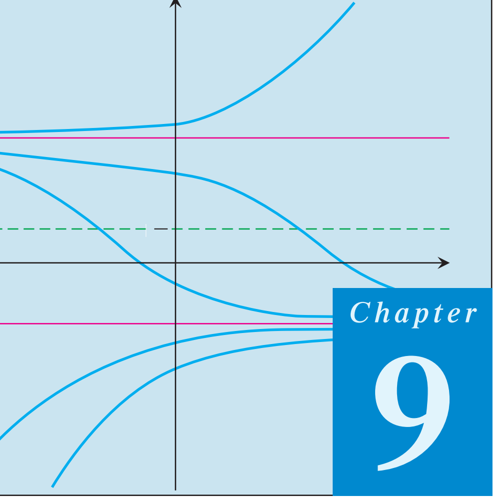


Chapter 9

Further Applications of Integration

Historical Biography

Carl Friedrich Gauss

(1777–1855)

Biography History

OVERVIEW In Section 4.8 we introduced differential equations of the form $dy/dx = f(x)$, where y is an unknown function being differentiated. For a continuous function f, we found the general solution $y(x)$ by integration: $y(x) = \int f(x)\,dx$. (Remember that the indefinite integral represents *all* the antiderivatives of f, so it contains an arbitrary constant $+C$ which must be shown once an antiderivative is found.) Many applications in the sciences, engineering, and economics involve a model formulated by even more general differential equations. In Section 7.5, for example, we found that exponential growth and decay is modeled by a differential equation of the form $dy/dx = ky$, for some constant $k \neq 0$. We have not yet considered differential equations such as $dy/dx = y - x$, yet such equations arise frequently in applications. In this chapter, we study several differential equations having the form $dy/dx = f(x, y)$, where f is a function of *both* the independent and dependent variables. We use the theory of indefinite integration to solve these differential equations, and investigate analytic, graphical, and numerical solution methods.

9.1 Slope Fields and Separable Differential Equations

Historical Biography

Jules Henri Poincaré

(1854–1912)

Biography

In calculating derivatives by implicit differentiation (Section 3.6), we found that the expression for the derivative dy/dx often contained both variables x and y, not just the independent variable x. We begin this section by considering the general differential equation $dy/dx = f(x, y)$ and what is meant by a solution to it. Then we investigate equations having a special form for which the function f can be expressed as a product of a function of x and a function of y.

General First-Order Differential Equations and Solutions

A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The equation is of *first-order* because it involves only the first derivative dy/dx (and not higher-order derivatives). We point out that the equations

$$y' = f(x, y) \quad \text{and} \quad \frac{d}{dx}y = f(x, y),$$

are equivalent to Equation (1) and all three forms will be used interchangeably in the text.

A **solution** of Equation (1) is a differentiable function $y = y(x)$ defined on an interval I of x-values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into Equation (1), the resulting equation is true for all x over the interval I. The **general solution** to a first-order differential equation is a solution that contains all possible solutions. The general solution always contains an arbitrary constant, but having this property doesn't mean a solution is the general solution. That is, a solution may contain an arbitrary constant without being the general solution. Establishing that a solution *is* the general solution may require deeper results from the theory of differential equations and is best studied in a more advanced course.

EXAMPLE 1 Verifying Solution Functions

Show that every member of the family of functions

$$y = \frac{C}{x} + 2$$

is a solution of the first-order differential equation

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$

on the interval $(0, \infty)$, where C is any constant.

Solution Differentiating $y = C/x + 2$ gives

$$\frac{dy}{dx} = C\frac{d}{dx}\left(\frac{1}{x}\right) + 0 = -\frac{C}{x^2}.$$

Thus we need only verify that for all $x \in (0, \infty)$,

$$-\frac{C}{x^2} = \frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right].$$

This last equation follows immediately by expanding the expression on the right side:

$$\frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right] = \frac{1}{x}\left(-\frac{C}{x}\right) = -\frac{C}{x^2}.$$

Therefore, for every value of C, the function $y = C/x + 2$ is a solution of the differential equation. ■

As was the case in finding antiderivatives, we often need a *particular* rather than the general solution to a first-order differential equation $y' = f(x, y)$. The **particular solution** satisfying the initial condition $y(x_0) = y_0$ is the solution $y = y(x)$ whose value is y_0 when $x = x_0$. Thus the graph of the particular solution passes through the point (x_0, y_0) in the xy-plane. A **first-order initial value problem** is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

Video Video

EXAMPLE 2 Verifying That a Function Is a Particular Solution

Show that the function

$$y = (x + 1) - \frac{1}{3}e^x$$

is a solution to the first-order initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$

Solution The equation

$$\frac{dy}{dx} = y - x$$

is a first-order differential equation with $f(x, y) = y - x$.

On the left:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + 1 - \frac{1}{3}e^x\right) = 1 - \frac{1}{3}e^x.$$

On the right:

$$y - x = (x + 1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x.$$

The function satisfies the initial condition because

$$y(0) = \left[(x + 1) - \frac{1}{3}e^x\right]_{x=0} = 1 - \frac{1}{3} = \frac{2}{3}.$$

The graph of the function is shown in Figure 9.1. ■

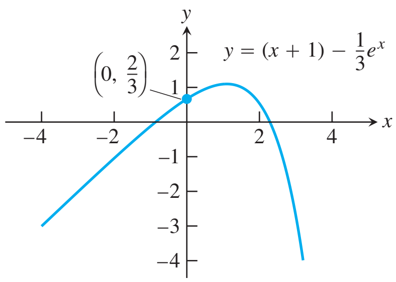


FIGURE 9.1 Graph of the solution $y = (x + 1) - \frac{1}{3}e^x$ to the differential equation $dy/dx = y - x$, with initial condition $y(0) = \frac{2}{3}$ (Example 2).

Slope Fields: Viewing Solution Curves

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the region of the xy-plane that constitutes the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. The resulting picture is called a **slope field** (or **direction field**) and gives a visualization of the general shape of the solution curves. Figure 9.2a shows a slope field, with a particular solution sketched into it in Figure 9.2b. We see how these line segments indicate the direction the solution curve takes at each point it passes through.

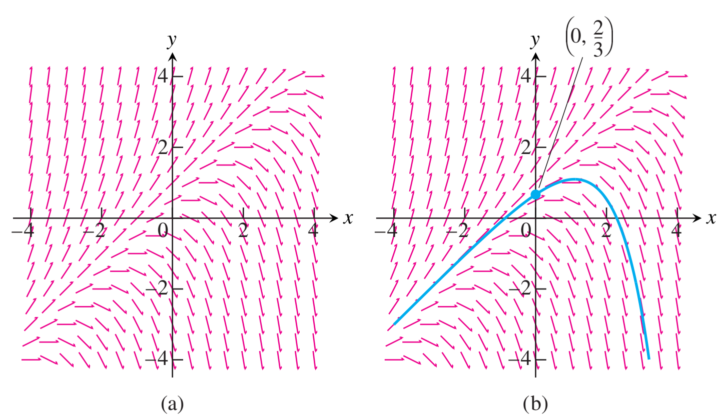
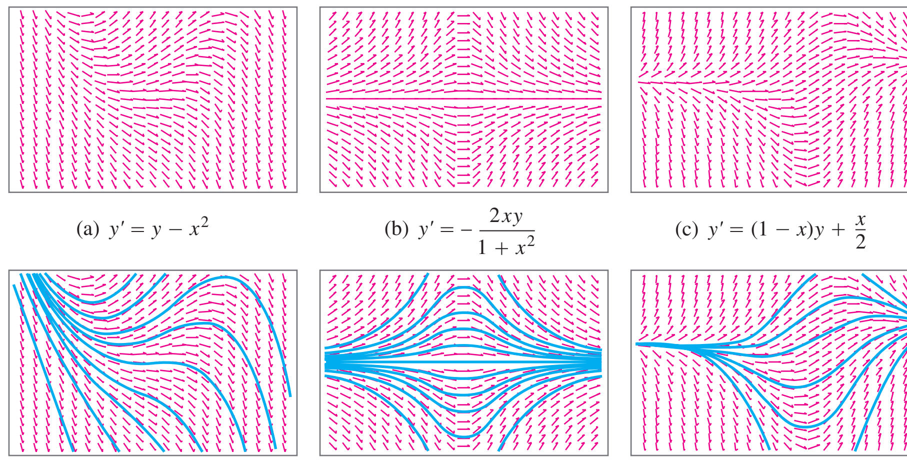


FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).

