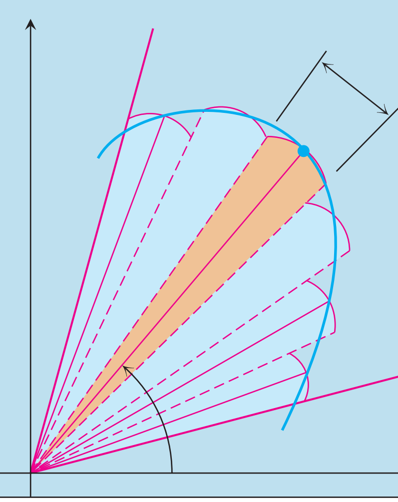


Chapter 10

# Conic Sections and Polar Coordinates

Biography

**OVERVIEW** In this chapter we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. These curves are called *conic sections*, or *conics*, and model the paths traveled by planets, satellites, and other bodies whose motions are driven by inverse square forces. In Chapter 13 we will see that once the path of a moving body is known to be a conic, we immediately have information about the body's velocity and the force that drives it. Planetary motion is best described with the help of polar coordinates, so we also investigate curves, derivatives, and integrals in this new coordinate system.

## 10.1 Conic Sections and Quadratic Equations

In Chapter 1 we defined a **circle** as the set of points in a plane whose distance from some fixed center point is a constant radius value. If the center is $(h, k)$ and the radius is $a$, the standard equation for the circle is $(x - h)^2 + (y - k)^2 = a^2$. It is an example of a conic section, which are the curves formed by cutting a double cone with a plane (Figure 10.1); hence the name *conic section.*

We now describe parabolas, ellipses, and hyperbolas as the graphs of quadratic equations in the coordinate plane.

### Parabolas

> **DEFINITIONS** **Parabola, Focus, Directrix**
> A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a **parabola**. The fixed point is the **focus** of the parabola. The fixed line is the **directrix**.

If the focus $F$ lies on the directrix $L$, the parabola is the line through $F$ perpendicular to $L$. We consider this to be a degenerate case and assume henceforth that $F$ does not lie on $L$.

A parabola has its simplest equation when its focus and directrix straddle one of the coordinate axes. For example, suppose that the focus lies at the point $F(0, p)$ on the positive $y$-axis and that the directrix is the line $y = -p$ (Figure 10.2). In the notation of the figure,

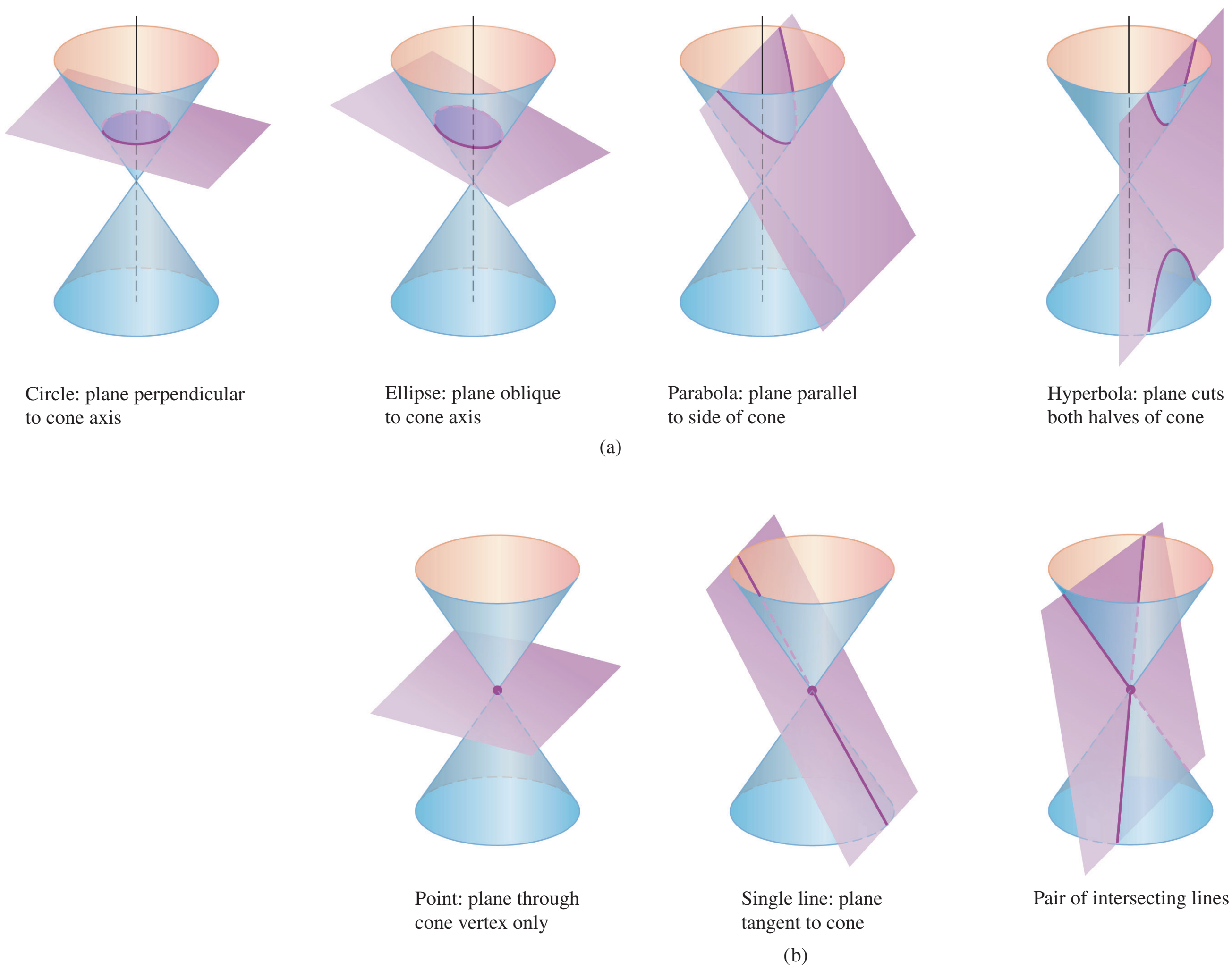



**FIGURE 10.1** The standard conic sections (a) are the curves in which a plane cuts a double cone. Hyperbolas come in two parts, called *branches*. The point and lines obtained by passing the plane through the cone's vertex (b) are *degenerate* conic sections.

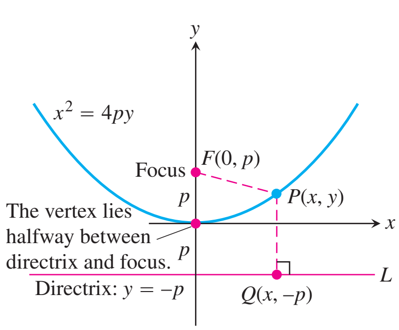



**FIGURE 10.2** The standard form of the parabola $x^2 = 4py,\ p > 0$.

a point $P(x, y)$ lies on the parabola if and only if $PF = PQ$. From the distance formula,

$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$$

$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(y+p)^2}.$$

When we equate these expressions, square, and simplify, we get

$$y = \frac{x^2}{4p} \quad \text{or} \quad x^2 = 4py. \qquad \text{Standard form} \qquad (1)$$

These equations reveal the parabola's symmetry about the $y$-axis. We call the $y$-axis the **axis** of the parabola (short for "axis of symmetry").

