Exercises 35–38 give foci and corresponding directrices of hyperbolas centered at the origin of the xy-plane. In each case, find the hyperbola's eccentricity. Then find the hyperbola's standard-form equation.

35. Focus: $(4, 0)$
Directrix: $x = 2$

36. Focus: $(\sqrt{10}, 0)$
Directrix: $x = \sqrt{2}$

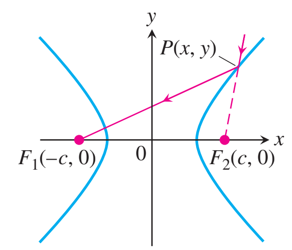
37. Focus: $(-2, 0)$
Directrix: $x = -\frac{1}{2}$

38. Focus: $(-6, 0)$
Directrix: $x = -2$

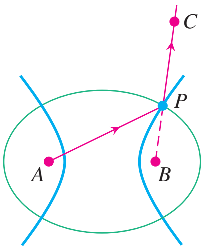
39. A hyperbola of eccentricity $3/2$ has one focus at $(1, -3)$. The corresponding directrix is the line $y = 2$. Find an equation for the hyperbola.

T **40. The effect of eccentricity on a hyperbola's shape** What happens to the graph of a hyperbola as its eccentricity increases? To find out, rewrite the equation $(x^2/a^2) - (y^2/b^2) = 1$ in terms of a and e instead of a and b. Graph the hyperbola for various values of e and describe what you find.

41. The reflective property of hyperbolas Show that a ray of light directed toward one focus of a hyperbolic mirror, as in the accompanying figure, is reflected toward the other focus. (*Hint:* Show that the tangent to the hyperbola at P bisects the angle made by segments PF_1 and PF_2.)



42. A confocal ellipse and hyperbola Show that an ellipse and a hyperbola that have the same foci A and B, as in the accompanying figure, cross at right angles at their point of intersection. (*Hint:* A ray of light from focus A that met the hyperbola at P would be reflected from the hyperbola as if it came directly from B (Exercise 41). The same ray would be reflected off the ellipse to pass through B (Exercise 22).)



10.3 Quadratic Equations and Rotations

In this section, we examine the Cartesian graph of any equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

in which A, B, and C are not all zero, and show that it is nearly always a conic section. The exceptions are the cases in which there is no graph at all or the graph consists of two parallel lines. It is conventional to call all graphs of Equation (1), curved or not, **quadratic curves**.

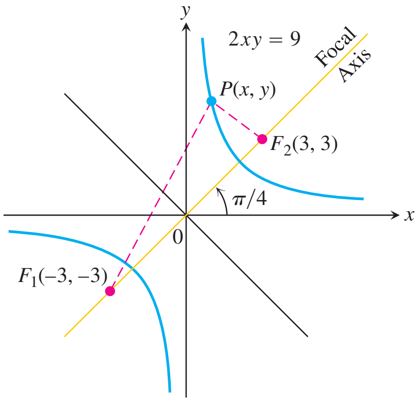


FIGURE 10.22 The focal axis of the hyperbola $2xy = 9$ makes an angle of $\pi/4$ radians with the positive x-axis.

The Cross Product Term

You may have noticed that the term Bxy did not appear in the equations for the conic sections in Section 10.1. This happened because the axes of the conic sections ran parallel to (in fact, coincided with) the coordinate axes.

To see what happens when the parallelism is absent, let us write an equation for a hyperbola with $a = 3$ and foci at $F_1(-3, -3)$ and $F_2(3, 3)$ (Figure 10.22). The equation $|PF_1 - PF_2| = 2a$ becomes $|PF_1 - PF_2| = 2(3) = 6$ and

$$\sqrt{(x+3)^2 + (y+3)^2} - \sqrt{(x-3)^2 + (y-3)^2} = \pm 6.$$

When we transpose one radical, square, solve for the radical that still appears, and square again, the equation reduces to

$$2xy = 9, \tag{2}$$

a case of Equation (1) in which the cross product term is present. The asymptotes of the hyperbola in Equation (2) are the x- and y-axes, and the focal axis makes an angle of $\pi/4$

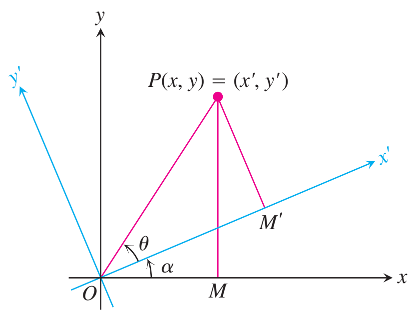


FIGURE 10.23 A counterclockwise rotation through angle α about the origin.

radians with the positive x-axis. As in this example, the cross product term is present in Equation (1) only when the axes of the conic are tilted.

To eliminate the xy-term from the equation of a conic, we rotate the coordinate axes to eliminate the "tilt" in the axes of the conic. The equations for the rotations we use are derived in the following way. In the notation of Figure 10.23, which shows a counterclockwise rotation about the origin through an angle α,

$$\begin{aligned} x &= OM = OP\cos(\theta+\alpha) = OP\cos\theta\cos\alpha - OP\sin\theta\sin\alpha \\ y &= MP = OP\sin(\theta+\alpha) = OP\cos\theta\sin\alpha + OP\sin\theta\cos\alpha. \end{aligned} \tag{3}$$

Since

$$OP\cos\theta = OM' = x'$$

and

$$OP\sin\theta = M'P = y',$$

Equations (3) reduce to the following.