Figure 9.3 shows three slope fields and we see how the solution curves behave by following the tangent line segments in these fields.



(a) $y' = y - x^2$ (b) $y' = -\dfrac{2xy}{1+x^2}$ (c) $y' = (1-x)y + \dfrac{x}{2}$

FIGURE 9.3 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with arrows, as they are here. This is not to be taken as an indication that slopes have directions, however, for they do not.

Constructing a slope field with pencil and paper can be quite tedious. All our examples were generated by a computer.

While general differential equations are difficult to solve, many important equations that arise in science and applications have special forms that make them solvable by special techniques. One such class is the separable equations.

Separable Equations

The equation $y' = f(x, y)$ is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y).$$

When we rewrite this equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}, \qquad H(y) = \frac{1}{h(y)}$$

its differential form allows us to collect all y terms with dy and all x terms with dx:

$$h(y)\,dy = g(x)\,dx.$$

Now we simply integrate both sides of this equation:

$$\int h(y)\,dy = \int g(x)\,dx. \tag{2}$$

After completing the integrations we obtain the solution y defined implicitly as a function of x.

The justification that we can simply integrate both sides in Equation (2) is based on the Substitution Rule (Section 5.5):

$$\int h(y)\,dy = \int h(y(x))\frac{dy}{dx}\,dx$$

$$= \int h(y(x))\frac{g(x)}{h(y(x))}\,dx \qquad \frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$= \int g(x)\,dx.$$

EXAMPLE 3 Solving a Separable Equation

Solve the differential equation

$$\frac{dy}{dx} = (1 + y^2)e^x.$$

Solution Since $1 + y^2$ is never zero, we can solve the equation by separating the variables.

$$\frac{dy}{dx} = (1 + y^2)e^x$$

$$dy = (1 + y^2)e^x\,dx \qquad \text{Treat } dy/dx \text{ as a quotient of differentials and multiply both sides by } dx.$$

$$\frac{dy}{1 + y^2} = e^x\,dx \qquad \text{Divide by } (1 + y^2).$$

$$\int \frac{dy}{1 + y^2} = \int e^x\,dx \qquad \text{Integrate both sides.}$$

$$\tan^{-1} y = e^x + C \qquad C \text{ represents the combined constants of integration.}$$

The equation $\tan^{-1} y = e^x + C$ gives y as an implicit function of x. When $-\pi/2 < e^x + C < \pi/2$, we can solve for y as an explicit function of x by taking the tangent of both sides:

$$\tan(\tan^{-1} y) = \tan(e^x + C)$$

$$y = \tan(e^x + C).$$

■

EXAMPLE 4 Solve the equation

$$(x + 1)\frac{dy}{dx} = x(y^2 + 1).$$

Solution We change to differential form, separate the variables, and integrate:

$$(x + 1)\,dy = x(y^2 + 1)\,dx$$

$$\frac{dy}{y^2 + 1} = \frac{x\,dx}{x + 1} \qquad x \neq -1$$

$$\int \frac{dy}{1 + y^2} = \int \left(1 - \frac{1}{x + 1}\right)dx$$

$$\tan^{-1} y = x - \ln|x + 1| + C.$$

■

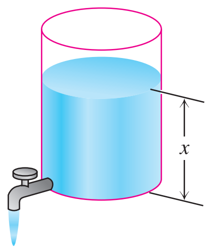


FIGURE 9.4 The rate at which water runs out is $k\sqrt{x}$, where k is a positive constant. In Example 5, $k = 1/2$ and x is measured in feet.

The initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

involves a separable differential equation, and the solution $y = y_0 e^{kt}$ gives the Law of Exponential Change (Section 7.5). We found this initial value problem to be a model for such phenomena as population growth, radioactive decay, and heat transfer. We now present an application involving a different separable first-order equation.

Torricelli's Law

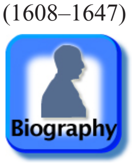
Torricelli's Law says that if you drain a tank like the one in Figure 9.4, the rate at which the water runs out is a constant times the square root of the water's depth x. The constant depends on the size of the drainage hole. In Example 5, we assume that the constant is $1/2$.

EXAMPLE 5 Draining a Tank

A right circular cylindrical tank with radius 5 ft and height 16 ft that was initially full of water is being drained at the rate of $0.5\sqrt{x}\ \text{ft}^3/\text{min}$. Find a formula for the depth and the amount of water in the tank at any time t. How long will it take to empty the tank?

HISTORICAL BIOGRAPHY

Evangelista Torricelli (1608–1647)



Solution The volume of a right circular cylinder with radius r and height h is $V = \pi r^2 h$, so the volume of water in the tank (Figure 9.4) is

$$V = \pi r^2 h = \pi(5)^2 x = 25\pi x.$$

Diffentiation leads to

$$\frac{dV}{dt} = 25\pi \frac{dx}{dt} \qquad \text{Negative because } V \text{ is decreasing and } dx/dt < 0$$

$$-0.5\sqrt{x} = 25\pi \frac{dx}{dt} \qquad \text{Torricelli's Law}$$

Thus we have the initial value problem

$$\frac{dx}{dt} = -\frac{\sqrt{x}}{50\pi},$$

$$x(0) = 16 \qquad \text{The water is 16 ft deep when } t = 0.$$

We solve the differential equation by separating the variables.

$$x^{-1/2}\,dx = -\frac{1}{50\pi}\,dt$$

$$\int x^{-1/2}\,dx = -\int \frac{1}{50\pi}\,dt \qquad \text{Integrate both sides.}$$

$$2x^{1/2} = -\frac{1}{50\pi}t + C \qquad \text{Constants combined}$$

The initial condition $x(0) = 16$ determines the value of C.

$$2(16)^{1/2} = -\frac{1}{50\pi}(0) + C$$

$$C = 8$$

With $C = 8$, we have

$$2x^{1/2} = -\frac{1}{50\pi}t + 8 \qquad \text{or} \qquad x^{1/2} = 4 - \frac{t}{100\pi}.$$

The formulas we seek are

$$x = \left(4 - \frac{t}{100\pi}\right)^2 \qquad \text{and} \qquad V = 25\pi x = 25\pi\left(4 - \frac{t}{100\pi}\right)^2.$$

At any time t, the water in the tank is $(4 - t/(100\pi))^2$ ft deep and the amount of water is $25\pi(4 - t/(100\pi))^2$ ft^3. At $t = 0$, we have $x = 16$ ft and $V = 400\pi$ ft^3, as required. The tank will empty $(V = 0)$ in $t = 400\pi$ minutes, which is about 21 hours. ■

EXERCISES 9.1

Verifying Solutions

In Exercises 1 and 2, show that each function $y = f(x)$ is a solution of the accompanying differential equation.

1. $2y' + 3y = e^{-x}$

a. $y = e^{-x}$ **b.** $y = e^{-x} + e^{-(3/2)x}$

c. $y = e^{-x} + Ce^{-(3/2)x}$

2. $y' = y^2$

a. $y = -\frac{1}{x}$ **b.** $y = -\frac{1}{x + 3}$ **c.** $y = -\frac{1}{x + C}$

In Exercises 3 and 4, show that the function $y = f(x)$ is a solution of the given differential equation.

3. $y = \frac{1}{x}\int_1^x \frac{e^t}{t}\,dt, \quad x^2y' + xy = e^x$

4. $y = \frac{1}{\sqrt{1 + x^4}}\int_1^x \sqrt{1 + t^4}\,dt, \quad y' + \frac{2x^3}{1 + x^4}y = 1$

In Exercises 5–8, show that each function is a solution of the given initial value problem.

	Differential equation	Initial condition	Solution candidate
5.	$y' + y = \frac{2}{1 + 4e^{2x}}$	$y(-\ln 2) = \frac{\pi}{2}$	$y = e^{-x}\tan^{-1}(2e^x)$
6.	$y' = e^{-x^2} - 2xy$	$y(2) = 0$	$y = (x - 2)e^{-x^2}$
7.	$xy' + y = -\sin x,$ $x > 0$	$y\left(\frac{\pi}{2}\right) = 0$	$y = \frac{\cos x}{x}$
8.	$x^2y' = xy - y^2,$ $x > 1$	$y(e) = e$	$y = \frac{x}{\ln x}$

Separable Equations

Solve the differential equation in Exercises 9–18.