The point where a parabola crosses its axis is the **vertex**. The vertex of the parabola $x^2 = 4py$ lies at the origin (Figure 10.2). The positive number $p$ is the parabola's **focal length**.

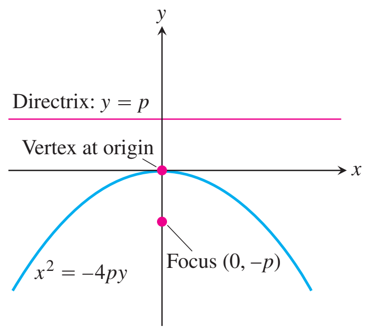



**FIGURE 10.3** The parabola $x^2 = -4py, p > 0$.

If the parabola opens downward, with its focus at $(0, -p)$ and its directrix the line $y = p$, then Equations (1) become

$$y = -\frac{x^2}{4p} \qquad \text{and} \qquad x^2 = -4py$$

(Figure 10.3). We obtain similar equations for parabolas opening to the right or to the left (Figure 10.4 and Table 10.1).

**TABLE 10.1** Standard-form equations for parabolas with vertices at the origin ($p > 0$)

| Equation | Focus | Directrix | Axis | Opens |
|---|---|---|---|---|
| $x^2 = 4py$ | $(0, p)$ | $y = -p$ | $y$-axis | Up |
| $x^2 = -4py$ | $(0, -p)$ | $y = p$ | $y$-axis | Down |
| $y^2 = 4px$ | $(p, 0)$ | $x = -p$ | $x$-axis | To the right |
| $y^2 = -4px$ | $(-p, 0)$ | $x = p$ | $x$-axis | To the left |

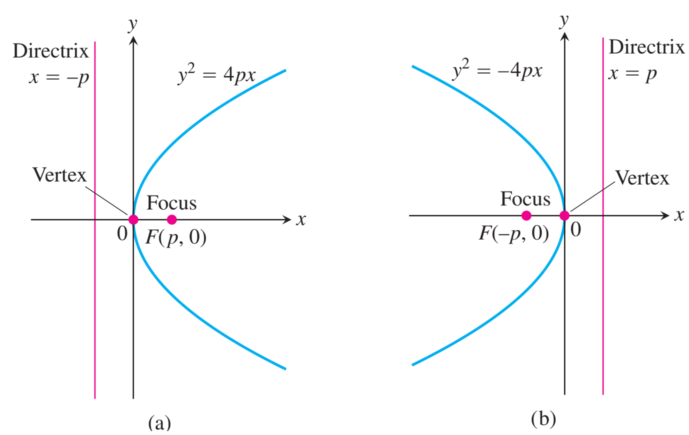



**FIGURE 10.4** (a) The parabola $y^2 = 4px$. (b) The parabola $y^2 = -4px$.

**EXAMPLE 1** Find the focus and directrix of the parabola $y^2 = 10x$.

**Solution** We find the value of $p$ in the standard equation $y^2 = 4px$:

$$4p = 10, \qquad \text{so} \qquad p = \frac{10}{4} = \frac{5}{2}.$$

Then we find the focus and directrix for this value of $p$:

$$\text{Focus:} \qquad (p, 0) = \left(\frac{5}{2}, 0\right)$$

$$\text{Directrix:} \qquad x = -p \qquad \text{or} \qquad x = -\frac{5}{2}.$$

■

The horizontal and vertical shift formulas in Section 1.5, can be applied to the equations in Table 10.1 to give equations for a variety of parabolas in other locations (see Exercises 39, 40, and 45–48).

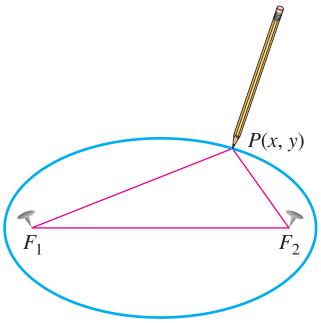



**FIGURE 10.5** One way to draw an ellipse uses two tacks and a loop of string to guide the pencil.

## Ellipses

> **DEFINITIONS Ellipse, Foci**
> An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.

The quickest way to construct an ellipse uses the definition. Put a loop of string around two tacks $F_1$ and $F_2$, pull the string taut with a pencil point $P$, and move the pencil around to trace a closed curve (Figure 10.5). The curve is an ellipse because the sum $PF_1 + PF_2$, being the length of the loop minus the distance between the tacks, remains constant. The ellipse's foci lie at $F_1$ and $F_2$.

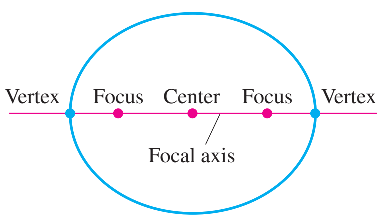



**FIGURE 10.6** Points on the focal axis of an ellipse.

> **DEFINITIONS Focal Axis, Center, Vertices**
> The line through the foci of an ellipse is the ellipse's **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and ellipse cross are the ellipse's **vertices** (Figure 10.6).

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Figure 10.7), and $PF_1 + PF_2$ is denoted by $2a$, then the coordinates of a point $P$ on the ellipse satisfy the equation

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{2}$$

Since $PF_1 + PF_2$ is greater than the length $F_1F_2$ (triangle inequality for triangle $PF_1F_2$), the number $2a$ is greater than $2c$. Accordingly, $a > c$ and the number $a^2 - c^2$ in Equation (2) is positive.

The algebraic steps leading to Equation (2) can be reversed to show that every point $P$ whose coordinates satisfy an equation of this form with $0 < c < a$ also satisfies the equation $PF_1 + PF_2 = 2a$. A point therefore lies on the ellipse if and only if its coordinates satisfy Equation (2).

If

$$b = \sqrt{a^2 - c^2}, \tag{3}$$

then $a^2 - c^2 = b^2$ and Equation (2) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{4}$$

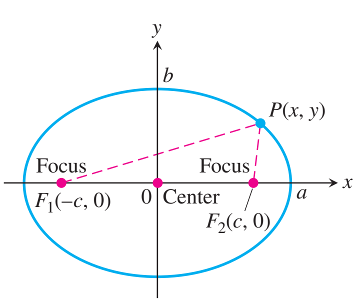



**FIGURE 10.7** The ellipse defined by the equation $PF_1 + PF_2 = 2a$ is the graph of the equation $(x^2/a^2) + (y^2/b^2) = 1$, where $b^2 = a^2 - c^2$.

Equation (4) reveals that this ellipse is symmetric with respect to the origin and both coordinate axes. It lies inside the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$. It crosses the axes at the points $(\pm a, 0)$ and $(0, \pm b)$. The tangents at these points are perpendicular to the axes because

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$ Obtained from Equation (4) by implicit differentiation

is zero if $x = 0$ and infinite if $y = 0$.

The **major axis** of the ellipse in Equation (4) is the line segment of length $2a$ joining the points $(\pm a, 0)$. The **minor axis** is the line segment of length $2b$ joining the points $(0, \pm b)$. The number $a$ itself is the **semimajor axis**, the number $b$ the **semiminor axis**. The number $c$, found from Equation (3) as

$$c = \sqrt{a^2 - b^2},$$

is the **center-to-focus distance** of the ellipse.