Equations for Rotating Coordinate Axes

$$\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha \\ y &= x'\sin\alpha + y'\cos\alpha \end{aligned} \tag{4}$$

You Try It

EXAMPLE 1 Finding an Equation for a Hyperbola

The x- and y-axes are rotated through an angle of $\pi/4$ radians about the origin. Find an equation for the hyperbola $2xy = 9$ in the new coordinates.

Solution Since $\cos\pi/4 = \sin\pi/4 = 1/\sqrt{2}$, we substitute

$$x = \frac{x'-y'}{\sqrt{2}}, \qquad y = \frac{x'+y'}{\sqrt{2}}$$

from Equations (4) into the equation $2xy = 9$ and obtain

$$\begin{aligned} 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) &= 9 \\ x'^2 - y'^2 &= 9 \\ \frac{x'^2}{9} - \frac{y'^2}{9} &= 1. \end{aligned}$$

See Figure 10.24. ■

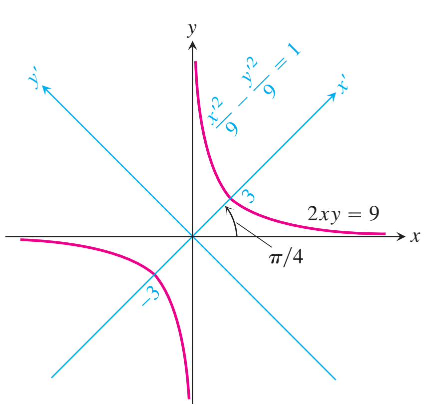


FIGURE 10.24 The hyperbola in Example 1 (x' and y' are the coordinates).

If we apply Equations (4) to the quadratic equation (1), we obtain a new quadratic equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0. \tag{5}$$

The new and old coefficients are related by the equations

$$
\begin{aligned}
A' &= A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha \\
B' &= B\cos 2\alpha + (C - A)\sin 2\alpha \\
C' &= A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha \\
D' &= D\cos\alpha + E\sin\alpha \\
E' &= -D\sin\alpha + E\cos\alpha \\
F' &= F.
\end{aligned}
\tag{6}
$$

These equations show, among other things, that if we start with an equation for a curve in which the cross product term is present ($B \neq 0$), we can find a rotation angle α that produces an equation in which no cross product term appears ($B' = 0$). To find α, we set $B' = 0$ in the second equation in (6) and solve the resulting equation,

$$B\cos 2\alpha + (C - A)\sin 2\alpha = 0,$$

for α. In practice, this means determining α from one of the two equations

Angle of Rotation

$$\cot 2\alpha = \frac{A - C}{B} \qquad \text{or} \qquad \tan 2\alpha = \frac{B}{A - C}. \tag{7}$$

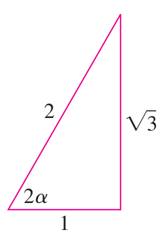


FIGURE 10.25 This triangle identifies $2\alpha = \cot^{-1}(1/\sqrt{3})$ as $\pi/3$ (Example 2).

EXAMPLE 2 Finding the Angle of Rotation

The coordinate axes are to be rotated through an angle α to produce an equation for the curve

$$2x^2 + \sqrt{3}\,xy + y^2 - 10 = 0$$

that has no cross product term. Find α and the new equation. Identify the curve.

Solution The equation $2x^2 + \sqrt{3}\,xy + y^2 - 10 = 0$ has $A = 2, B = \sqrt{3}$, and $C = 1$. We substitute these values into Equation (7) to find α:

$$\cot 2\alpha = \frac{A - C}{B} = \frac{2 - 1}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

From the right triangle in Figure 10.25, we see that one appropriate choice of angle is $2\alpha = \pi/3$, so we take $\alpha = \pi/6$. Substituting $\alpha = \pi/6$, $A = 2$, $B = \sqrt{3}$, $C = 1$, $D = E = 0$, and $F = -10$ into Equations (6) gives

$$A' = \frac{5}{2}, \qquad B' = 0, \qquad C' = \frac{1}{2}, \qquad D' = E' = 0, \qquad F' = -10.$$

Equation (5) then gives

$$\frac{5}{2}x'^2 + \frac{1}{2}y'^2 - 10 = 0, \qquad \text{or} \qquad \frac{x'^2}{4} + \frac{y'^2}{20} = 1.$$

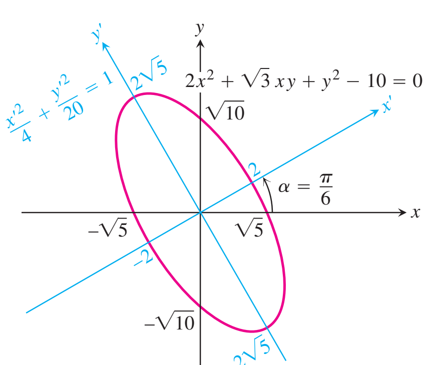


FIGURE 10.26 The conic section in Example 2.

The curve is an ellipse with foci on the new y'-axis (Figure 10.26). ■

Possible Graphs of Quadratic Equations

We now return to the graph of the general quadratic equation.

Since axes can always be rotated to eliminate the cross product term, there is no loss of generality in assuming that this has been done and that our equation has the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \tag{8}$$

Equation (8) represents

(a) a *circle* if $A = C \neq 0$ (special cases: the graph is a point or there is no graph at all);

(b) a *parabola* if Equation (8) is quadratic in one variable and linear in the other;

(c) an *ellipse* if A and C are both positive or both negative (special cases: circles, a single point, or no graph at all);

(d) a *hyperbola* if A and C have opposite signs (special case: a pair of intersecting lines);

(e) a *straight line* if A and C are zero and at least one of D and E is different from zero;

(f) *one or two straight lines* if the left-hand side of Equation (8) can be factored into the product of two linear factors.

