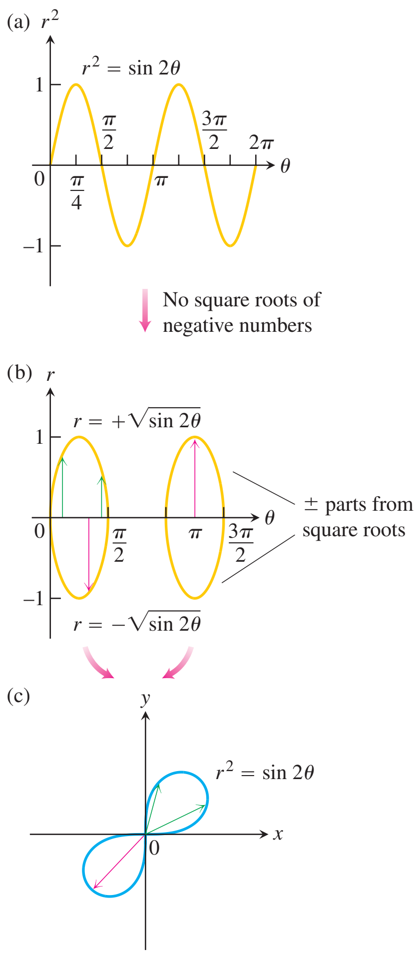


FIGURE 10.46 To plot $r = f(\theta)$ in the Cartesian $r\theta$-plane in (b), we first plot $r^2 = \sin 2\theta$ in the $r^2\theta$-plane in (a) and then ignore the values of θ for which $\sin 2\theta$ is negative. The radii from the sketch in (b) cover the polar graph of the lemniscate in (c) twice (Example 3).

Finding Points Where Polar Graphs Intersect

The fact that we can represent a point in different ways in polar coordinates makes extra care necessary in deciding when a point lies on the graph of a polar equation and in determining the points in which polar graphs intersect. The problem is that a point of intersection may satisfy the equation of one curve with polar coordinates that are different from the ones with which it satisfies the equation of another curve. Thus, solving the equations of two curves simultaneously may not identify all their points of intersection. One sure way to identify all the points of intersection is to graph the equations.

EXAMPLE 4 Deceptive Polar Coordinates

Show that the point $(2, \pi/2)$ lies on the curve $r = 2\cos 2\theta$.

Solution It may seem at first that the point $(2, \pi/2)$ does not lie on the curve because substituting the given coordinates into the equation gives

$$2 = 2\cos 2\left(\frac{\pi}{2}\right) = 2\cos \pi = -2,$$

which is not a true equality. The magnitude is right, but the sign is wrong. This suggests looking for a pair of coordinates for the same given point in which r is negative, for example, $(-2, -(\pi/2))$. If we try these in the equation $r = 2\cos 2\theta$, we find

$$-2 = 2\cos 2\left(-\frac{\pi}{2}\right) = 2(-1) = -2,$$

and the equation is satisfied. The point $(2, \pi/2)$ does lie on the curve. ■

EXAMPLE 5 Elusive Intersection Points

Find the points of intersection of the curves

$$r^2 = 4\cos\theta \qquad \text{and} \qquad r = 1 - \cos\theta .$$

HISTORICAL BIOGRAPHY

Johannes Kepler
(1571–1630)

Solution In Cartesian coordinates, we can always find the points where two curves cross by solving their equations simultaneously. In polar coordinates, the story is different. Simultaneous solution may reveal some intersection points without revealing others. In this example, simultaneous solution reveals only two of the four intersection points. The others are found by graphing. (Also, see Exercise 49.)

If we substitute $\cos\theta = r^2/4$ in the equation $r = 1 - \cos\theta$, we get

$$r = 1 - \cos\theta = 1 - \frac{r^2}{4}$$

$$4r = 4 - r^2$$

$$r^2 + 4r - 4 = 0$$

$$r = -2 \pm 2\sqrt{2}. \qquad \text{Quadratic formula}$$

The value $r = -2 - 2\sqrt{2}$ has too large an absolute value to belong to either curve. The values of θ corresponding to $r = -2 + 2\sqrt{2}$ are

$$\theta = \cos^{-1}(1 - r) \qquad \text{From } r = 1 - \cos\theta$$

$$= \cos^{-1}\left(1 - \left(2\sqrt{2} - 2\right)\right) \qquad \text{Set } r = 2\sqrt{2} - 2.$$

$$= \cos^{-1}\left(3 - 2\sqrt{2}\right)$$

$$= \pm 80^\circ. \qquad \text{Rounded to the nearest degree}$$

We have thus identified two intersection points: $(r, \theta) = (2\sqrt{2} - 2, \pm 80^\circ)$.