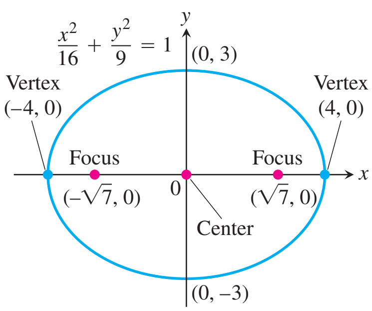



**FIGURE 10.8** An ellipse with its major axis horizontal (Example 2).

## EXAMPLE 2 Major Axis Horizontal

The ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \tag{5}$$

(Figure 10.8) has

Semimajor axis: $a = \sqrt{16} = 4$, Semiminor axis: $b = \sqrt{9} = 3$

Center-to-focus distance: $c = \sqrt{16 - 9} = \sqrt{7}$

Foci: $(\pm c, 0) = \left(\pm\sqrt{7}, 0\right)$

Vertices: $(\pm a, 0) = (\pm 4, 0)$

Center: $(0, 0)$. ■

## EXAMPLE 3 Major Axis Vertical

The ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1, \tag{6}$$

obtained by interchanging $x$ and $y$ in Equation (5), has its major axis vertical instead of horizontal (Figure 10.9). With $a^2$ still equal to 16 and $b^2$ equal to 9, we have

Semimajor axis: $a = \sqrt{16} = 4$, Semiminor axis: $b = \sqrt{9} = 3$

Center-to-focus distance: $c = \sqrt{16 - 9} = \sqrt{7}$

Foci: $(0, \pm c) = \left(0, \pm\sqrt{7}\right)$

Vertices: $(0, \pm a) = (0, \pm 4)$

Center: $(0, 0)$. ■

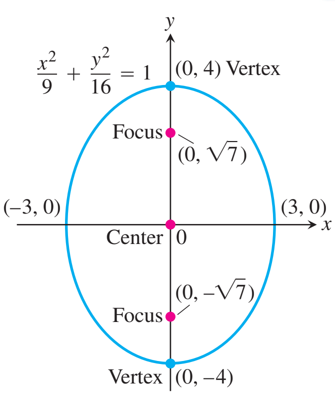



**FIGURE 10.9** An ellipse with its major axis vertical (Example 3).

There is never any cause for confusion in analyzing Equations (5) and (6). We simply find the intercepts on the coordinate axes; then we know which way the major axis runs because it is the longer of the two axes. The center always lies at the origin and the foci and vertices lie on the major axis.

**Standard-Form Equations for Ellipses Centered at the Origin**

*Foci on the x-axis:* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

*Foci on the y-axis:* $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

In each case, $a$ is the semimajor axis and $b$ is the semiminor axis.

## Hyperbolas

**DEFINITIONS Hyperbola, Foci**

A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the **foci** of the hyperbola.

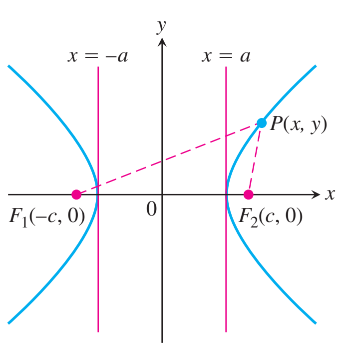



**FIGURE 10.10** Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown here, $PF_1 - PF_2 = 2a$. For points on the left-hand branch, $PF_2 - PF_1 = 2a$. We then let $b = \sqrt{c^2 - a^2}$.

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Figure 10.10) and the constant difference is $2a$, then a point $(x, y)$ lies on the hyperbola if and only if

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a. \tag{7}$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{8}$$

So far, this looks just like the equation for an ellipse. But now $a^2 - c^2$ is negative because $2a$, being the difference of two sides of triangle $PF_1F_2$, is less than $2c$, the third side.

The algebraic steps leading to Equation (8) can be reversed to show that every point $P$ whose coordinates satisfy an equation of this form with $0 < a < c$ also satisfies Equation (7). A point therefore lies on the hyperbola if and only if its coordinates satisfy Equation (8).

If we let $b$ denote the positive square root of $c^2 - a^2$,

$$b = \sqrt{c^2 - a^2}, \tag{9}$$

then $a^2 - c^2 = -b^2$ and Equation (8) takes the more compact form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{10}$$

The differences between Equation (10) and the equation for an ellipse (Equation 4) are the minus sign and the new relation

$$c^2 = a^2 + b^2. \qquad \text{From Equation (9)}$$

Like the ellipse, the hyperbola is symmetric with respect to the origin and coordinate axes. It crosses the $x$-axis at the points $(\pm a, 0)$. The tangents at these points are vertical because

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y} \qquad \text{Obtained from Equation (10) by implicit differentiation}$$

is infinite when $y = 0$. The hyperbola has no $y$-intercepts; in fact, no part of the curve lies between the lines $x = -a$ and $x = a$.

**DEFINITIONS Focal Axis, Center, Vertices**
The line through the foci of a hyperbola is the **focal axis**. The point on the axis halfway between the foci is the hyperbola's **center**. The points where the focal axis and hyperbola cross are the **vertices** (Figure 10.11).

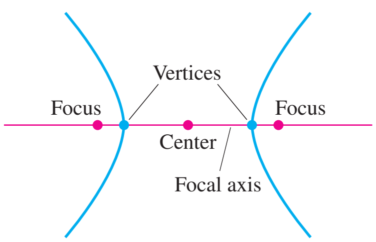


**FIGURE 10.11** Points on the focal axis of a hyperbola.

## Asymptotes of Hyperbolas and Graphing

If we solve Equation (10) for $y$ we obtain

$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$$

$$= \frac{b^2}{a^2}x^2\left(1 - \frac{a^2}{x^2}\right)$$

or, taking square roots,

$$y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}}.$$

As $x \to \pm\infty$, the factor $\sqrt{1 - a^2/x^2}$ approaches 1, and the factor $\pm(b/a)x$ is dominant. Thus the lines

$$y = \pm\frac{b}{a}x$$

are the two **asymptotes** of the hyperbola defined by Equation (10). The asymptotes give the guidance we need to graph hyperbolas quickly. The fastest way to find the equations of the asymptotes is to replace the 1 in Equation (10) by 0 and solve the new equation for $y$:

$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{hyperbola}} \rightarrow \underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0}_{\text{0 for 1}} \rightarrow \underbrace{y = \pm\frac{b}{a}x.}_{\text{asymptotes}}$$

**Standard-Form Equations for Hyperbolas Centered at the Origin**

*Foci on the x-axis*: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

Asymptotes: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$ or $y = \pm \dfrac{b}{a}x$

*Foci on the y-axis*: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

Asymptotes: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 0$ or $y = \pm \dfrac{a}{b}x$

Notice the difference in the asymptote equations ($b/a$ in the first, $a/b$ in the second).