9. $2\sqrt{xy}\,\frac{dy}{dx} = 1, \quad x, y > 0$

10. $\frac{dy}{dx} = x^2\sqrt{y}, \quad y > 0$

11. $\frac{dy}{dx} = e^{x-y}$

12. $\frac{dy}{dx} = 3x^2\,e^{-y}$

13. $\frac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y}$

14. $\sqrt{2xy}\,\frac{dy}{dx} = 1$

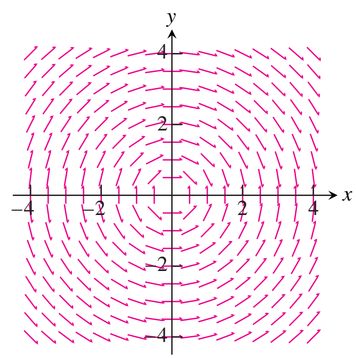
15. $\sqrt{x}\,\frac{dy}{dx} = e^{y+\sqrt{x}}, \quad x > 0$

16. $(\sec x)\frac{dy}{dx} = e^{y+\sin x}$

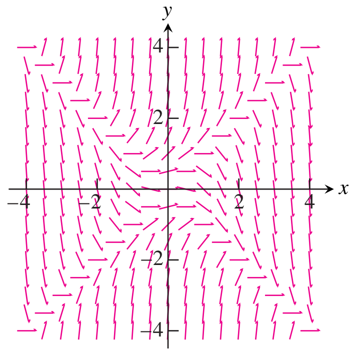
17. $\frac{dy}{dx} = 2x\sqrt{1 - y^2}, \quad -1 < y < 1$

18. $\frac{dy}{dx} = \frac{e^{2x-y}}{e^{x+y}}$

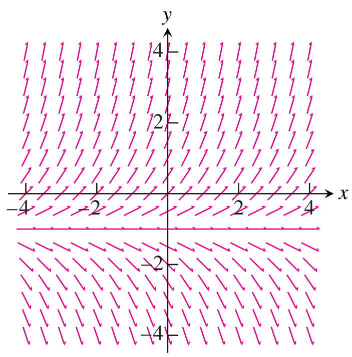
In Exercises 19–22, match the differential equations with their slope fields, graphed here.



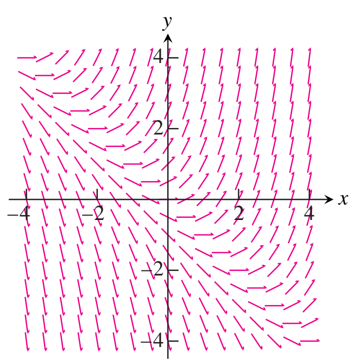
(a)



(b)



(c)



(d)

19. $y' = x + y$

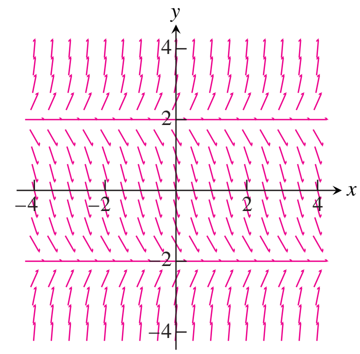
20. $y' = y + 1$

21. $y' = -\frac{x}{y}$

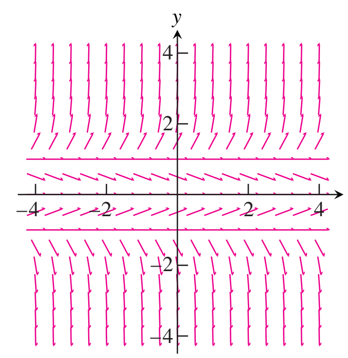
22. $y' = y^2 - x^2$

In Exercises 23 and 24, copy the slope fields and sketch in some of the solution curves.

23. $y' = (y + 2)(y - 2)$



24. $y' = y(y + 1)(y - 1)$



COMPUTER EXPLORATIONS

Slope Fields and Solution Curves

In Exercises 25–30, obtain a slope field and add to it graphs of the solution curves passing through the given points.

25. $y' = y$ with

a. $(0, 1)$ **b.** $(0, 2)$ **c.** $(0, -1)$

26. $y' = 2(y - 4)$ with

a. $(0, 1)$ **b.** $(0, 4)$ **c.** $(0, 5)$

27. $y' = y(x + y)$ with

a. $(0, 1)$ **b.** $(0, -2)$ **c.** $(0, 1/4)$ **d.** $(-1, -1)$

28. $y' = y^2$ with

a. $(0, 1)$ **b.** $(0, 2)$ **c.** $(0, -1)$ **d.** $(0, 0)$

29. $y' = (y - 1)(x + 2)$ with

a. $(0, -1)$ **b.** $(0, 1)$ **c.** $(0, 3)$ **d.** $(1, -1)$

30. $y' = \dfrac{xy}{x^2 + 4}$ with

a. $(0, 2)$ **b.** $(0, -6)$ **c.** $\left(-2\sqrt{3}, -4\right)$

In Exercises 31 and 32, obtain a slope field and graph the particular solution over the specified interval. Use your CAS DE solver to find the general solution of the differential equation.

31. A logistic equation $y' = y(2 - y), \quad y(0) = 1/2;$
$0 \leq x \leq 4, \quad 0 \leq y \leq 3$

32. $y' = (\sin x)(\sin y), \quad y(0) = 2; \quad -6 \leq x \leq 6, \quad -6 \leq y \leq 6$

Exercises 33 and 34 have no explicit solution in terms of elementary functions. Use a CAS to explore graphically each of the differential equations.

33. $y' = \cos(2x - y), \quad y(0) = 2; \quad 0 \leq x \leq 5, \quad 0 \leq y \leq 5;$
$y(2)$

34. A Gompertz equation $y' = y(1/2 - \ln y), \quad y(0) = 1/3;$
$0 \leq x \leq 4, \quad 0 \leq y \leq 3; \quad y(3)$

35. Use a CAS to find the solutions of $y' + y = f(x)$ subject to the initial condition $y(0) = 0$, if $f(x)$ is

a. $2x$ **b.** $\sin 2x$ **c.** $3e^{x/2}$ **d.** $2e^{-x/2}\cos 2x$.

Graph all four solutions over the interval $-2 \leq x \leq 6$ to compare the results.

36. a. Use a CAS to plot the slope field of the differential equation

$$y' = \frac{3x^2 + 4x + 2}{2(y - 1)}$$

over the region $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.

b. Separate the variables and use a CAS integrator to find the general solution in implicit form.

c. Using a CAS implicit function grapher, plot solution curves for the arbitrary constant values $C = -6, -4, -2, 0, 2, 4, 6$.

d. Find and graph the solution that satisfies the initial condition $y(0) = -1$.

9.2 First-Order Linear Differential Equations

The exponential growth/decay equation $dy/dx = ky$ (Section 7.5) is a separable differential equation. It is also a special case of a differential equation having a *linear form*. Linear differential equations model a number of real-world phenomena, including electrical circuits and chemical mixture problems.

A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**.

Since the exponential growth/decay equation can be put in the standard form

$$\frac{dy}{dx} - ky = 0,$$

we see it is a linear equation with $P(x) = -k$ and $Q(x) = 0$. Equation (1) is *linear* (in y) because y and its derivative dy/dx occur only to the first power, are not multiplied together, nor do they appear as the argument of a function (such as $\sin y$, e^y, or $\sqrt{dy/dx}$).

EXAMPLE 1 Finding the Standard Form

Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution

$$x\frac{dy}{dx} = x^2 + 3y$$

$$\frac{dy}{dx} = x + \frac{3}{x}y \qquad \text{Divide by } x$$

$$\frac{dy}{dx} - \frac{3}{x}y = x \qquad \text{Standard form with } P(x) = -3/x \text{ and } Q(x) = x$$

Notice that $P(x)$ is $-3/x$, not $+3/x$. The standard form is $y' + P(x)y = Q(x)$, so the minus sign is part of the formula for $P(x)$. ■

Solving Linear Equations

We solve the equation

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{2}$$

by multiplying both sides by a *positive* function $v(x)$ that transforms the left side into the derivative of the product $v(x) \cdot y$. We will show how to find v in a moment, but first we want to show how, once found, it provides the solution we seek.

Here is why multiplying by $v(x)$ works:

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{Original equation is in standard form.}$$

$$v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x) \qquad \text{Multiply by positive } v(x).$$

$$\frac{d}{dx}(v(x) \cdot y) = v(x)Q(x) \qquad v(x) \text{ is chosen to make } v\frac{dy}{dx} + Pvy = \frac{d}{dx}(v \cdot y).$$

$$v(x) \cdot y = \int v(x)Q(x)\, dx \qquad \text{Integrate with respect to } x.$$

$$y = \frac{1}{v(x)}\int v(x)Q(x)\, dx \tag{3}$$

Equation (3) expresses the solution of Equation (2) in terms of the function $v(x)$ and $Q(x)$. We call $v(x)$ an **integrating factor** for Equation (2) because its presence makes the equation integrable.