See Table 10.3 for examples.

TABLE 10.3 Examples of quadratic curves $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

	A	B	C	D	E	F	**Equation**	**Remarks**
Circle	1		1			-4	$x^2 + y^2 = 4$	$A = C$; $F < 0$
Parabola			1	-9			$y^2 = 9x$	Quadratic in y, linear in x
Ellipse	4		9			-36	$4x^2 + 9y^2 = 36$	A, C have same sign, $A \neq C$; $F < 0$
Hyperbola	1		-1			-1	$x^2 - y^2 = 1$	A, C have opposite signs
One line (still a conic section)	1						$x^2 = 0$	y-axis
Intersecting lines (still a conic section)		1		1	-1	-1	$xy + x - y - 1 = 0$	Factors to $(x - 1)(y + 1) = 0$, so $x = 1$, $y = -1$
Parallel lines (not a conic section)	1			-3		2	$x^2 - 3x + 2 = 0$	Factors to $(x - 1)(x - 2) = 0$, so $x = 1$, $x = 2$
Point	1		1				$x^2 + y^2 = 0$	The origin
No graph	1					1	$x^2 = -1$	No graph

The Discriminant Test

We do not need to eliminate the xy-term from the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{9}$$

to tell what kind of conic section the equation represents. If this is the only information we want, we can apply the following test instead.

As we have seen, if $B \neq 0$, then rotating the coordinate axes through an angle α that satisfies the equation

$$\cot 2\alpha = \frac{A - C}{B} \tag{10}$$

will change Equation (9) into an equivalent form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \tag{11}$$

without a cross product term.

Now, the graph of Equation (11) is a (real or degenerate)

(a) *parabola* if A' or $C' = 0$; that is, if $A'C' = 0$;

(b) *ellipse* if A' and C' have the same sign; that is, if $A'C' > 0$;

(c) *hyperbola* if A' and C' have opposite signs; that is, if $A'C' < 0$.

It can also be verified from Equations (6) that for any rotation of axes,

$$B^2 - 4AC = B'^2 - 4A'C'. \tag{12}$$

This means that the quantity $B^2 - 4AC$ is not changed by a rotation. But when we rotate through the angle α given by Equation (10), B' becomes zero, so

$$B^2 - 4AC = -4A'C'.$$

Since the curve is a parabola if $A'C' = 0$, an ellipse if $A'C' > 0$, and a hyperbola if $A'C' < 0$, the curve must be a parabola if $B^2 - 4AC = 0$, an ellipse if $B^2 - 4AC < 0$, and a hyperbola if $B^2 - 4AC > 0$. The number $B^2 - 4AC$ is called the **discriminant** of Equation (9).

The Discriminant Test

With the understanding that occasional degenerate cases may arise, the quadratic curve $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

(a) a **parabola** if $B^2 - 4AC = 0$,
(b) an **ellipse** if $B^2 - 4AC < 0$,
(c) a **hyperbola** if $B^2 - 4AC > 0$.

You Try It You Try It

EXAMPLE 3 Applying the Discriminant Test

(a) $3x^2 - 6xy + 3y^2 + 2x - 7 = 0$ represents a parabola because

$$B^2 - 4AC = (-6)^2 - 4 \cdot 3 \cdot 3 = 36 - 36 = 0.$$

(b) $x^2 + xy + y^2 - 1 = 0$ represents an ellipse because

$$B^2 - 4AC = (1)^2 - 4 \cdot 1 \cdot 1 = -3 < 0.$$

(c) $xy - y^2 - 5y + 1 = 0$ represents a hyperbola because

$$B^2 - 4AC = (1)^2 - 4(0)(-1) = 1 > 0.$$

■

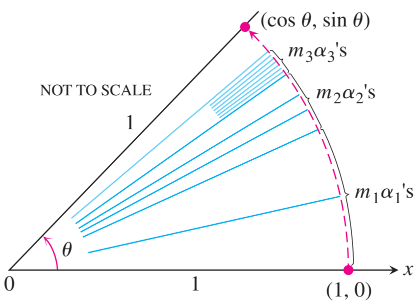


FIGURE 10.27 To calculate the sine and cosine of an angle θ between 0 and 2π, the calculator rotates the point (1, 0) to an appropriate location on the unit circle and displays the resulting coordinates.

USING TECHNOLOGY How Calculators Use Rotations to Evaluate Sines and Cosines

Some calculators use rotations to calculate sines and cosines of arbitrary angles. The procedure goes something like this: The calculator has, stored,

1. ten angles or so, say

$$\alpha_1 = \sin^{-1}(10^{-1}), \qquad \alpha_2 = \sin^{-1}(10^{-2}), \qquad \dots, \qquad \alpha_{10} = \sin^{-1}(10^{-10}),$$

and

2. twenty numbers, the sines and cosines of the angles $\alpha_1, \alpha_2, \dots, \alpha_{10}$.

To calculate the sine and cosine of an arbitrary angle θ, we enter θ (in radians) into the calculator. The calculator subtracts or adds multiples of 2π to θ to replace θ by the angle between 0 and 2π that has the same sine and cosine as θ (we continue to call the angle θ). The calculator then "writes" θ as a sum of multiples of α_1 (as many as possible without overshooting) plus multiples of α_2 (again, as many as possible), and so on, working its way to α_{10}. This gives

$$\theta \approx m_1\alpha_1 + m_2\alpha_2 + \cdots + m_{10}\alpha_{10}.$$

The calculator then rotates the point (1, 0) through m_1 copies of α_1 (through α_1, m_1 times in succession), plus m_2 copies of α_2, and so on, finishing off with m_{10} copies of α_{10} (Figure 10.27). The coordinates of the final position of (1, 0) on the unit circle are the values the calculator gives for $(\cos\theta, \sin\theta)$.