## EXAMPLE 4 Foci on the *x*-axis

The equation

$$\frac{x^2}{4} - \frac{y^2}{5} = 1 \tag{11}$$

is Equation (10) with $a^2 = 4$ and $b^2 = 5$ (Figure 10.12). We have

Center-to-focus distance: $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$

Foci: $(\pm c, 0) = (\pm 3, 0)$, Vertices: $(\pm a, 0) = (\pm 2, 0)$

Center: $(0, 0)$

Asymptotes: $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 0$ or $y = \pm \dfrac{\sqrt{5}}{2}x$. ■

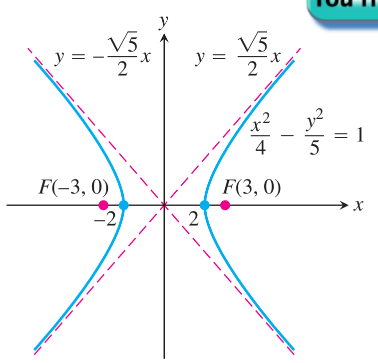



**FIGURE 10.12** The hyperbola and its asymptotes in Example 4.

## EXAMPLE 5 Foci on the *y*-axis

The hyperbola

$$\frac{y^2}{4} - \frac{x^2}{5} = 1,$$

obtained by interchanging $x$ and $y$ in Equation (11), has its vertices on the $y$-axis instead of the $x$-axis (Figure 10.13). With $a^2$ still equal to 4 and $b^2$ equal to 5, we have

Center-to-focus distance: $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$

Foci: $(0, \pm c) = (0, \pm 3)$, Vertices: $(0, \pm a) = (0, \pm 2)$

Center: $(0, 0)$

Asymptotes: $\dfrac{y^2}{4} - \dfrac{x^2}{5} = 0$ or $y = \pm \dfrac{2}{\sqrt{5}}x$. ■

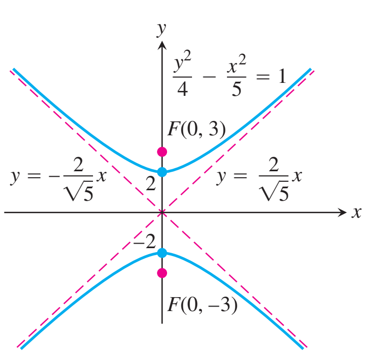



**FIGURE 10.13** The hyperbola and its asymptotes in Example 5.

## Reflective Properties

The chief applications of parabolas involve their use as reflectors of light and radio waves. Rays originating at a parabola's focus are reflected out of the parabola parallel to the parabola's axis (Figure 10.14 and Exercise 90). Moreover, the time any ray takes from the focus to a line parallel to the parabola's directrix (thus perpendicular to its axis) is the same for each of the rays. These properties are used by flashlight, headlight, and spotlight reflectors and by microwave broadcast antennas.

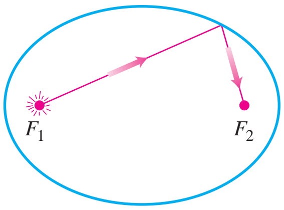



**FIGURE 10.15** An elliptical mirror (shown here in profile) reflects light from one focus to the other.

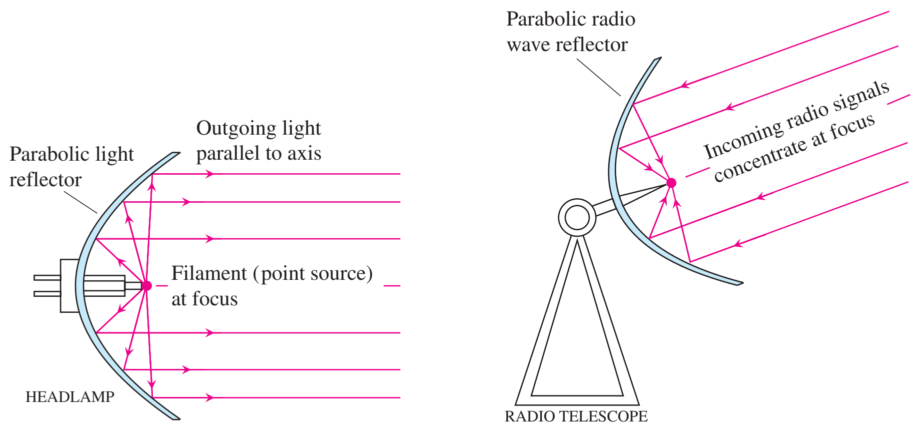



**FIGURE 10.14** Parabolic reflectors can generate a beam of light parallel to the parabola's axis from a source at the focus; or they can receive rays parallel to the axis and concentrate them at the focus.

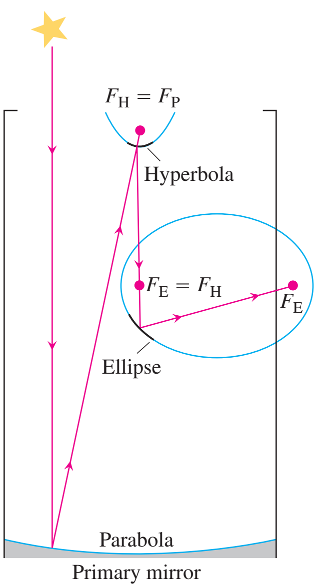



**FIGURE 10.16** Schematic drawing of a reflecting telescope.

If an ellipse is revolved about its major axis to generate a surface (the surface is called an *ellipsoid*) and the interior is silvered to produce a mirror, light from one focus will be reflected to the other focus (Figure 10.15). Ellipsoids reflect sound the same way, and this property is used to construct *whispering galleries*, rooms in which a person standing at one focus can hear a whisper from the other focus. (Statuary Hall in the U.S. Capitol building is a whispering gallery.)

Light directed toward one focus of a hyperbolic mirror is reflected toward the other focus. This property of hyperbolas is combined with the reflective properties of parabolas and ellipses in designing some modern telescopes. In Figure 10.16 starlight reflects off a primary parabolic mirror toward the mirror's focus $F_P$. It is then reflected by a small hyperbolic mirror, whose focus is $F_H = F_P$, toward the second focus of the hyperbola, $F_E = F_H$. Since this focus is shared by an ellipse, the light is reflected by the elliptical mirror to the ellipse's second focus to be seen by an observer.

# EXERCISES 10.1

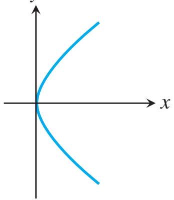
## Identifying Graphs

Match the parabolas in Exercises 1–4 with the following equations:

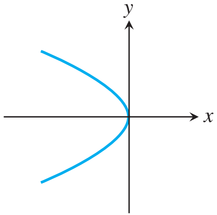
$$x^2 = 2y, \quad x^2 = -6y, \quad y^2 = 8x, \quad y^2 = -4x.$$

Then find the parabola's focus and directrix.

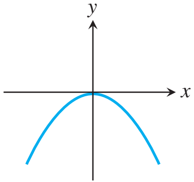
**1.**



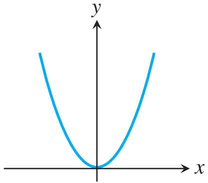

**2.**




**3.**




**4.**

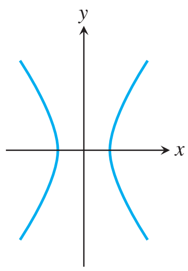


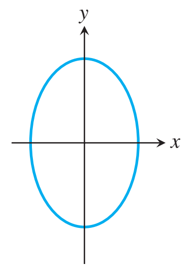

Match each conic section in Exercises 5–8 with one of these equations:

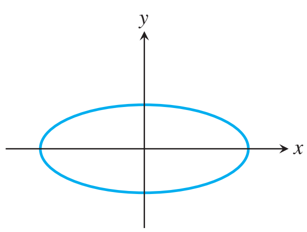
$$\frac{x^2}{4} + \frac{y^2}{9} = 1, \qquad \frac{x^2}{2} + y^2 = 1,$$

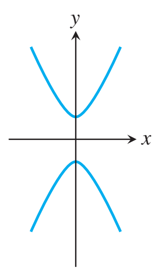
$$\frac{y^2}{4} - x^2 = 1, \qquad \frac{x^2}{4} - \frac{y^2}{9} = 1.$$

Then find the conic section's foci and vertices. If the conic section is a hyperbola, find its asymptotes as well.