If we graph the equations $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$ together (Figure 10.47), as we can now do by combining the graphs in Figures 10.44 and 10.45, we see that the curves also intersect at the point $(2, \pi)$ and the origin. Why weren't the r-values of these points revealed by the simultaneous solution? The answer is that the points $(0, 0)$ and $(2, \pi)$ are not on the curves "simultaneously." They are not reached at the same value of θ. On the curve $r = 1 - \cos\theta$, the point $(2, \pi)$ is reached when $\theta = \pi$. On the curve $r^2 = 4\cos\theta$, it is reached when $\theta = 0$, where it is identified not by the coordinates $(2, \pi)$, which do not satisfy the equation, but by the coordinates $(-2, 0)$, which do. Similarly, the cardioid reaches the origin when $\theta = 0$, but the curve $r^2 = 4\cos\theta$ reaches the origin when $\theta = \pi/2$. ■

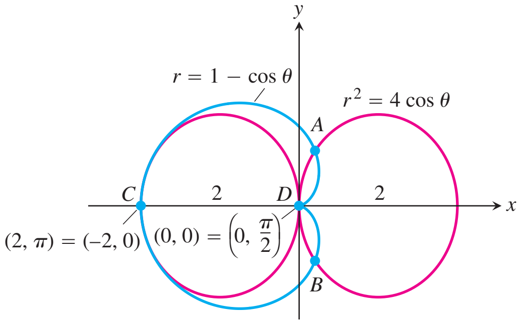


FIGURE 10.47 The four points of intersection of the curves $r = 1 - \cos\theta$ and $r^2 = 4\cos\theta$ (Example 5). Only A and B were found by simultaneous solution. The other two were disclosed by graphing.

USING TECHNOLOGY **Graphing Polar Curves Parametrically**

For complicated polar curves we may need to use a graphing calculator or computer to graph the curve. If the device does not plot polar graphs directly, we can convert $r = f(\theta)$ into parametric form using the equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then we use the device to draw a parametrized curve in the Cartesian xy-plane. It may be required to use the parameter t rather than θ for the graphing device.

EXERCISES 10.6

Symmetries and Polar Graphs

Identify the symmetries of the curves in Exercises 1–12. Then sketch the curves.

1. $r = 1 + \cos\theta$

2. $r = 2 - 2\cos\theta$

3. $r = 1 - \sin\theta$

4. $r = 1 + \sin\theta$

5. $r = 2 + \sin\theta$

6. $r = 1 + 2\sin\theta$

7. $r = \sin(\theta/2)$

8. $r = \cos(\theta/2)$

9. $r^2 = \cos\theta$

10. $r^2 = \sin\theta$

11. $r^2 = -\sin\theta$

12. $r^2 = -\cos\theta$

Graph the lemniscates in Exercises 13–16. What symmetries do these curves have?

13. $r^2 = 4\cos 2\theta$

14. $r^2 = 4\sin 2\theta$

15. $r^2 = -\sin 2\theta$

16. $r^2 = -\cos 2\theta$

Slopes of Polar Curves

Find the slopes of the curves in Exercises 17–20 at the given points. Sketch the curves along with their tangents at these points.

17. Cardioid $r = -1 + \cos\theta; \quad \theta = \pm\pi/2$

18. Cardioid $r = -1 + \sin\theta; \quad \theta = 0, \pi$

19. Four-leaved rose $r = \sin 2\theta; \quad \theta = \pm\pi/4, \pm 3\pi/4$

20. Four-leaved rose $r = \cos 2\theta; \quad \theta = 0, \pm\pi/2, \pi$

Limaçons

Graph the limaçons in Exercises 21–24. Limaçon ("*lee*-ma-sahn") is Old French for "snail." You will understand the name when you graph the limaçons in Exercise 21. Equations for limaçons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. There are four basic shapes.

21. Limaçons with an inner loop

a. $r = \dfrac{1}{2} + \cos\theta$ **b.** $r = \dfrac{1}{2} + \sin\theta$

22. Cardioids

a. $r = 1 - \cos\theta$ **b.** $r = -1 + \sin\theta$

23. Dimpled limaçons

a. $r = \dfrac{3}{2} + \cos\theta$ **b.** $r = \dfrac{3}{2} - \sin\theta$

24. Oval limaçons

a. $r = 2 + \cos\theta$ **b.** $r = -2 + \sin\theta$

Graphing Polar Inequalities

25. Sketch the region defined by the inequalities $-1 \le r \le 2$ and $-\pi/2 \le \theta \le \pi/2$.

26. Sketch the region defined by the inequalities $0 \le r \le 2\sec\theta$ and $-\pi/4 \le \theta \le \pi/4$.

In Exercises 27 and 28, sketch the region defined by the inequality.

27. $0 \le r \le 2 - 2\cos\theta$

28. $0 \le r^2 \le \cos\theta$

Intersections

29. Show that the point $(2, 3\pi/4)$ lies on the curve $r = 2\sin 2\theta$.

30. Show that $(1/2, 3\pi/2)$ lies on the curve $r = -\sin(\theta/3)$.

Find the points of intersection of the pairs of curves in Exercises 31–38.

31. $r = 1 + \cos\theta, \quad r = 1 - \cos\theta$

32. $r = 1 + \sin\theta, \quad r = 1 - \sin\theta$

33. $r = 2\sin\theta, \quad r = 2\sin 2\theta$

34. $r = \cos\theta, \quad r = 1 - \cos\theta$

35. $r = \sqrt{2}, \quad r^2 = 4\sin\theta$

36. $r^2 = \sqrt{2}\sin\theta, \quad r^2 = \sqrt{2}\cos\theta$

37. $r = 1, \quad r^2 = 2\sin 2\theta$

38. $r^2 = \sqrt{2}\cos 2\theta, \quad r^2 = \sqrt{2}\sin 2\theta$

T Find the points of intersection of the pairs of curves in Exercises 39–42.

39. $r^2 = \sin 2\theta, \quad r^2 = \cos 2\theta$

40. $r = 1 + \cos\dfrac{\theta}{2}, \quad r = 1 - \sin\dfrac{\theta}{2}$

41. $r = 1, \quad r = 2\sin 2\theta$

42. $r = 1, \quad r^2 = 2\sin 2\theta$