EXERCISES 10.3

Using the Discriminant

Use the discriminant $B^2 - 4AC$ to decide whether the equations in Exercises 1–16 represent parabolas, ellipses, or hyperbolas.

1. $x^2 - 3xy + y^2 - x = 0$

2. $3x^2 - 18xy + 27y^2 - 5x + 7y = -4$

3. $3x^2 - 7xy + \sqrt{17}\,y^2 = 1$

4. $2x^2 - \sqrt{15}\,xy + 2y^2 + x + y = 0$

5. $x^2 + 2xy + y^2 + 2x - y + 2 = 0$

6. $2x^2 - y^2 + 4xy - 2x + 3y = 6$

7. $x^2 + 4xy + 4y^2 - 3x = 6$

8. $x^2 + y^2 + 3x - 2y = 10$ **9.** $xy + y^2 - 3x = 5$

10. $3x^2 + 6xy + 3y^2 - 4x + 5y = 12$

11. $3x^2 - 5xy + 2y^2 - 7x - 14y = -1$

12. $2x^2 - 4.9xy + 3y^2 - 4x = 7$

13. $x^2 - 3xy + 3y^2 + 6y = 7$

14. $25x^2 + 21xy + 4y^2 - 350x = 0$

15. $6x^2 + 3xy + 2y^2 + 17y + 2 = 0$

16. $3x^2 + 12xy + 12y^2 + 435x - 9y + 72 = 0$

Rotating Coordinate Axes

In Exercises 17–26, rotate the coordinate axes to change the given equation into an equation that has no cross product (xy) term. Then identify the graph of the equation. (The new equations will vary with the size and direction of the rotation you use.)

17. $xy = 2$ **18.** $x^2 + xy + y^2 = 1$

19. $3x^2 + 2\sqrt{3}\,xy + y^2 - 8x + 8\sqrt{3}\,y = 0$

20. $x^2 - \sqrt{3}\,xy + 2y^2 = 1$ **21.** $x^2 - 2xy + y^2 = 2$

22. $3x^2 - 2\sqrt{3}\,xy + y^2 = 1$

23. $\sqrt{2}\,x^2 + 2\sqrt{2}\,xy + \sqrt{2}\,y^2 - 8x + 8y = 0$

24. $xy - y - x + 1 = 0$

25. $3x^2 + 2xy + 3y^2 = 19$

26. $3x^2 + 4\sqrt{3}\,xy - y^2 = 7$

27. Find the sine and cosine of an angle in Quadrant I through which the coordinate axes can be rotated to eliminate the cross product term from the equation

$$14x^2 + 16xy + 2y^2 - 10x + 26{,}370y - 17 = 0.$$

Do not carry out the rotation.

28. Find the sine and cosine of an angle in Quadrant II through which the coordinate axes can be rotated to eliminate the cross product term from the equation

$$4x^2 - 4xy + y^2 - 8\sqrt{5}\,x - 16\sqrt{5}\,y = 0.$$

Do not carry out the rotation.

The conic sections in Exercises 17–26 were chosen to have rotation angles that were "nice" in the sense that once we knew $\cot 2\alpha$ or $\tan 2\alpha$ we could identify 2α and find $\sin \alpha$ and $\cos \alpha$ from familiar triangles.

In Exercises 29–34, use a calculator to find an angle α through which the coordinate axes can be rotated to change the given equation into a quadratic equation that has no cross product term. Then find $\sin \alpha$ and $\cos \alpha$ to two decimal places and use Equations (6) to find the coefficients of the new equation to the nearest decimal place. In each case, say whether the conic section is an ellipse, a hyperbola, or a parabola.

29. $x^2 - xy + 3y^2 + x - y - 3 = 0$

30. $2x^2 + xy - 3y^2 + 3x - 7 = 0$

31. $x^2 - 4xy + 4y^2 - 5 = 0$

32. $2x^2 - 12xy + 18y^2 - 49 = 0$

33. $3x^2 + 5xy + 2y^2 - 8y - 1 = 0$

34. $2x^2 + 7xy + 9y^2 + 20x - 86 = 0$

Theory and Examples

35. What effect does a 90° rotation about the origin have on the equations of the following conic sections? Give the new equation in each case.

a. The ellipse $(x^2/a^2) + (y^2/b^2) = 1 \quad (a > b)$

b. The hyperbola $(x^2/a^2) = (y^2/b^2) = 1$

c. The circle $x^2 + y^2 = a^2$

d. The line $y = mx$ **e.** The line $y = mx + b$

36. What effect does a 180° rotation about the origin have on the equations of the following conic sections? Give the new equation in each case.

a. The ellipse $(x^2/a^2) + (y^2/b^2) = 1 \quad (a > b)$

b. The hyperbola $(x^2/a^2) = (y^2/b^2) = 1$

c. The circle $x^2 + y^2 = a^2$

d. The line $y = mx$ **e.** The line $y = mx + b$

37. The Hyperbola $xy = a$ The hyperbola $xy = 1$ is one of many hyperbolas of the form $xy = a$ that appear in science and mathematics.

a. Rotate the coordinate axes through an angle of 45° to change the equation $xy = 1$ into an equation with no xy-term. What is the new equation?

b. Do the same for the equation $xy = a$.

38. Find the eccentricity of the hyperbola $xy = 2$.

39. Can anything be said about the graph of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ if $AC < 0$? Give reasons for your answer.