**5.**



**6.**



**7.**



**8.**



## Parabolas

Exercises 9–16 give equations of parabolas. Find each parabola's focus and directrix. Then sketch the parabola. Include the focus and directrix in your sketch.

**9.** $y^2 = 12x$ **10.** $x^2 = 6y$ **11.** $x^2 = -8y$

**12.** $y^2 = -2x$ **13.** $y = 4x^2$ **14.** $y = -8x^2$

**15.** $x = -3y^2$ **16.** $x = 2y^2$

## Ellipses

Exercises 17–24 give equations for ellipses. Put each equation in standard form. Then sketch the ellipse. Include the foci in your sketch.

**17.** $16x^2 + 25y^2 = 400$ **18.** $7x^2 + 16y^2 = 112$

**19.** $2x^2 + y^2 = 2$ **20.** $2x^2 + y^2 = 4$

**21.** $3x^2 + 2y^2 = 6$ **22.** $9x^2 + 10y^2 = 90$

**23.** $6x^2 + 9y^2 = 54$ **24.** $169x^2 + 25y^2 = 4225$

Exercises 25 and 26 give information about the foci and vertices of ellipses centered at the origin of the $xy$-plane. In each case, find the ellipse's standard-form equation from the given information.

**25.** Foci: $\left(\pm\sqrt{2}, 0\right)$ Vertices: $(\pm 2, 0)$

**26.** Foci: $(0, \pm 4)$ Vertices: $(0, \pm 5)$

## Hyperbolas

Exercises 27–34 give equations for hyperbolas. Put each equation in standard form and find the hyperbola's asymptotes. Then sketch the hyperbola. Include the asymptotes and foci in your sketch.

**27.** $x^2 - y^2 = 1$ **28.** $9x^2 - 16y^2 = 144$

**29.** $y^2 - x^2 = 8$ **30.** $y^2 - x^2 = 4$

**31.** $8x^2 - 2y^2 = 16$ **32.** $y^2 - 3x^2 = 3$

**33.** $8y^2 - 2x^2 = 16$ **34.** $64x^2 - 36y^2 = 2304$

Exercises 35–38 give information about the foci, vertices, and asymptotes of hyperbolas centered at the origin of the $xy$-plane. In each case, find the hyperbola's standard-form equation from the information given.

**35.** Foci: $\left(0, \pm\sqrt{2}\right)$ Asymptotes: $y = \pm x$

**36.** Foci: $(\pm 2, 0)$ Asymptotes: $y = \pm\frac{1}{\sqrt{3}}x$

**37.** Vertices: $(\pm 3, 0)$ Asymptotes: $y = \pm\frac{4}{3}x$

**38.** Vertices: $(0, \pm 2)$ Asymptotes: $y = \pm\frac{1}{2}x$

## Shifting Conic Sections

**39.** The parabola $y^2 = 8x$ is shifted down 2 units and right 1 unit to generate the parabola $(y + 2)^2 = 8(x - 1)$.

**a.** Find the new parabola's vertex, focus, and directrix.

**b.** Plot the new vertex, focus, and directrix, and sketch in the parabola.

**40.** The parabola $x^2 = -4y$ is shifted left 1 unit and up 3 units to generate the parabola $(x + 1)^2 = -4(y - 3)$.

**a.** Find the new parabola's vertex, focus, and directrix.

**b.** Plot the new vertex, focus, and directrix, and sketch in the parabola.

**41.** The ellipse $(x^2/16) + (y^2/9) = 1$ is shifted 4 units to the right and 3 units up to generate the ellipse

$$\frac{(x-4)^2}{16} + \frac{(y-3)^2}{9} = 1.$$

**a.** Find the foci, vertices, and center of the new ellipse.

**b.** Plot the new foci, vertices, and center, and sketch in the new ellipse.

**42.** The ellipse $(x^2/9) + (y^2/25) = 1$ is shifted 3 units to the left and 2 units down to generate the ellipse

$$\frac{(x+3)^2}{9} + \frac{(y+2)^2}{25} = 1.$$

**a.** Find the foci, vertices, and center of the new ellipse.

**b.** Plot the new foci, vertices, and center, and sketch in the new ellipse.

**43.** The hyperbola $(x^2/16) - (y^2/9) = 1$ is shifted 2 units to the right to generate the hyperbola

$$\frac{(x-2)^2}{16} - \frac{y^2}{9} = 1.$$

**a.** Find the center, foci, vertices, and asymptotes of the new hyperbola.

**b.** Plot the new center, foci, vertices, and asymptotes, and sketch in the hyperbola.

**44.** The hyperbola $(y^2/4) - (x^2/5) = 1$ is shifted 2 units down to generate the hyperbola

$$\frac{(y+2)^2}{4} - \frac{x^2}{5} = 1.$$

**a.** Find the center, foci, vertices, and asymptotes of the new hyperbola.

**b.** Plot the new center, foci, vertices, and asymptotes, and sketch in the hyperbola.

Exercises 45–48 give equations for parabolas and tell how many units up or down and to the right or left each parabola is to be shifted. Find an equation for the new parabola, and find the new vertex, focus, and directrix.

**45.** $y^2 = 4x$, left 2, down 3 **46.** $y^2 = -12x$, right 4, up 3

**47.** $x^2 = 8y$, right 1, down 7 **48.** $x^2 = 6y$, left 3, down 2

Exercises 49–52 give equations for ellipses and tell how many units up or down and to the right or left each ellipse is to be shifted. Find an equation for the new ellipse, and find the new foci, vertices, and center.

**49.** $\dfrac{x^2}{6} + \dfrac{y^2}{9} = 1$, left 2, down 1

**50.** $\dfrac{x^2}{2} + y^2 = 1$, right 3, up 4

**51.** $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$, right 2, up 3

**52.** $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$, left 4, down 5

Exercises 53–56 give equations for hyperbolas and tell how many units up or down and to the right or left each hyperbola is to be shifted. Find an equation for the new hyperbola, and find the new center, foci, vertices, and asymptotes.

**53.** $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$, right 2, up 2

**54.** $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$, left 2, down 1

**55.** $y^2 - x^2 = 1$, left 1, down 1

**56.** $\dfrac{y^2}{3} - x^2 = 1$, right 1, up 3

Find the center, foci, vertices, asymptotes, and radius, as appropriate, of the conic sections in Exercises 57–68.

**57.** $x^2 + 4x + y^2 = 12$

**58.** $2x^2 + 2y^2 - 28x + 12y + 114 = 0$

**59.** $x^2 + 2x + 4y - 3 = 0$ **60.** $y^2 - 4y - 8x - 12 = 0$

**61.** $x^2 + 5y^2 + 4x = 1$ **62.** $9x^2 + 6y^2 + 36y = 0$

**63.** $x^2 + 2y^2 - 2x - 4y = -1$

**64.** $4x^2 + y^2 + 8x - 2y = -1$

**65.** $x^2 - y^2 - 2x + 4y = 4$ **66.** $x^2 - y^2 + 4x - 6y = 6$

**67.** $2x^2 - y^2 + 6y = 3$ **68.** $y^2 - 4x^2 + 16x = 24$

## Inequalities

Sketch the regions in the $xy$-plane whose coordinates satisfy the inequalities or pairs of inequalities in Exercises 69–74.