Why doesn't the formula for $P(x)$ appear in the solution as well? It does, but indirectly, in the construction of the positive function $v(x)$. We have

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + Pvy \qquad \text{Condition imposed on } v$$

$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + Pvy \qquad \text{Product Rule for derivatives}$$

$$y\frac{dv}{dx} = Pvy \qquad \text{The terms } v\frac{dy}{dx} \text{ cancel.}$$

This last equation will hold if

$$\frac{dv}{dx} = Pv$$

$$\frac{dv}{v} = P\, dx \qquad \text{Variables separated, } v > 0$$

$$\int \frac{dv}{v} = \int P\, dx \qquad \text{Integrate both sides.}$$

$$\ln v = \int P\, dx \qquad \text{Since } v > 0, \text{ we do not need absolute value signs in } \ln v.$$

$$e^{\ln v} = e^{\int P\, dx} \qquad \text{Exponentiate both sides to solve for } v.$$

$$v = e^{\int P\, dx} \tag{4}$$

Thus a formula for the general solution to Equation (1) is given by Equation (3), where $v(x)$ is given by Equation (4). However, rather than memorizing the formula, just remember how to find the integrating factor once you have the standard form so $P(x)$ is correctly identified.

> To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

When you integrate the left-side product in this procedure, you always obtain the product $v(x)y$ of the integrating factor and solution function y because of the way v is defined.

HISTORICAL BIOGRAPHY

Adrien Marie Legendre (1752–1833)

EXAMPLE 2 Solving a First-Order Linear Differential Equation

Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution First we put the equation in standard form (Example 1):

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

so $P(x) = -3/x$ is identified.

The integrating factor is

$$\begin{aligned} v(x) = e^{\int P(x)\,dx} &= e^{\int (-3/x)\,dx} \\ &= e^{-3\ln|x|} && \text{Constant of integration is 0, so } v \text{ is as simple as possible.} \\ &= e^{-3\ln x} && x > 0 \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

Next we multiply both sides of the standard form by $v(x)$ and integrate:

$$\begin{aligned} \frac{1}{x^3}\cdot\left(\frac{dy}{dx} - \frac{3}{x}y\right) &= \frac{1}{x^3}\cdot x \\ \frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y &= \frac{1}{x^2} \\ \frac{d}{dx}\left(\frac{1}{x^3}y\right) &= \frac{1}{x^2} && \text{Left side is } \tfrac{d}{dx}(v\cdot y). \\ \frac{1}{x^3}y &= \int \frac{1}{x^2}\,dx && \text{Integrate both sides.} \\ \frac{1}{x^3}y &= -\frac{1}{x} + C. \end{aligned}$$

Solving this last equation for y gives the general solution:

$$y = x^3\left(-\frac{1}{x} + C\right) = -x^2 + Cx^3, \qquad x > 0.$$

■

EXAMPLE 3 Solving a First-Order Linear Initial Value Problem

Solve the equation

$$xy' = x^2 + 3y, \qquad x > 0,$$

given the initial condition $y(1) = 2$.

Solution We first solve the differential equation (Example 2), obtaining

$$y = -x^2 + Cx^3, \qquad x > 0.$$

We then use the initial condition to find C:

$$\begin{aligned} y &= -x^2 + Cx^3 \\ 2 &= -(1)^2 + C(1)^3 && y = 2 \text{ when } x = 1 \\ C &= 2 + (1)^2 = 3. \end{aligned}$$

The solution of the initial value problem is the function $y = -x^2 + 3x^3$. ■

EXAMPLE 4 Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.

Solution With $x > 0$, we write the equation in standard form:

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}.$$

Then the integrating factor is given by

$$v = e^{\int -dx/3x} = e^{(-1/3)\ln x} = x^{-1/3}. \qquad x > 0$$

Thus

$$x^{-1/3}y = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx. \qquad \text{Left side is } vy.$$

Integration by parts of the right side gives

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) + \int x^{-4/3}\,dx + C.$$

Therefore

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C$$

or, solving for y,

$$y = -(\ln x + 4) + Cx^{1/3}.$$

When $x = 1$ and $y = -2$ this last equation becomes

$$-2 = -(0 + 4) + C,$$

so

$$C = 2.$$

Substitution into the equation for y gives the particular solution

$$y = 2x^{1/3} - \ln x - 4. \qquad \blacksquare$$

In solving the linear equation in Example 2, we integrated both sides of the equation after multiplying each side by the integrating factor. However, we can shorten the amount of work, as in Example 4, by remembering that the left side *always* integrates into the product $v(x) \cdot y$ of the integrating factor times the solution function. From Equation (3) this means that

$$v(x)y = \int v(x)Q(x)\, dx.$$

We need only integrate the product of the integrating factor $v(x)$ with the right side $Q(x)$ of Equation (1) and then equate the result with $v(x)y$ to obtain the general solution. Nevertheless, to emphasize the role of $v(x)$ in the solution process, we sometimes follow the complete procedure as illustrated in Example 2.

Observe that if the function $Q(x)$ is identically zero in the standard form given by Equation (1), the linear equation is separable:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} + P(x)y = 0 \qquad Q(x) \equiv 0$$

$$dy = -P(x)\, dx \qquad \text{Separating the variables}$$

We now present two applied problems modeled by a first-order linear differential equation.

RL Circuits

The diagram in Figure 9.5 represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source of V volts.

Ohm's Law, $V = RI$, has to be modified for such a circuit. The modified form is

$$L\frac{di}{dt} + Ri = V, \tag{5}$$

where i is the intensity of the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current will flow after the switch is closed.

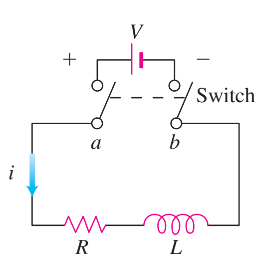


FIGURE 9.5 The *RL* circuit in Example 5.

EXAMPLE 5 Electric Current Flow

The switch in the *RL* circuit in Figure 9.5 is closed at time $t = 0$. How will the current flow as a function of time?

Solution Equation (5) is a first-order linear differential equation for i as a function of t. Its standard form is

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}, \tag{6}$$

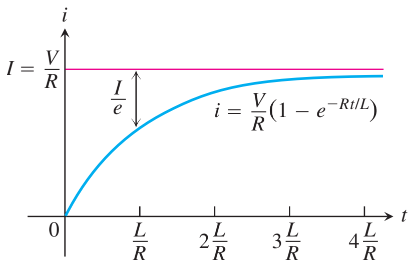


FIGURE 9.6 The growth of the current in the *RL* circuit in Example 5. *I* is the current's steady-state value. The number $t = L/R$ is the time constant of the circuit. The current gets to within 5% of its steady-state value in 3 time constants (Exercise 31).

and the corresponding solution, given that $i = 0$ when $t = 0$, is

$$i = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t} \tag{7}$$

(Exercise 32). Since R and L are positive, $-(R/L)$ is negative and $e^{-(R/L)t} \to 0$ as $t \to \infty$. Thus,

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-(R/L)t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \frac{V}{R}.$$

At any given time, the current is theoretically less than V/R, but as time passes, the current approaches the **steady-state value** V/R. According to the equation

$$L\frac{di}{dt} + Ri = V,$$

$I = V/R$ is the current that will flow in the circuit if either $L = 0$ (no inductance) or $di/dt = 0$ (steady current, $i =$ constant) (Figure 9.6).