40. Degenerate conics Does any nondegenerate conic section $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ have all of the following properties?

a. It is symmetric with respect to the origin.

b. It passes through the point (1, 0).

c. It is tangent to the line $y = 1$ at the point $(-2, 1)$.

Give reasons for your answer.

41. Show that the equation $x^2 + y^2 = a^2$ becomes $x'^2 + y'^2 = a^2$ for every choice of the angle α in the rotation equations (4).

42. Show that rotating the axes through an angle of $\pi/4$ radians will eliminate the xy-term from Equation (1) whenever $A = C$.

43. a. Decide whether the equation

$$x^2 + 4xy + 4y^2 + 6x + 12y + 9 = 0$$

represents an ellipse, a parabola, or a hyperbola.

b. Show that the graph of the equation in part (a) is the line $2y = -x - 3$.

44. a. Decide whether the conic section with equation

$$9x^2 + 6xy + y^2 - 12x - 4y + 4 = 0$$

represents a parabola, an ellipse, or a hyperbola.

b. Show that the graph of the equation in part (a) is the line $y = -3x + 2$.

45. a. What kind of conic section is the curve $xy + 2x - y = 0$?

b. Solve the equation $xy + 2x - y = 0$ for y and sketch the curve as the graph of a rational function of x.

c. Find equations for the lines parallel to the line $y = -2x$ that are normal to the curve. Add the lines to your sketch.

46. Prove or find counterexamples to the following statements about the graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

a. If $AC > 0$, the graph is an ellipse.

b. If $AC > 0$, the graph is a hyperbola.

c. If $AC < 0$, the graph is a hyperbola.

47. A nice area formula for ellipses When $B^2 - 4AC$ is negative, the equation

$$Ax^2 + Bxy + Cy^2 = 1$$

represents an ellipse. If the ellipse's semi-axes are a and b, its area is πab (a standard formula). Show that the area is also given by the formula $2\pi/\sqrt{4AC - B^2}$. (*Hint:* Rotate the coordinate axes to eliminate the xy-term and apply Equation (12) to the new equation.)

48. Other invariants We describe the fact that $B'^2 - 4A'C'$ equals $B^2 - 4AC$ after a rotation about the origin by saying that the discriminant of a quadratic equation is an *invariant* of the equation.

Use Equations (6) to show that the numbers **(a)** $A + C$ and **(b)** $D^2 + E^2$ are also invariants, in the sense that

$$A' + C' = A + C \quad \text{and} \quad D'^2 + E'^2 = D^2 + E^2.$$

We can use these equalities to check against numerical errors when we rotate axes.

49. A proof that $B'^2 - 4A'C' = B^2 - 4AC$ Use Equations (6) to show that $B'^2 - 4A'C' = B^2 - 4AC$ for any rotation of axes about the origin.

10.4 Conics and Parametric Equations; The Cycloid

Curves in the Cartesian plane defined by parametric equations, and the calculation of their derivatives, were introduced in Section 3.5. There we studied parametrizations of lines, circles, and ellipses. In this section we discuss parametrization of parabolas, hyperbolas, cycloids, brachistocrones, and tautocrones.

Parabolas and Hyperbolas

In Section 3.5 we used the parametrization

$$x = \sqrt{t}, \qquad y = t, \qquad t > 0$$

to describe the motion of a particle moving along the right branch of the parabola $y = x^2$. In the following example we obtain a parametrization of the entire parabola, not just its right branch.

EXAMPLE 1 An Entire Parabola

You Try It

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = t, \qquad y = t^2, \qquad -\infty < t < \infty.$$

Identify the particle's path and describe the motion.

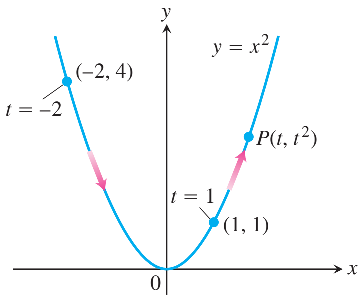


FIGURE 10.28 The path defined by $x = t, y = t^2, -\infty < t < \infty$ is the entire parabola $y = x^2$ (Example 1).

Solution We identify the path by eliminating t between the equations $x = t$ and $y = t^2$, obtaining

$$y = (t)^2 = x^2.$$

The particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along this curve.

In contrast to Example 10 in Section 3.5, the particle now traverses the entire parabola. As t increases from $-\infty$ to ∞, the particle comes down the left-hand side, passes through the origin, and moves up the right-hand side (Figure 10.28). ■

As Example 1 illustrates, any curve $y = f(x)$ has the parametrization $x = t$, $y = f(t)$. This is so simple we usually do not use it, but the point of view is occasionally helpful.

EXAMPLE 2 A Parametrization of the Right-hand Branch of the Hyperbola $x^2 - y^2 = 1$

You Try It

Describe the motion of the particle whose position $P(x, y)$ at time t is given by

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

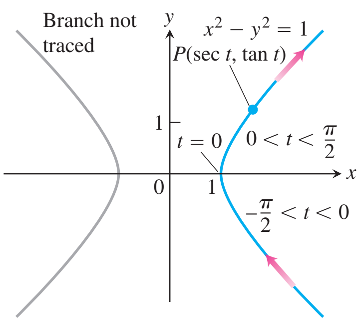


FIGURE 10.29 The equations $x = \sec t$, $y = \tan t$ and interval $-\pi/2 < t < \pi/2$ describe the right-hand branch of the hyperbola $x^2 - y^2 = 1$ (Example 2).