**69.** $9x^2 + 16y^2 \le 144$

**70.** $x^2 + y^2 \ge 1$ and $4x^2 + y^2 \le 4$

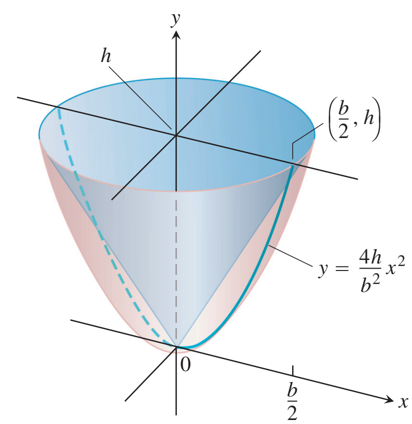
**71.** $x^2 + 4y^2 \ge 4$ and $4x^2 + 9y^2 \le 36$

**72.** $(x^2 + y^2 - 4)(x^2 + 9y^2 - 9) \le 0$

**73.** $4y^2 - x^2 \ge 4$ **74.** $|x^2 - y^2| \le 1$

## Theory and Examples

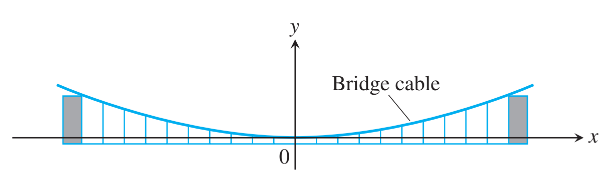
**75. Archimedes' formula for the volume of a parabolic solid** The region enclosed by the parabola $y = (4h/b^2)x^2$ and the line $y = h$ is revolved about the $y$-axis to generate the solid shown here. Show that the volume of the solid is $3/2$ the volume of the corresponding cone.




**76. Suspension bridge cables hang in parabolas** The suspension bridge cable shown here supports a uniform load of $w$ pounds per horizontal foot. It can be shown that if $H$ is the horizontal tension of the cable at the origin, then the curve of the cable satisfies the equation

$$\frac{dy}{dx} = \frac{w}{H}x.$$

Show that the cable hangs in a parabola by solving this differential equation subject to the initial condition that $y = 0$ when $x = 0$.



**77.** Find an equation for the circle through the points (1, 0), (0, 1), and (2, 2).

**78.** Find an equation for the circle through the points (2, 3), (3, 2), and $(-4, 3)$.

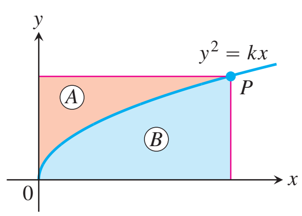
**79.** Find an equation for the circle centered at $(-2, 1)$ that passes through the point (1, 3). Is the point (1.1, 2.8) inside, outside, or on the circle?

**80.** Find equations for the tangents to the circle $(x - 2)^2 + (y - 1)^2 = 5$ at the points where the circle crosses the coordinate axes. (*Hint:* Use implicit differentiation.)

**81.** If lines are drawn parallel to the coordinate axes through a point $P$ on the parabola $y^2 = kx$, $k > 0$, the parabola partitions the rectangular region bounded by these lines and the coordinate axes into two smaller regions, $A$ and $B$.

**a.** If the two smaller regions are revolved about the $y$-axis, show that they generate solids whose volumes have the ratio 4:1.

**b.** What is the ratio of the volumes generated by revolving the regions about the $x$-axis?




**82.** Show that the tangents to the curve $y^2 = 4px$ from any point on the line $x = -p$ are perpendicular.

**83.** Find the dimensions of the rectangle of largest area that can be inscribed in the ellipse $x^2 + 4y^2 = 4$ with its sides parallel to the coordinate axes. What is the area of the rectangle?

**84.** Find the volume of the solid generated by revolving the region enclosed by the ellipse $9x^2 + 4y^2 = 36$ about the **(a)** $x$-axis, **(b)** $y$-axis.

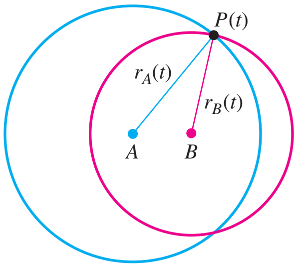
**85.** The "triangular" region in the first quadrant bounded by the $x$-axis, the line $x = 4$, and the hyperbola $9x^2 - 4y^2 = 36$ is revolved about the $x$-axis to generate a solid. Find the volume of the solid.

**86.** The region bounded on the left by the $y$-axis, on the right by the hyperbola $x^2 - y^2 = 1$, and above and below by the lines $y = \pm 3$ is revolved about the $y$-axis to generate a solid. Find the volume of the solid.

**87.** Find the centroid of the region that is bounded below by the $x$-axis and above by the ellipse $(x^2/9) + (y^2/16) = 1$.

**88.** The curve $y = \sqrt{x^2 + 1}$, $0 \le x \le \sqrt{2}$, which is part of the upper branch of the hyperbola $y^2 - x^2 = 1$, is revolved about the $x$-axis to generate a surface. Find the area of the surface.

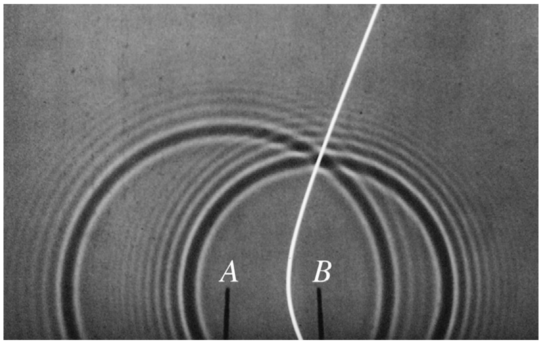
**89.** The circular waves in the photograph here were made by touching the surface of a ripple tank, first at $A$ and then at $B$. As the waves expanded, their point of intersection appeared to trace a hyperbola. Did it really do that? To find out, we can model the waves with circles centered at $A$ and $B$.




At time $t$, the point $P$ is $r_A(t)$ units from $A$ and $r_B(t)$ units from $B$. Since the radii of the circles increase at a constant rate, the rate at which the waves are traveling is

$$\frac{dr_A}{dt} = \frac{dr_B}{dt}.$$

Conclude from this equation that $r_A - r_B$ has a constant value, so that $P$ must lie on a hyperbola with foci at $A$ and $B$.




**90. The reflective property of parabolas** The figure here shows a typical point $P(x_0, y_0)$ on the parabola $y^2 = 4px$. The line $L$ is tangent to the parabola at $P$. The parabola's focus lies at $F(p, 0)$. The ray $L'$ extending from $P$ to the right is parallel to the $x$-axis. We show that light from $F$ to $P$ will be reflected out along $L'$ by showing that $\beta$ equals $\alpha$. Establish this equality by taking the following steps.

**a.** Show that $\tan \beta = 2p/y_0$.

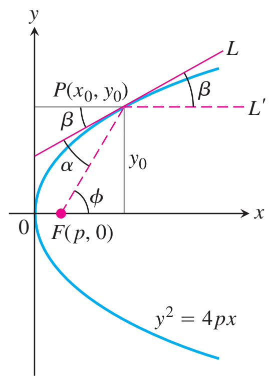
**b.** Show that $\tan \phi = y_0/(x_0 - p)$.