Equation (7) expresses the solution of Equation (6) as the sum of two terms: a **steady-state solution** V/R and a **transient solution** $-(V/R)e^{-(R/L)t}$ that tends to zero as $t \to \infty$. ■

Mixture Problems

A chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process, it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula

$$\begin{matrix}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{matrix} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs.}\end{pmatrix} \tag{8}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\begin{aligned}\text{Departure rate} &= \frac{y(t)}{V(t)}\cdot(\text{outflow rate})\\ &= \begin{pmatrix}\text{concentration in}\\ \text{container at time } t\end{pmatrix}\cdot(\text{outflow rate}).\end{aligned} \tag{9}$$

Accordingly, Equation (8) becomes

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)}\cdot(\text{outflow rate}). \tag{10}$$

If, say, y is measured in pounds, V in gallons, and t in minutes, the units in Equation (10) are

$$\frac{\text{pounds}}{\text{minutes}} = \frac{\text{pounds}}{\text{minutes}} - \frac{\text{pounds}}{\text{gallons}}\cdot\frac{\text{gallons}}{\text{minutes}}.$$

Video

EXAMPLE 6 Oil Refinery Storage Tank

In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of

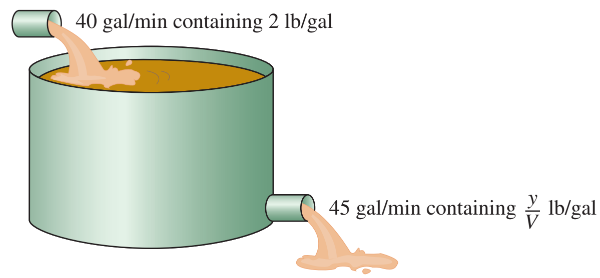


FIGURE 9.7 The storage tank in Example 6 mixes input liquid with stored liquid to produce an output liquid.

additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 9.7)?

Solution Let y be the amount (in pounds) of additive in the tank at time t. We know that $y = 100$ when $t = 0$. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$\begin{aligned} V(t) &= 2000 \text{ gal} + \left(40 \frac{\text{gal}}{\text{min}} - 45 \frac{\text{gal}}{\text{min}}\right)(t \text{ min}) \\ &= (2000 - 5t) \text{ gal}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Rate out} &= \frac{y(t)}{V(t)} \cdot \text{outflow rate} && \text{Eq. (9)} \\ &= \left(\frac{y}{2000 - 5t}\right) 45 && \text{Outflow rate is 45 gal/min. and } v = 2000 - 5t. \\ &= \frac{45y}{2000 - 5t} \frac{\text{lb}}{\text{min}}. \end{aligned}$$

Also,

$$\begin{aligned} \text{Rate in} &= \left(2 \frac{\text{lb}}{\text{gal}}\right)\left(40 \frac{\text{gal}}{\text{min}}\right) \\ &= 80 \frac{\text{lb}}{\text{min}}. && \text{Eq. (10)} \end{aligned}$$

The differential equation modeling the mixture process is

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t}$$

in pounds per minute.

To solve this differential equation, we first write it in standard form:

$$\frac{dy}{dt} + \frac{45}{2000 - 5t} y = 80.$$

Thus, $P(t) = 45/(2000 - 5t)$ and $Q(t) = 80$.

The integrating factor is

$$\begin{aligned} v(t) &= e^{\int P\,dt} = e^{\int \frac{45}{2000-5t}\,dt} \\ &= e^{-9\ln(2000-5t)} \qquad 2000 - 5t > 0 \\ &= (2000-5t)^{-9}. \end{aligned}$$

Multiplying both sides of the standard equation by $v(t)$ and integrating both sides gives,

$$\begin{aligned} (2000-5t)^{-9}\cdot\left(\frac{dy}{dt} + \frac{45}{2000-5t}\,y\right) &= 80(2000-5t)^{-9} \\ (2000-5t)^{-9}\frac{dy}{dt} + 45(2000-5t)^{-10}y &= 80(2000-5t)^{-9} \\ \frac{d}{dt}\left[(2000-5t)^{-9}y\right] &= 80(2000-5t)^{-9} \\ (2000-5t)^{-9}y &= \int 80(2000-5t)^{-9}\,dt \\ (2000-5t)^{-9}y &= 80\cdot\frac{(2000-5t)^{-8}}{(-8)(-5)} + C. \end{aligned}$$

The general solution is

$$y = 2(2000-5t) + C(2000-5t)^{9}.$$

Because $y = 100$ when $t = 0$, we can determine the value of C:

$$\begin{aligned} 100 &= 2(2000-0) + C(2000-0)^{9} \\ C &= -\frac{3900}{(2000)^{9}}. \end{aligned}$$

The particular solution of the initial value problem is

$$y = 2(2000-5t) - \frac{3900}{(2000)^{9}}(2000-5t)^{9}.$$

The amount of additive 20 min after the pumping begins is

$$y(20) = 2[2000-5(20)] - \frac{3900}{(2000)^{9}}[2000-5(20)]^{9} \approx 1342 \text{ lb}.$$ ■

EXERCISES 9.2

First-Order Linear Equations

Solve the differential equations in Exercises 1–14.

1. $x \frac{dy}{dx} + y = e^x, \quad x > 0$

2. $e^x \frac{dy}{dx} + 2e^x y = 1$

3. $xy' + 3y = \frac{\sin x}{x^2}, \quad x > 0$

4. $y' + (\tan x)y = \cos^2 x, \quad -\pi/2 < x < \pi/2$

5. $x \frac{dy}{dx} + 2y = 1 - \frac{1}{x}, \quad x > 0$

6. $(1 + x)y' + y = \sqrt{x}$

7. $2y' = e^{x/2} + y$

8. $e^{2x} y' + 2e^{2x} y = 2x$

9. $xy' - y = 2x \ln x$

10. $x \frac{dy}{dx} = \frac{\cos x}{x} - 2y, \quad x > 0$

11. $(t - 1)^3 \frac{ds}{dt} + 4(t - 1)^2 s = t + 1, \quad t > 1$

12. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

13. $\sin\theta \dfrac{dr}{d\theta} + (\cos\theta)r = \tan\theta, \quad 0 < \theta < \pi/2$

14. $\tan\theta \dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \pi/2$

Solving Initial Value Problems

Solve the initial value problems in Exercises 15–20.

15. $\dfrac{dy}{dt} + 2y = 3, \quad y(0) = 1$

16. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

17. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y(\pi/2) = 1$

18. $\theta\dfrac{dy}{d\theta} - 2y = \theta^3\sec\theta\tan\theta, \quad \theta > 0, \quad y(\pi/3) = 2$

19. $(x+1)\dfrac{dy}{dx} - 2(x^2+x)y = \dfrac{e^{x^2}}{x+1}, \quad x > -1, \quad y(0) = 5$

20. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

21. Solve the exponential growth/decay initial value problem for y as a function of t thinking of the differential equation as a first-order linear equation with $P(x) = -k$ and $Q(x) = 0$:

$$\frac{dy}{dt} = ky \quad (k \text{ constant}), \quad y(0) = y_0$$

22. Solve the following initial value problem for u as a function of t:

$$\frac{du}{dt} + \frac{k}{m}u = 0 \quad (k \text{ and } m \text{ positive constants}), \quad u(0) = u_0$$

a. as a first-order linear equation.

b. as a separable equation.

Theory and Examples

23. Is either of the following equations correct? Give reasons for your answers.

a. $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + C$ **b.** $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + Cx$

24. Is either of the following equations correct? Give reasons for your answers.

a. $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + C$

b. $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + \dfrac{C}{\cos x}$

25. Salt mixture A tank initially contains 100 gal of brine in which 50 lb of salt are dissolved. A brine containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 4 gal/min.

a. At what rate (pounds per minute) does salt enter the tank at time t?

b. What is the volume of brine in the tank at time t?

c. At what rate (pounds per minute) does salt leave the tank at time t?

d. Write down and solve the initial value problem describing the mixing process.

e. Find the concentration of salt in the tank 25 min after the process starts.

26. Mixture problem A 200-gal tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 lb/gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.

a. At what time will the tank be full?

b. At the time the tank is full, how many pounds of concentrate will it contain?

27. Fertilizer mixture A tank contains 100 gal of fresh water. A solution containing 1 lb/gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at the rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

28. Carbon monoxide pollution An executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft^3/min. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 ft^3/min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

29. Current in a closed *RL* circuit How many seconds after the switch in an RL circuit is closed will it take the current i to reach half of its steady state value? Notice that the time depends on R and L and not on how much voltage is applied.

30. Current in an open *RL* circuit If the switch is thrown open after the current in an RL circuit has built up to its steady-state value $I = V/R$, the decaying current (graphed here) obeys the equation

$$L\frac{di}{dt} + Ri = 0,$$

which is Equation (5) with $V = 0$.

a. Solve the equation to express i as a function of t.

b. How long after the switch is thrown will it take the current to fall to half its original value?

c. Show that the value of the current when $t = L/R$ is I/e. (The significance of this time is explained in the next exercise.)

9.5 Applications of First-Order Differential Equations

We now look at three applications of the differential equations we have been studying. The first application analyzes an object moving along a straight line while subject to a force opposing its motion. The second is a model of population growth which takes into account factors in the environment placing limits on growth, such as the availability of food or other vital resources. The last application considers a curve or curves intersecting each curve in a second family of curves *orthogonally* (that is, at right angles).