Solution We find a Cartesian equation for the coordinates of P by eliminating t between the equations

$$\sec t = x, \qquad \tan t = y.$$

We accomplish this with the identity $\sec^2 t - \tan^2 t = 1$, which yields

$$x^2 - y^2 = 1.$$

Since the particle's coordinates (x, y) satisfy the equation $x^2 - y^2 = 1$, the motion takes place somewhere on this hyperbola. As t runs between $-\pi/2$ and $\pi/2$, $x = \sec t$ remains positive and $y = \tan t$ runs between $-\infty$ and ∞, so P traverses the hyperbola's right-hand branch. It comes in along the branch's lower half as $t \to 0^-$, reaches $(1, 0)$ at $t = 0$, and moves out into the first quadrant as t increases toward $\pi/2$ (Figure 10.29). ■

Cycloids

The problem with a pendulum clock whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing. The wider the swing, the longer it takes the bob to return to center (its lowest position).

This does not happen if the bob can be made to swing in a *cycloid*. In 1673, Christiaan Huygens designed a pendulum clock whose bob would swing in a cycloid, a curve we define in Example 3. He hung the bob from a fine wire constrained by guards that caused it to draw up as it swung away from center (Figure 10.30).

HISTORICAL BIOGRAPHY

Christiaan Huygens (1629–1695)

Biography

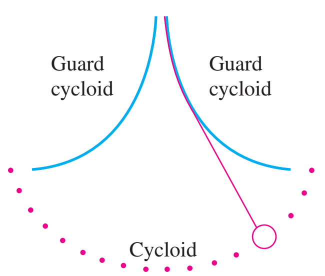


FIGURE 10.30 In Huygens' pendulum clock, the bob swings in a cycloid, so the frequency is independent of the amplitude.

EXAMPLE 3 Parametrizing a Cycloid

A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a **cycloid**.

Solution We take the line to be the x-axis, mark a point P on the wheel, start the wheel with P at the origin, and roll the wheel to the right. As parameter, we use the angle t through which the wheel turns, measured in radians. Figure 10.31 shows the wheel a short while later, when its base lies at units from the origin. The wheel's center C lies at (at, a) and the coordinates of P are

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta.$$

To express θ in terms of t, we observe that $t + \theta = 3\pi/2$ in the figure, so that

$$\theta = \frac{3\pi}{2} - t.$$

This makes

$$\cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t, \qquad \sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t.$$

The equations we seek are

$$x = at - a\sin t, \qquad y = a - a\cos t.$$

These are usually written with the a factored out:

$$x = a(t - \sin t), \qquad y = a(1 - \cos t). \tag{1}$$

Figure 10.32 shows the first arch of the cycloid and part of the next. ■

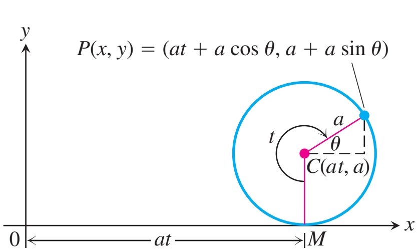


FIGURE 10.31 The position of $P(x, y)$ on the rolling wheel at angle t (Example 3).

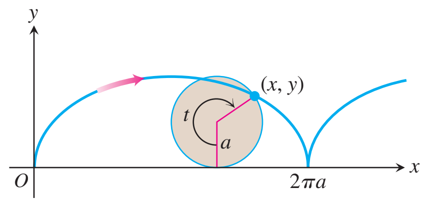


FIGURE 10.32 The cycloid $x = a(t - \sin t), y = a(1 - \cos t)$, for $t \geq 0$.

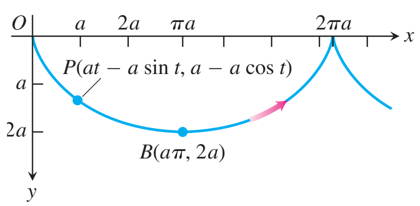


FIGURE 10.33 To study motion along an upside-down cycloid under the influence of gravity, we turn Figure 10.32 upside down. This points the y-axis in the direction of the gravitational force and makes the downward y-coordinates positive. The equations and parameter interval for the cycloid are still

$$x = a(t - \sin t),$$
$$y = a(1 - \cos t), \quad t \geq 0.$$

The arrow shows the direction of increasing t.

Brachistochrones and Tautochrones

If we turn Figure 10.32 upside down, Equations (1) still apply and the resulting curve (Figure 10.33) has two interesting physical properties. The first relates to the origin O and the point B at the bottom of the first arch. Among all smooth curves joining these points, the cycloid is the curve along which a frictionless bead, subject only to the force of gravity, will slide from O to B the fastest. This makes the cycloid a **brachistochrone** ("brah-*kiss*-toe-krone"), or shortest time curve for these points. The second property is that even if you start the bead partway down the curve toward B, it will still take the bead the same amount of time to reach B. This makes the cycloid a **tautochrone** ("*taw*-toe-krone"), or same-time curve for O and B.

Are there any other brachistochrones joining O and B, or is the cycloid the only one? We can formulate this as a mathematical question in the following way. At the start, the kinetic energy of the bead is zero, since its velocity is zero. The work done by gravity in moving the bead from $(0, 0)$ to any other point (x, y) in the plane is mgy, and this must equal the change in kinetic energy. That is,

$$mgy = \frac{1}{2}mv^2 - \frac{1}{2}m(0)^2.$$

Thus, the velocity of the bead when it reaches (x, y) has to be

$$v = \sqrt{2gy}.$$

That is,

$$\frac{ds}{dt} = \sqrt{2gy}$$

ds is the arc length differential along the bead's path.

or

$$dt = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{1 + (dy/dx)^2}\,dx}{\sqrt{2gy}}.$$

The time T_f it takes the bead to slide along a particular path $y = f(x)$ from O to $B(a\pi, 2a)$ is

$$T_f = \int_{x=0}^{x=a\pi} \sqrt{\frac{1 + (dy/dx)^2}{2gy}}\,dx. \tag{2}$$

What curves $y = f(x)$, if any, minimize the value of this integral?