**c.** Use the identity

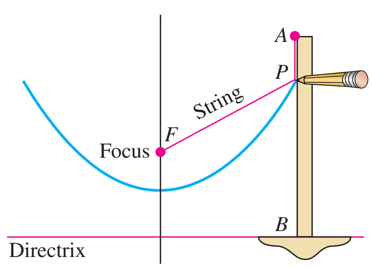
$$\tan \alpha = \frac{\tan \phi - \tan \beta}{1 + \tan \phi \tan \beta}$$

to show that $\tan \alpha = 2p/y_0$.

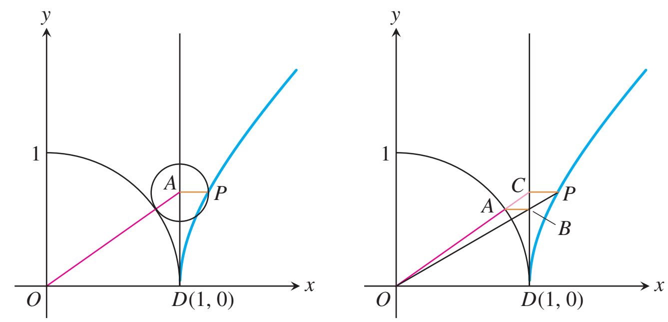
Since $\alpha$ and $\beta$ are both acute, $\tan\beta = \tan\alpha$ implies $\beta = \alpha$.




**91. How the astronomer Kepler used string to draw parabolas** Kepler's method for drawing a parabola (with more modern tools) requires a string the length of a T square and a table whose edge can serve as the parabola's directrix. Pin one end of the string to the point where you want the focus to be and the other end to the upper end of the T square. Then, holding the string taut against the T square with a pencil, slide the T square along the table's edge. As the T square moves, the pencil will trace a parabola. Why?




**92. Construction of a hyperbola** The following diagrams appeared (unlabeled) in Ernest J. Eckert, "Constructions Without Words," *Mathematics Magazine*, Vol. 66, No. 2, Apr. 1993, p. 113. Explain the constructions by finding the coordinates of the point $P$.




(a) (b)

**93. The width of a parabola at the focus** Show that the number $4p$ is the *width* of the parabola $x^2 = 4py$ $(p > 0)$ at the focus by showing that the line $y = p$ cuts the parabola at points that are $4p$ units apart.

**94. The asymptotes of $(x^2/a^2) - (y^2/b^2) = 1$** Show that the vertical distance between the line $y = (b/a)x$ and the upper half of the right-hand branch $y = (b/a)\sqrt{x^2 - a^2}$ of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ approaches 0 by showing that

$$\lim_{x\to\infty}\left(\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}\right) = \frac{b}{a}\lim_{x\to\infty}\left(x - \sqrt{x^2 - a^2}\right) = 0.$$

Similar results hold for the remaining portions of the hyperbola and the lines $y = \pm(b/a)x$.

# 10.2 Classifying Conic Sections by Eccentricity

We now show how to associate with each conic section a number called the conic section's *eccentricity*. The eccentricity reveals the conic section's type (circle, ellipse, parabola, or hyperbola) and, in the case of ellipses and hyperbolas, describes the conic section's general proportions.

## Eccentricity

Although the center-to-focus distance $c$ does not appear in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (a > b)$$

for an ellipse, we can still determine $c$ from the equation $c = \sqrt{a^2 - b^2}$. If we fix $a$ and vary $c$ over the interval $0 \leq c \leq a$, the resulting ellipses will vary in shape (Figure 10.17). They are circles if $c = 0$ (so that $a = b$) and flatten as $c$ increases. If $c = a$, the foci and vertices overlap and the ellipse degenerates into a line segment.

We use the ratio of $c$ to $a$ to describe the various shapes the ellipse can take. We call this ratio the ellipse's eccentricity.

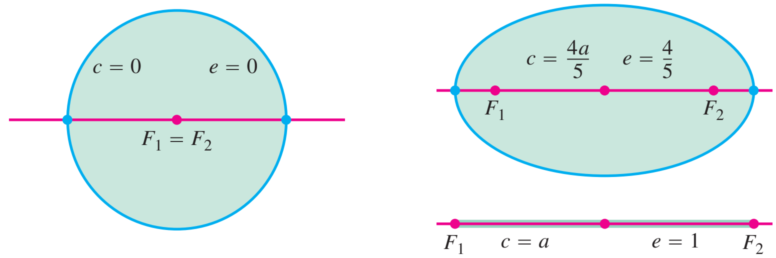



**FIGURE 10.17** The ellipse changes from a circle to a line segment as $c$ increases from 0 to $a$.

**DEFINITION Eccentricity of an Ellipse**

The **eccentricity** of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ $(a > b)$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

**TABLE 10.2** Eccentricities of planetary orbits

| | | | |
|---|---|---|---|
| Mercury | 0.21 | Saturn | 0.06 |
| Venus | 0.01 | Uranus | 0.05 |
| Earth | 0.02 | Neptune | 0.01 |
| Mars | 0.09 | Pluto | 0.25 |
| Jupiter | 0.05 | | |

The planets in the solar system revolve around the sun in (approximate) elliptical orbits with the sun at one focus. Most of the orbits are nearly circular, as can be seen from the eccentricities in Table 10.2. Pluto has a fairly eccentric orbit, with $e = 0.25$, as does Mercury, with $e = 0.21$. Other members of the solar system have orbits that are even more eccentric. Icarus, an asteroid about 1 mile wide that revolves around the sun every 409 Earth days, has an orbital eccentricity of 0.83 (Figure 10.18).

You Try It

## EXAMPLE 1 Halley's Comet

The orbit of Halley's comet is an ellipse 36.18 astronomical units long by 9.12 astronomical units wide. (One *astronomical unit* [AU] is 149,597,870 km, the semimajor axis of Earth's orbit.) Its eccentricity is

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{(36.18/2)^2 - (9.12/2)^2}}{(1/2)(36.18)} = \frac{\sqrt{(18.09)^2 - (4.56)^2}}{18.09} \approx 0.97. \quad \blacksquare$$

HISTORICAL BIOGRAPHY

Edmund Halley
(1656–1742)

Biography

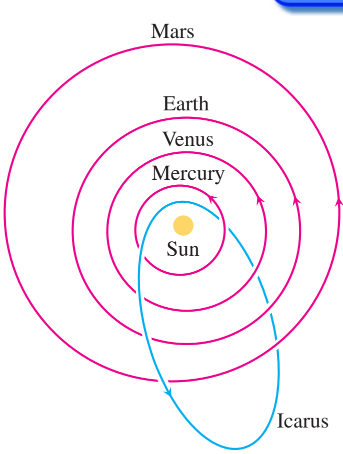



**FIGURE 10.18** The orbit of the asteroid Icarus is highly eccentric. Earth's orbit is so nearly circular that its foci lie inside the sun.

Whereas a parabola has one focus and one directrix, each **ellipse** has two foci and two **directrices**. These are the lines perpendicular to the major axis at distances $\pm a/e$ from the center. The parabola has the property that

$$PF = 1 \cdot PD \tag{1}$$

for any point $P$ on it, where $F$ is the focus and $D$ is the point nearest $P$ on the directrix. For an ellipse, it can be shown that the equations that replace Equation (1) are

$$PF_1 = e \cdot PD_1, \qquad PF_2 = e \cdot PD_2. \tag{2}$$

Here, $e$ is the eccentricity, $P$ is any point on the ellipse, $F_1$ and $F_2$ are the foci, and $D_1$ and $D_2$ are the points on the directrices nearest $P$ (Figure 10.19).