Resistance Proportional to Velocity

In some cases it is reasonable to assume that the resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The faster the object moves, the more its forward progress is resisted by the air through which it passes. To describe this in mathematical terms, we picture the object as a mass m moving along a coordinate line with position function s and velocity v at time t. From Newton's second law of motion, the resisting force opposing the motion is

$$\text{Force} = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}.$$

We can express the assumption that the resisting force is proportional to velocity by writing

$$m\frac{dv}{dt} = -kv \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{k}{m}v \qquad (k > 0).$$

This is a separable differential equation representing exponential change. The solution to the equation with initial condition $v = v_0$ at $t = 0$ is (Section 7.5)

$$v = v_0 e^{-(k/m)t}. \tag{1}$$

What can we learn from Equation (1)? For one thing, we can see that if m is something large, like the mass of a 20,000-ton ore boat in Lake Erie, it will take a long time for the velocity to approach zero (because t must be large in the exponent of the equation in order to make kt/m large enough for v to be small). We can learn even more if we integrate Equation (1) to find the position s as a function of time t.

Suppose that a body is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast? To find out, we start with Equation (1) and solve the initial value problem

$$\frac{ds}{dt} = v_0 e^{-(k/m)t}, \qquad s(0) = 0.$$

Integrating with respect to t gives

$$s = -\frac{v_0 m}{k}e^{-(k/m)t} + C.$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \qquad \text{and} \qquad C = \frac{v_0 m}{k}.$$

The body's position at time t is therefore

$$s(t) = -\frac{v_0 m}{k} e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right). \tag{2}$$

To find how far the body will coast, we find the limit of $s(t)$ as $t \to \infty$. Since $-(k/m) < 0$, we know that $e^{-(k/m)t} \to 0$ as $t \to \infty$, so that

$$\begin{aligned}\lim_{t\to\infty} s(t) &= \lim_{t\to\infty} \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right)\\ &= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.\end{aligned}$$

Thus,

$$\text{Distance coasted} = \frac{v_0 m}{k}. \tag{3}$$

This is an ideal figure, of course. Only in mathematics can time stretch to infinity. The number $v_0 m/k$ is only an upper bound (albeit a useful one). It is true to life in one respect, at least: if m is large, it will take a lot of energy to stop the body. That is why ocean liners have to be docked by tugboats. Any liner of conventional design entering a slip with enough speed to steer would smash into the pier before it could stop.

EXAMPLE 1 A Coasting Ice Skater

For a 192-lb ice skater, the k in Equation (1) is about $1/3$ slug/sec and $m = 192/32 = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

In the English system, where weight is measured in pounds, mass is measured in **slugs**. Thus,

$$\text{Pounds} = \text{slugs} \times 32,$$

assuming the gravitational constant is 32 ft/sec^2.

Solution We answer the first question by solving Equation (1) for t:

$$\begin{aligned}11e^{-t/18} &= 1 \qquad \text{Eq. (1) with } k = 1/3,\ m = 6,\ v_0 = 11,\ v = 1\\ e^{-t/18} &= 1/11\\ -t/18 &= \ln(1/11) = -\ln 11\\ t &= 18 \ln 11 \approx 43 \text{ sec}.\end{aligned}$$

We answer the second question with Equation (3):

$$\begin{aligned}\text{Distance coasted} &= \frac{v_0 m}{k} = \frac{11 \cdot 6}{1/3}\\ &= 198 \text{ ft}.\end{aligned}$$

■

Modeling Population Growth

In Section 7.5 we modeled population growth with the Law of Exponential Change:

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

where P is the population at time t, $k > 0$ is a constant growth rate, and P_0 is the size of the population at time $t = 0$. In Section 7.5 we found the solution $P = P_0 e^{kt}$ to this model. However, an issue to be addressed is "how good is the model?"

To begin an assessment of the model, notice that the exponential growth differential equation says that

$$\frac{dP/dt}{P} = k \tag{4}$$

is constant. This rate is called the **relative growth rate**. Now, Table 9.4 gives the world population at midyear for the years 1980 to 1989. Taking $dt = 1$ and $dP \approx \Delta P$, we see from the table that the relative growth rate in Equation (4) is approximately the constant 0.017. Thus, based on the tabled data with $t = 0$ representing 1980, $t = 1$ representing 1981, and so forth, the world population could be modeled by

$$\text{Differential equation:} \quad \frac{dP}{dt} = 0.017P$$

$$\text{Initial condition:} \quad P(0) = 4454.$$

TABLE 9.4 World population (midyear)

Year	Population (millions)	$\Delta P/P$
1980	4454	$76/4454 \approx 0.0171$
1981	4530	$80/4530 \approx 0.0177$
1982	4610	$80/4610 \approx 0.0174$
1983	4690	$80/4690 \approx 0.0171$
1984	4770	$81/4770 \approx 0.0170$
1985	4851	$82/4851 \approx 0.0169$
1986	4933	$85/4933 \approx 0.0172$
1987	5018	$87/5018 \approx 0.0173$
1988	5105	$85/5105 \approx 0.0167$
1989	5190	

Source: U.S. Bureau of the Census (Sept., 1999): www.census.gov/ipc/www/worldpop.html.

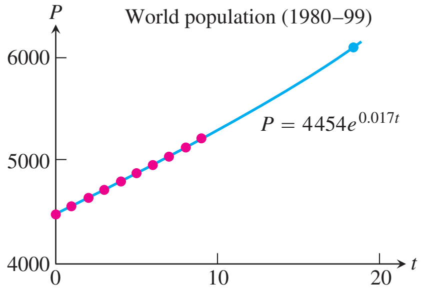


FIGURE 9.23 Notice that the value of the solution $P = 4454e^{0.017t}$ is 6152.16 when $t = 19$, which is slightly higher than the actual population in 1999.

The solution to this initial value problem gives the population function $P = 4454e^{0.017t}$. In year 1999 (so $t = 19$), the solution predicts the world population in midyear to be about 6152 million, or 6.15 billion (Figure 9.23), which is more than the actual population of 6001 million given by the U.S. Bureau of the Census (Table 9.5). Let's examine more recent data to see if there is a change in the growth rate.

Table 9.5 shows the world population for the years 1990 to 2002. From the table we see that the relative growth rate is positive but decreases as the population increases due to environmental, economic, and other factors. On average, the growth rate decreases by about 0.0003 per year over the years 1990 to 2002. That is, the graph of k in Equation (4) is closer to being a line with a negative slope $-r = -0.0003$. In Example 5 of Section 9.4 we proposed the more realistic **logistic growth model**

$$\frac{dP}{dt} = r(M - P)P, \tag{5}$$

where M is the maximum population, or **carrying capacity**, that the environment is capable of sustaining in the long run. Comparing Equation (5) with the exponential model, we see that $k = r(M - P)$ is a linearly decreasing function of the population rather than a constant. The graphical solution curves to the logistic model of Equation (5) were obtained in Section 9.4 and are displayed (again) in Figure 9.24. Notice from the graphs that if $P < M$, the population grows toward M; if $P > M$, the growth rate will be negative (as $r > 0$, $M > 0$) and the population decreasing.

TABLE 9.5 Recent world population

Year	Population (millions)	$\Delta P/P$
1990	5275	$84/5275 \approx 0.0159$
1991	5359	$84/5359 \approx 0.0157$
1992	5443	$81/5443 \approx 0.0149$
1993	5524	$81/5524 \approx 0.0147$
1994	5605	$80/5605 \approx 0.0143$
1995	5685	$79/5685 \approx 0.0139$
1996	5764	$80/5764 \approx 0.0139$
1997	5844	$79/5844 \approx 0.0135$
1998	5923	$78/5923 \approx 0.0132$
1999	6001	$78/6001 \approx 0.0130$
2000	6079	$73/6079 \approx 0.0120$
2001	6152	$76/6152 \approx 0.0124$
2002	6228	?
2003	?	

Source: U.S. Bureau of the Census (Sept., 2003): www.census.gov/ipc/www/worldpop.html.