At first sight, we might guess that the straight line joining O and B would give the shortest time, but perhaps not. There might be some advantage in having the bead fall vertically at first to build up its velocity faster. With a higher velocity, the bead could travel a longer path and still reach B first. Indeed, this is the right idea. The solution, from a branch of mathematics known as the *calculus of variations,* is that the original cycloid from O to B is the one and only brachistochrone for O and B.

While the solution of the brachistrochrone problem is beyond our present reach, we can still show why the cycloid is a tautochrone. For the cycloid, Equation (2) takes the form

$$\begin{aligned} T_{\text{cycloid}} &= \int_{x=0}^{x=a\pi} \sqrt{\frac{dx^2 + dy^2}{2gy}} \\ &= \int_{t=0}^{t=\pi} \sqrt{\frac{a^2(2 - 2\cos t)}{2ga(1 - \cos t)}}\,dt \\ &= \int_0^{\pi} \sqrt{\frac{a}{g}}\,dt = \pi\sqrt{\frac{a}{g}}. \end{aligned}$$

From Equations (1), $dx = a(1 - \cos t)\,dt$, $dy = a\sin t\,dt$, and $y = a(1 - \cos t)$

Thus, the amount of time it takes the frictionless bead to slide down the cycloid to B after it is released from rest at O is $\pi\sqrt{a/g}$.

Suppose that instead of starting the bead at O we start it at some lower point on the cycloid, a point (x_0, y_0) corresponding to the parameter value $t_0 > 0$. The bead's velocity at any later point (x, y) on the cycloid is

$$v = \sqrt{2g(y - y_0)} = \sqrt{2ga(\cos t_0 - \cos t)}. \qquad y = a(1 - \cos t)$$

Accordingly, the time required for the bead to slide from (x_0, y_0) down to B is

$$\begin{aligned}
T &= \int_{t_0}^{\pi} \sqrt{\frac{a^2(2 - 2\cos t)}{2ga(\cos t_0 - \cos t)}}\, dt = \sqrt{\frac{a}{g}} \int_{t_0}^{\pi} \sqrt{\frac{1 - \cos t}{\cos t_0 - \cos t}}\, dt \\
&= \sqrt{\frac{a}{g}} \int_{t_0}^{\pi} \sqrt{\frac{2\sin^2(t/2)}{(2\cos^2(t_0/2) - 1) - (2\cos^2(t/2) - 1)}}\, dt \\
&= \sqrt{\frac{a}{g}} \int_{t_0}^{\pi} \frac{\sin(t/2)\, dt}{\sqrt{\cos^2(t_0/2) - \cos^2(t/2)}} \\
&= \sqrt{\frac{a}{g}} \int_{t=t_0}^{t=\pi} \frac{-2\, du}{\sqrt{a^2 - u^2}} \qquad \begin{aligned} u &= \cos(t/2) \\ -2\, du &= \sin(t/2)\, dt \\ c &= \cos(t_0/2) \end{aligned} \\
&= 2\sqrt{\frac{a}{g}} \left[-\sin^{-1}\frac{u}{c}\right]_{t=t_0}^{t=\pi} \\
&= 2\sqrt{\frac{a}{g}} \left[-\sin^{-1}\frac{\cos(t/2)}{\cos(t_0/2)}\right]_{t_0}^{\pi} \\
&= 2\sqrt{\frac{a}{g}}(-\sin^{-1} 0 + \sin^{-1} 1) = \pi\sqrt{\frac{a}{g}}.
\end{aligned}$$

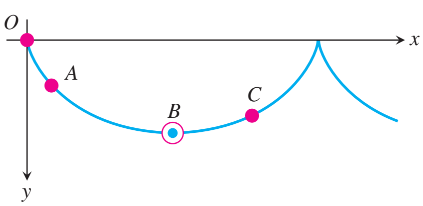


FIGURE 10.34 Beads released simultaneously on the cycloid at O, A, and C will reach B at the same time.

This is precisely the time it takes the bead to slide to B from O. It takes the bead the same amount of time to reach B no matter where it starts. Beads starting simultaneously from O, A, and C in Figure 10.34, for instance, will all reach B at the same time. This is the reason that Huygens' pendulum clock is independent of the amplitude of the swing.

EXERCISES 10.4

Parametric Equations for Conics

Exercises 1–12 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

1. $x = \cos t, \quad y = \sin t, \quad 0 \le t \le \pi$

2. $x = \sin(2\pi(1 - t)), \quad y = \cos(2\pi(1 - t)); \quad 0 \le t \le 1$

3. $x = 4\cos t, \quad y = 5\sin t; \quad 0 \le t \le \pi$

4. $x = 4\sin t, \quad y = 5\cos t; \quad 0 \le t \le 2\pi$

5. $x = t, \quad y = \sqrt{t}; \quad t \ge 0$

6. $x = \sec^2 t - 1, \quad y = \tan t; \quad -\pi/2 < t < \pi/2$

7. $x = -\sec t, \quad y = \tan t; \quad -\pi/2 < t < \pi/2$

8. $x = \csc t, \quad y = \cot t; \quad 0 < t < \pi$

9. $x = t, \quad y = \sqrt{4 - t^2}; \quad 0 \le t \le 2$

10. $x = t^2, \quad y = \sqrt{t^4 + 1}; \quad t \ge 0$

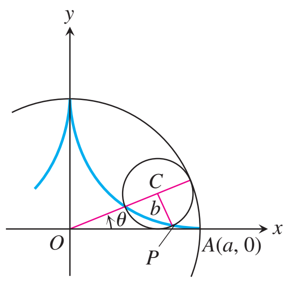
11. $x = -\cosh t, \quad y = \sinh t; \quad -\infty < t < \infty$