In both Equations (2) the directrix and focus must correspond; that is, if we use the distance from $P$ to $F_1$, we must also use the distance from $P$ to the directrix at the same end of the ellipse. The directrix $x = -a/e$ corresponds to $F_1(-c, 0)$, and the directrix $x = a/e$ corresponds to $F_2(c, 0)$.

The eccentricity of a hyperbola is also $e = c/a$, only in this case $c$ equals $\sqrt{a^2 + b^2}$ instead of $\sqrt{a^2 - b^2}$. In contrast to the eccentricity of an ellipse, the eccentricity of a hyperbola is always greater than 1.

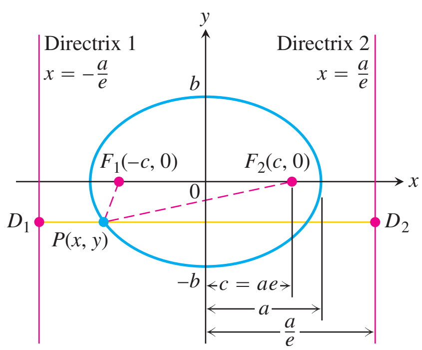



**FIGURE 10.19** The foci and directrices of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Directrix 1 corresponds to focus $F_1$, and directrix 2 to focus $F_2$.

**DEFINITION Eccentricity of a Hyperbola**

The **eccentricity** of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

In both ellipse and hyperbola, the eccentricity is the ratio of the distance between the foci to the distance between the vertices (because $c/a = 2c/2a$).

$$\text{Eccentricity} = \frac{\text{distance between foci}}{\text{distance between vertices}}$$

In an ellipse, the foci are closer together than the vertices and the ratio is less than 1. In a hyperbola, the foci are farther apart than the vertices and the ratio is greater than 1.

## EXAMPLE 2 Finding the Vertices of an Ellipse

Locate the vertices of an ellipse of eccentricity 0.8 whose foci lie at the points $(0, \pm 7)$.

**Solution** Since $e = c/a$, the vertices are the points $(0, \pm a)$ where

$$a = \frac{c}{e} = \frac{7}{0.8} = 8.75,$$

or $(0, \pm 8.75)$. ■

## EXAMPLE 3 Eccentricity of a Hyperbola

Find the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$.

**Solution** We divide both sides of the hyperbola's equation by 144 to put it in standard form, obtaining

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1 \quad \text{and} \quad \frac{x^2}{16} - \frac{y^2}{9} = 1.$$

With $a^2 = 16$ and $b^2 = 9$, we find that $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$, so

$$e = \frac{c}{a} = \frac{5}{4}.$$ ■

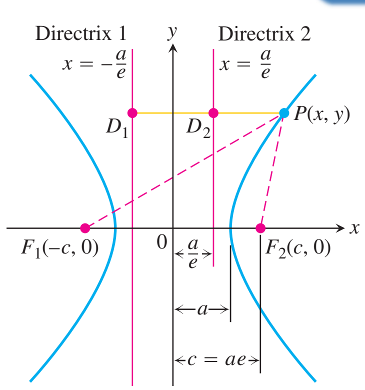



**FIGURE 10.20** The foci and directrices of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$. No matter where $P$ lies on the hyperbola, $PF_1 = e \cdot PD_1$ and $PF_2 = e \cdot PD_2$.

As with the ellipse, it can be shown that the lines $x = \pm a/e$ act as **directrices** for the **hyperbola** and that

$$PF_1 = e \cdot PD_1 \quad \text{and} \quad PF_2 = e \cdot PD_2. \tag{3}$$

Here $P$ is any point on the hyperbola, $F_1$ and $F_2$ are the foci, and $D_1$ and $D_2$ are the points nearest $P$ on the directrices (Figure 10.20).

To complete the picture, we define the eccentricity of a parabola to be $e = 1$. Equations (1) to (3) then have the common form $PF = e \cdot PD$.

**DEFINITION Eccentricity of a Parabola**

The **eccentricity** of a parabola is $e = 1$.

The "focus–directrix" equation $PF = e \cdot PD$ unites the parabola, ellipse, and hyperbola in the following way. Suppose that the distance $PF$ of a point $P$ from a fixed point $F$ (the focus) is a constant multiple of its distance from a fixed line (the directrix). That is, suppose

$$PF = e \cdot PD, \tag{4}$$

where $e$ is the constant of proportionality. Then the path traced by $P$ is

**(a)** a *parabola* if $e = 1$,

**(b)** an *ellipse* of eccentricity $e$ if $e < 1$, and

**(c)** a *hyperbola* of eccentricity $e$ if $e > 1$.

There are no coordinates in Equation (4) and when we try to translate it into coordinate form it translates in different ways, depending on the size of $e$. At least, that is what happens in Cartesian coordinates. However, in polar coordinates, as we will see in Section 10.8, the equation $PF = e \cdot PD$ translates into a single equation regardless of the value of $e$, an equation so simple that it has been the equation of choice of astronomers and space scientists for nearly 300 years.

Given the focus and corresponding directrix of a hyperbola centered at the origin and with foci on the $x$-axis, we can use the dimensions shown in Figure 10.20 to find $e$. Knowing $e$, we can derive a Cartesian equation for the hyperbola from the equation $PF = e \cdot PD$, as in the next example. We can find equations for ellipses centered at the origin and with foci on the $x$-axis in a similar way, using the dimensions shown in Figure 10.19.

You Try It

## EXAMPLE 4 Cartesian Equation for a Hyperbola

Find a Cartesian equation for the hyperbola centered at the origin that has a focus at (3, 0) and the line $x = 1$ as the corresponding directrix.

**Solution** We first use the dimensions shown in Figure 10.20 to find the hyperbola's eccentricity. The focus is

$$(c, 0) = (3, 0) \qquad \text{so} \qquad c = 3.$$

The directrix is the line

$$x = \frac{a}{e} = 1, \qquad \text{so} \qquad a = e.$$

When combined with the equation $e = c/a$ that defines eccentricity, these results give

$$e = \frac{c}{a} = \frac{3}{e}, \qquad \text{so} \qquad e^2 = 3 \quad \text{and} \quad e = \sqrt{3}.$$

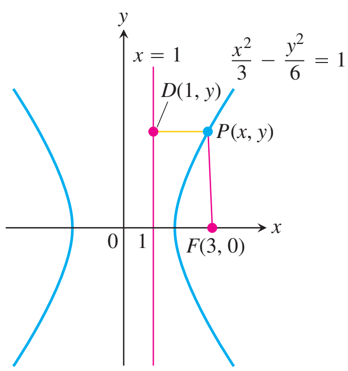



**FIGURE 10.21** The hyperbola and directrix in Example 4.

Knowing $e$, we can now derive the equation we want from the equation $PF = e \cdot PD$. In the notation of Figure 10.21, we have

$$PF = e \cdot PD \qquad \text{Equation (4)}$$

$$\sqrt{(x-3)^2 + (y-0)^2} = \sqrt{3}\,|x-1| \qquad e = \sqrt{3}$$

$$x^2 - 6x + 9 + y^2 = 3(x^2 - 2x + 1)$$

$$2x^2 - y^2 = 6$$

$$\frac{x^2}{3} - \frac{y^2}{6} = 1.$$

■