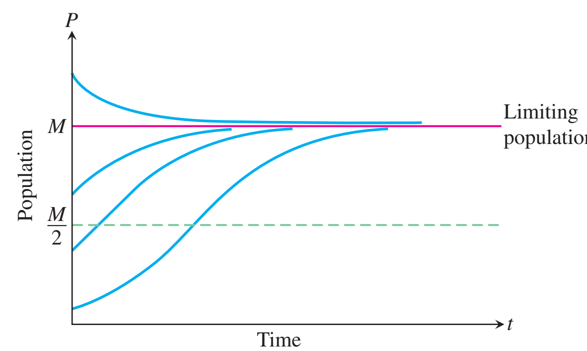


FIGURE 9.24 Solution curves to the logistic population model $dP/dt = r(M - P)P$.

EXAMPLE 2 Modeling a Bear Population

A national park is known to be capable of supporting 100 grizzly bears, but no more. Ten bears are in the park at present. We model the population with a logistic differential equation with $r = 0.001$ (although the model may not give reliable results for very small population levels).

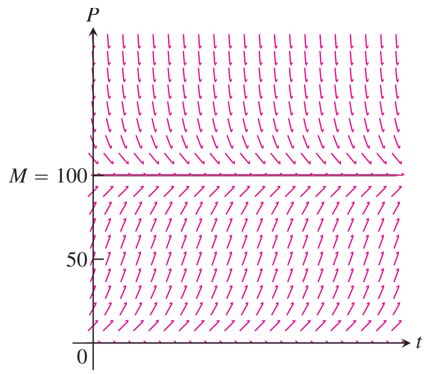


FIGURE 9.25 A slope field for the logistic differential equation $dP/dt = 0.001(100 - P)P$ (Example 2).

(a) Draw and describe a slope field for the differential equation.

(b) Use Euler's method with step size $dt = 1$ to estimate the population size in 20 years.

(c) Find a logistic growth analytic solution $P(t)$ for the population and draw its graph.

(d) When will the bear population reach 50?

Solution

(a) *Slope field.* The carrying capacity is 100, so $M = 100$. The solution we seek is a solution to the following differential equation.

$$\frac{dP}{dt} = 0.001(100 - P)P$$

Figure 9.25 shows a slope field for this differential equation. There appears to be a horizontal asymptote at $P = 100$. The solution curves fall toward this level from above and rise toward it from below.

(b) *Euler's method.* With step size $dt = 1$, $t_0 = 0$, $P(0) = 10$, and

$$\frac{dP}{dt} = f(t, P) = 0.001(100 - P)P,$$

we obtain the approximations in Table 9.6, using the iteration formula

$$P_n = P_{n-1} + 0.001(100 - P_{n-1})P_{n-1}.$$

TABLE 9.6 Euler solution of $dP/dt = 0.001(100 - P)P$, $P(0) = 10$, step size $dt = 1$

t	P (Euler)	t	P (Euler)
0	10		
1	10.9	11	24.3629
2	11.8712	12	26.2056
3	12.9174	13	28.1395
4	14.0423	14	30.1616
5	15.2493	15	32.2680
6	16.5417	16	34.4536
7	17.9222	17	36.7119
8	19.3933	18	39.0353
9	20.9565	19	41.4151
10	22.6130	20	43.8414

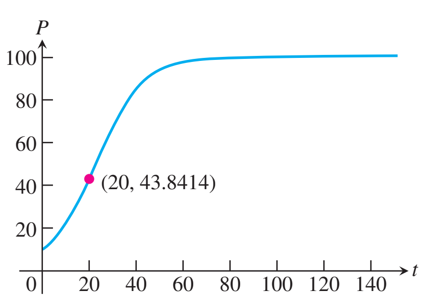


FIGURE 9.26 Euler approximations of the solution to $dP/dt = 0.001(100 - P)P$, $P(0) = 10$, step size $dt = 1$.

There are approximately 44 grizzly bears after 20 years. Figure 9.26 shows a graph of the Euler approximation over the interval $0 \leq t \leq 150$ with step size $dt = 1$. It looks like the lower curves we sketched in Figure 9.24.

(c) *Analytic solution*. We can assume that $t = 0$ when the bear population is 10, so $P(0) = 10$. The logistic growth model we seek is the solution to the following initial value problem.

$$\text{Differential equation:} \quad \frac{dP}{dt} = 0.001(100 - P)P$$

$$\text{Initial condition:} \quad P(0) = 10$$

To prepare for integration, we rewrite the differential equation in the form

$$\frac{1}{P(100 - P)} \frac{dP}{dt} = 0.001.$$

Using partial fraction decomposition on the left-hand side and multiplying both sides by 100, we get

$$\left(\frac{1}{P} + \frac{1}{100 - P}\right)\frac{dP}{dt} = 0.1$$

$$\ln|P| - \ln|100 - P| = 0.1t + C \qquad \text{Integrate with respect to } t.$$

$$\ln\left|\frac{P}{100 - P}\right| = 0.1t + C$$

$$\ln\left|\frac{100 - P}{P}\right| = -0.1t - C \qquad \ln\frac{a}{b} = -\ln\frac{b}{a}$$

$$\left|\frac{100 - P}{P}\right| = e^{-0.1t - C} \qquad \text{Exponentiate.}$$

$$\frac{100 - P}{P} = (\pm e^{-C})e^{-0.1t}$$

$$\frac{100}{P} - 1 = Ae^{-0.1t} \qquad \text{Let } A = \pm e^{-c}.$$

$$P = \frac{100}{1 + Ae^{-0.1t}}. \qquad \text{Solve for } P.$$

This is the general solution to the differential equation. When $t = 0$, $P = 10$, and we obtain

$$10 = \frac{100}{1 + Ae^0}$$

$$1 + A = 10$$

$$A = 9.$$

Thus, the logistic growth model is

$$P = \frac{100}{1 + 9e^{-0.1t}}.$$

Its graph (Figure 9.27) is superimposed on the slope field from Figure 9.25.

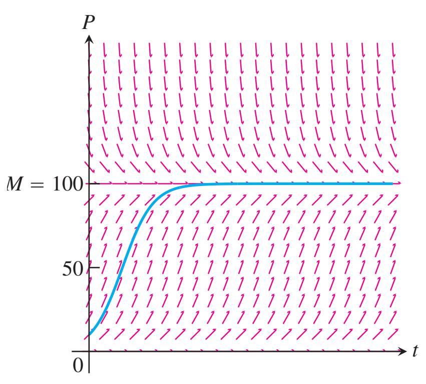


FIGURE 9.27 The graph of

$$P = \frac{100}{1 + 9e^{-0.1t}}$$

superimposed on the slope field in Figure 9.25 (Example 2).

(d) When will the bear population reach 50? For this model,

$$\begin{aligned} 50 &= \frac{100}{1 + 9e^{-0.1t}} \\ 1 + 9e^{-0.1t} &= 2 \\ e^{-0.1t} &= \frac{1}{9} \\ e^{0.1t} &= 9 \\ t &= \frac{\ln 9}{0.1} \approx 22 \text{ years.} \end{aligned}$$

The solution of the general logistic differential equation

$$\frac{dP}{dt} = r(M - P)P$$

can be obtained as in Example 2. In Exercise 10, we ask you to show that the solution is

$$P = \frac{M}{1 + Ae^{-rMt}}.$$

The value of A is determined by an appropriate initial condition.

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family at right angles, or *orthogonally* (Figure 9.28). For instance, each straight line through the origin is an orthogonal trajectory of the family of circles $x^2 + y^2 = a^2$, centered at the origin (Figure 9.29). Such mutually orthogonal systems of curves are of particular importance in physical problems related to electrical potential, where the curves in one family correspond to flow of electric current and those in the other family correspond to curves of constant potential. They also occur in hydrodynamics and heat-flow problems.

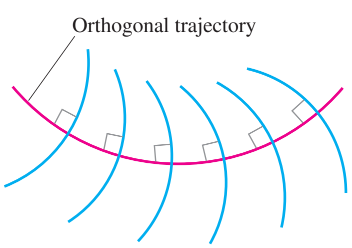


FIGURE 9.28 An orthogonal trajectory intersects the family of curves at right angles, or orthogonally.

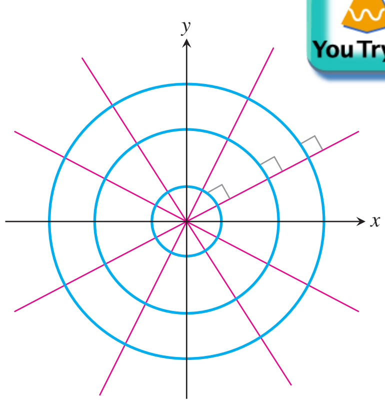


FIGURE 9.29 Every straight line through the origin is orthogonal to the family of circles centered at the origin.