12. $x = 2\sinh t, \quad y = 2\cosh t; \quad -\infty < t < \infty$

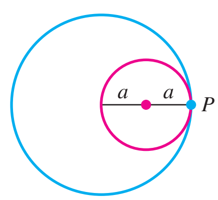
13. Hypocycloids When a circle rolls on the inside of a fixed circle, any point P on the circumference of the rolling circle describes a *hypocycloid*. Let the fixed circle be $x^2 + y^2 = a^2$, let the radius of the rolling circle be b, and let the initial position of the tracing point P be $A(a, 0)$. Find parametric equations for the hypocycloid, using as the parameter the angle θ from the positive x-axis to the line joining the circles' centers. In particular, if

$b = a/4$, as in the accompanying figure, show that the hypocycloid is the astroid

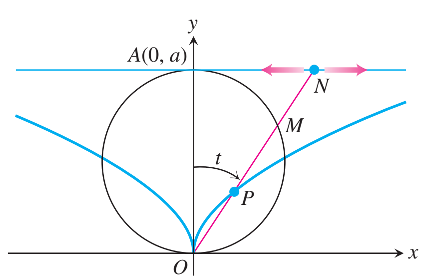
$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$



14. More about hypocycloids The accompanying figure shows a circle of radius a tangent to the inside of a circle of radius $2a$. The point P, shown as the point of tangency in the figure, is attached to the smaller circle. What path does P trace as the smaller circle rolls around the inside of the larger circle?



15. As the point N moves along the line $y = a$ in the accompanying figure, P moves in such a way that $OP = MN$. Find parametric equations for the coordinates of P as functions of the angle t that the line ON makes with the positive y-axis.



16. Trochoids A wheel of radius a rolls along a horizontal straight line without slipping. Find parametric equations for the curve traced out by a point P on a spoke of the wheel b units from its center. As parameter, use the angle θ through which the wheel turns. The curve is called a *trochoid*, which is a cycloid when $b = a$.

Distance Using Parametric Equations

17. Find the point on the parabola $x = t, y = t^2, -\infty < t < \infty$, closest to the point $(2, 1/2)$. (*Hint:* Minimize the square of the distance as a function of t.)

18. Find the point on the ellipse $x = 2\cos t, y = \sin t, 0 \le t \le 2\pi$ closest to the point $(3/4, 0)$. (*Hint:* Minimize the square of the distance as a function of t.)

T GRAPHER EXPLORATIONS

If you have a parametric equation grapher, graph the following equations over the given intervals.

19. Ellipse $x = 4\cos t, \quad y = 2\sin t,$ over

a. $0 \le t \le 2\pi$ **b.** $0 \le t \le \pi$

c. $-\pi/2 \le t \le \pi/2$.

20. Hyperbola branch $x = \sec t$ (enter as $1/\cos(t)$), $y = \tan t$ (enter as $\sin(t)/\cos(t)$), over

a. $-1.5 \le t \le 1.5$ **b.** $-0.5 \le t \le 0.5$

c. $-0.1 \le t \le 0.1$.

21. Parabola $x = 2t + 3, \quad y = t^2 - 1, \quad -2 \le t \le 2$

22. Cycloid $x = t - \sin t, \quad y = 1 - \cos t,$ over

a. $0 \le t \le 2\pi$ **b.** $0 \le t \le 4\pi$

c. $\pi \le t \le 3\pi$.

23. A nice curve (a deltoid)

$$x = 2\cos t + \cos 2t, \quad y = 2\sin t - \sin 2t; \quad 0 \le t \le 2\pi$$

What happens if you replace 2 with -2 in the equations for x and y? Graph the new equations and find out.

24. An even nicer curve

$$x = 3\cos t + \cos 3t, \quad y = 3\sin t - \sin 3t; \quad 0 \le t \le 2\pi$$

What happens if you replace 3 with -3 in the equations for x and y? Graph the new equations and find out.

25. Three beautiful curves

a. *Epicycloid:*

$$x = 9\cos t - \cos 9t, \quad y = 9\sin t - \sin 9t; \quad 0 \le t \le 2\pi$$

b. *Hypocycloid:*

$$x = 8\cos t + 2\cos 4t, \quad y = 8\sin t - 2\sin 4t; \quad 0 \le t \le 2\pi$$

c. *Hypotrochoid:*

$$x = \cos t + 5\cos 3t, \quad y = 6\cos t - 5\sin 3t; \quad 0 \le t \le 2\pi$$

26. More beautiful curves

a. $x = 6\cos t + 5\cos 3t, \quad y = 6\sin t - 5\sin 3t;$
$0 \le t \le 2\pi$

b. $x = 6\cos 2t + 5\cos 6t, \quad y = 6\sin 2t - 5\sin 6t;$
$0 \le t \le \pi$

c. $x = 6\cos t + 5\cos 3t, \quad y = 6\sin 2t - 5\sin 3t;$
$0 \le t \le 2\pi$

d. $x = 6\cos 2t + 5\cos 6t, \quad y = 6\sin 4t - 5\sin 6t;$
$0 \le t \le \pi$