EXAMPLE 3 Finding Orthogonal Trajectories

Find the orthogonal trajectories of the family of curves $xy = a$, where $a \neq 0$ is an arbitrary constant.

Solution The curves $xy = a$ form a family of hyperbolas with asymptotes $y = \pm x$. First we find the slopes of each curve in this family, or their dy/dx values. Differentiating $xy = a$ implicitly gives

$$x\frac{dy}{dx} + y = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

Thus the slope of the tangent line at any point (x, y) on one of the hyperbolas $xy = a$ is $y' = -y/x$. On an orthogonal trajectory the slope of the tangent line at this same point must be the negative reciprocal, or x/y. Therefore, the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

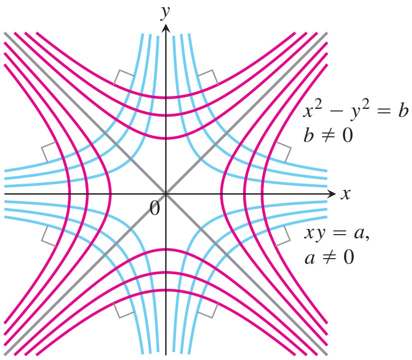


FIGURE 9.30 Each curve is orthogonal to every curve it meets in the other family (Example 3).

This differential equation is separable and we solve it as in Section 9.1:

$$y\,dy = x\,dx \qquad \text{Separate variables.}$$

$$\int y\,dy = \int x\,dx \qquad \text{Integrate both sides.}$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 - x^2 = b, \tag{6}$$

where $b = 2C$ is an arbitrary constant. The orthogonal trajectories are the family of hyperbolas given by Equation (6) and sketched in Figure 9.30. ■

EXERCISES 9.5

Exercises

1. **Coasting bicycle** A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The k in Equation (1) is about 3.9 kg/sec.
   - **a.** About how far will the cyclist coast before reaching a complete stop?
   - **b.** How long will it take the cyclist's speed to drop to 1 m/sec?
2. **Coasting battleship** Suppose that an Iowa class battleship has mass around 51,000 metric tons (51,000,000 kg) and a k value in Equation (1) of about 59,000 kg/sec. Assume that the ship loses power when it is moving at a speed of 9 m/sec.
   - **a.** About how far will the ship coast before it is dead in the water?
   - **b.** About how long will it take the ship's speed to drop to 1 m/sec?
3. The data in Table 9.7 were collected with a motion detector and a CBL™ by Valerie Sharritts, a mathematics teacher at St. Francis DeSales High School in Columbus, Ohio. The table shows the distance s (meters) coasted on in-line skates in t sec by her daughter Ashley when she was 10 years old. Find a model for Ashley's position given by the data in Table 9.7 in the form of Equation (2). Her initial velocity was $v_0 = 2.75$ m/sec, her mass $m = 39.92$ kg (she weighed 88 lb), and her total coasting distance was 4.91 m.
4. **Coasting to a stop** Table 9.8 shows the distance s (meters) coasted on in-line skates in terms of time t (seconds) by Kelly Schmitzer. Find a model for her position in the form of Equation (2). Her initial velocity was $v_0 = 0.80$ m/sec, her mass $m = 49.90$ kg (110 lb), and her total coasting distance was 1.32 m.
5. **Guppy population** A 2000-gal tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0015(150 - P)P,$$

where time t is in weeks.

TABLE 9.7 Ashley Sharritts skating data

t (sec)	*s* (m)	*t* (sec)	*s* (m)	*t* (sec)	*s* (m)
0	0	2.24	3.05	4.48	4.77
0.16	0.31	2.40	3.22	4.64	4.82
0.32	0.57	2.56	3.38	4.80	4.84
0.48	0.80	2.72	3.52	4.96	4.86
0.64	1.05	2.88	3.67	5.12	4.88
0.80	1.28	3.04	3.82	5.28	4.89
0.96	1.50	3.20	3.96	5.44	4.90
1.12	1.72	3.36	4.08	5.60	4.90
1.28	1.93	3.52	4.18	5.76	4.91
1.44	2.09	3.68	4.31	5.92	4.90
1.60	2.30	3.84	4.41	6.08	4.91
1.76	2.53	4.00	4.52	6.24	4.90
1.92	2.73	4.16	4.63	6.40	4.91
2.08	2.89	4.32	4.69	6.56	4.91

   - **a.** Find a formula for the guppy population in terms of t.
   - **b.** How long will it take for the guppy population to be 100? 125?
6. **Gorilla population** A certain wild animal preserve can support no more than 250 lowland gorillas. Twenty-eight gorillas were known to be in the preserve in 1970. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0004(250 - P)P,$$

where time t is in years.

TABLE 9.8 Kelly Schmitzer skating data

t (sec)	*s* (m)	*t* (sec)	*s* (m)	*t* (sec)	*s* (m)
0	0	1.5	0.89	3.1	1.30
0.1	0.07	1.7	0.97	3.3	1.31
0.3	0.22	1.9	1.05	3.5	1.32
0.5	0.36	2.1	1.11	3.7	1.32
0.7	0.49	2.3	1.17	3.9	1.32
0.9	0.60	2.5	1.22	4.1	1.32
1.1	0.71	2.7	1.25	4.3	1.32
1.3	0.81	2.9	1.28	4.5	1.32

a. Find a formula for the gorilla population in terms of t.

b. How long will it take for the gorilla population to reach the carrying capacity of the preserve?

7. Pacific halibut fishery The Pacific halibut fishery has been modeled by the logistic equation

$$\frac{dy}{dt} = r(M - y)y$$

where $y(t)$ is the total weight of the halibut population in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7$ kg, and $r = 0.08875 \times 10^{-7}$ per year.

a. If $y(0) = 1.6 \times 10^7$ kg, what is the total weight of the halibut population after 1 year?

b. When will the total weight in the halibut fishery reach 4×10^7 kg?

8. Modified logistic model Suppose that the logistic differential equation in Example 2 is modified to

$$\frac{dP}{dt} = 0.001(100 - P)P - c$$

for some constant c.

a. Explain the meaning of the constant c. What values for c might be realistic for the grizzly bear population?

b. Draw a direction field for the differential equation when $c = 1$. What are the equilibrium solutions (Section 9.4)?

c. Sketch several solution curves in your direction field from part (a). Describe what happens to the grizzly bear population for various initial populations.

9. Exact solutions Find the exact solutions to the following initial value problems.

a. $y' = 1 + y, \quad y(0) = 1$

b. $y' = 0.5(400 - y)y, \quad y(0) = 2$

10. Logistic differential equation Show that the solution of the differential equation

$$\frac{dP}{dt} = r(M - P)P$$

is

$$P = \frac{M}{1 + Ae^{-rMt}},$$

where A is an arbitrary constant.

11. Catastrophic solution Let k and P_0 be positive constants.

a. Solve the initial value problem?

$$\frac{dP}{dt} = kP^2, \quad P(0) = P_0$$

b. Show that the graph of the solution in part (a) has a vertical asymptote at a positive value of t. What is that value of t?

12. Extinct populations Consider the population model

$$\frac{dP}{dt} = r(M - P)(P - m),$$

where $r > 0$, M is the maximum sustainable population, and m is the minimum population below which the species becomes extinct.

a. Let $m = 100$, and $M = 1200$, and assume that $m < P < M$. Show that the differential equation can be rewritten in the form

$$\left[\frac{1}{1200 - P} + \frac{1}{P - 100}\right]\frac{dP}{dt} = 1100r$$

and solve this separable equation.

b. Find the solution to part (a) that satisfies $P(0) = 300$.

c. Solve the differential equation with the restriction $m < P < M$.

Orthogonal Trajectories

In Exercises 13–18, find the orthogonal trajectories of the family of curves. Sketch several members of each family.

13. $y = mx$ **14.** $y = cx^2$

15. $kx^2 + y^2 = 1$ **16.** $2x^2 + y^2 = c^2$

17. $y = ce^{-x}$ **18.** $y = e^{kx}$

19. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal.

20. Find the family of solutions of the given differential equation and the family of orthogonal trajectories. Sketch both families.

a. $x\,dx + y\,dy = 0$ **b.** $x\,dy - 2y\,dx = 0$

21. Suppose a and b are positive numbers. Sketch the parabolas

$$y^2 = 4a^2 - 4ax \quad \text{and} \quad y^2 = 4b^2 + 4bx$$

in the same diagram. Show that they intersect at $\left(a - b, \pm 2\sqrt{ab}\right)$, and that each "$a$-parabola" is orthogonal to every "b-parabola."