# 8.6 Integral Tables and Computer Algebra Systems

As we have studied, the basic techniques of integration are substitution and integration by parts. We apply these techniques to transform unfamiliar integrals into integrals whose forms we recognize or can find in a table. But where do the integrals in the tables come from? They come from applying substitutions and integration by parts, or by differentiating important functions that arise in practice or applications and tabling the results (as we did in creating Table 8.1). When an integral matches an integral in the table or can be changed into one of the tabulated integrals with some appropriate combination of algebra, trigonometry, substitution, and calculus, the matched result can be used to solve the integration problem at hand.

Computer Algebra Systems (CAS) can also be used to evaluate an integral, if such a system is available. However, beware that there are many relatively simple functions, like $\sin(x^2)$ or $1/\ln x$ for which even the most powerful computer algebra systems cannot find explicit antiderivative formulas because no such formulas exist.

In this section we discuss how to use tables and computer algebra systems to evaluate integrals.

## Integral Tables

A Brief Table of Integrals is provided at the back of the book, after the index. (More extensive tables appear in compilations such as *CRC Mathematical Tables*, which contain thousands of integrals.) The integration formulas are stated in terms of constants $a$, $b$, $c$, $m$, $n$, and so on. These constants can usually assume any real value and need not be integers. Occasional limitations on their values are stated with the formulas. Formula 5 requires $n \neq -1$, for example, and Formula 11 requires $n \neq 2$.

The formulas also assume that the constants do not take on values that require dividing by zero or taking even roots of negative numbers. For example, Formula 8 assumes that $a \neq 0$, and Formulas 13(a) and (b) cannot be used unless $b$ is positive.

The integrals in Examples 1–5 of this section can be evaluated using algebraic manipulation, substitution, or integration by parts. Here we illustrate how the integrals are found using the Brief Table of Integrals.

**EXAMPLE 1** Find

$$\int x(2x+5)^{-1}\,dx.$$

**Solution** We use Formula 8 (not 7, which requires $n \neq -1$):

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x+5)^{-1}\,dx = \frac{x}{2} - \frac{5}{4}\ln|2x+5| + C. \qquad \blacksquare$$

**EXAMPLE 2** Find

You Try It

$$\int \frac{dx}{x\sqrt{2x+4}}.$$

**Solution** We use Formula 13(b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}} \right| + C, \qquad \text{if } b > 0.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x+4}} = \frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{2x+4}-\sqrt{4}}{\sqrt{2x+4}+\sqrt{4}} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\sqrt{2x+4}-2}{\sqrt{2x+4}+2} \right| + C.$$

Formula 13(a), which would require $b < 0$ here, is not appropriate in Example 2. It *is* appropriate, however, in the next example.

**EXAMPLE 3** Find

$$\int \frac{dx}{x\sqrt{2x-4}}.$$

**Solution** We use Formula 13(a):

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}} \tan^{-1} \sqrt{\frac{ax-b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}} \tan^{-1} \sqrt{\frac{2x-4}{4}} + C = \tan^{-1} \sqrt{\frac{x-2}{2}} + C.$$

**EXAMPLE 4** Find

$$\int \frac{dx}{x^2\sqrt{2x-4}}.$$

**Solution** We begin with Formula 15:

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{ax+b}} + C.$$

With $a = 2$ and $b = -4$, we have

$$\int \frac{dx}{x^2\sqrt{2x-4}} = -\frac{\sqrt{2x-4}}{-4x} + \frac{2}{2 \cdot 4} \int \frac{dx}{x\sqrt{2x-4}} + C.$$

We then use Formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x-4}} = \frac{\sqrt{2x-4}}{4x} + \frac{1}{4} \tan^{-1} \sqrt{\frac{x-2}{2}} + C.$$

**EXAMPLE 5** Find

$$\int x \sin^{-1} x \, dx.$$

**Solution** We use Formula 99:

$$\int x^n \sin^{-1} ax\, dx = \frac{x^{n+1}}{n+1}\sin^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x\, dx = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{x^2\,dx}{\sqrt{1-x^2}}.$$

The integral on the right is found in the table as Formula 33:

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}x\sqrt{a^2-x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C.$$

The combined result is

$$\begin{aligned}\int x \sin^{-1} x\, dx &= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C\right)\\ &= \left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + C'.\end{aligned}$$

## Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying formulas like

$$\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx \tag{1}$$

$$\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx \tag{2}$$

$$\int \sin^n x \cos^m x\, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2} x \cos^m x\, dx \qquad (n \neq -m). \tag{3}$$

Formulas like these are called **reduction formulas** because they replace an integral containing some power of a function with an integral of the same form with the power reduced. By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly.

### EXAMPLE 6 Using a Reduction Formula

You Try It

Find

$$\int \tan^5 x\, dx.$$

**Solution** We apply Equation (1) with $n = 5$ to get

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx.$$

We then apply Equation (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \int \tan x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

The combined result is

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C'.$$ ■

As their form suggests, reduction formulas are derived by integration by parts.

## EXAMPLE 7 Deriving a Reduction Formula

Show that for any positive integer $n$,

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

**Solution** We use the integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

with

$$u = (\ln x)^n, \qquad du = n(\ln x)^{n-1}\frac{dx}{x}, \qquad dv = dx, \qquad v = x,$$

to obtain

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$ ■

Sometimes two reduction formulas come into play.

## EXAMPLE 8 Find

$$\int \sin^2 x \cos^3 x\,dx.$$

**Solution 1** We apply Equation (3) with $n = 2$ and $m = 3$ to get

$$\begin{aligned}\int \sin^2 x\cos^3 x\,dx &= -\frac{\sin x\cos^4 x}{2+3} + \frac{1}{2+3}\int \sin^0 x\cos^3 x\,dx \\ &= -\frac{\sin x\cos^4 x}{5} + \frac{1}{5}\int \cos^3 x\,dx.\end{aligned}$$

We can evaluate the remaining integral with Formula 61 (another reduction formula):

$$\int \cos^n ax\,dx = \frac{\cos^{n-1} ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx.$$

With $n = 3$ and $a = 1$, we have

$$\int \cos^3 x\,dx = \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx$$
$$= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C.$$

The combined result is

$$\int \sin^2 x \cos^3 x\,dx = -\frac{\sin x \cos^4 x}{5} + \frac{1}{5}\left(\frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C\right)$$
$$= -\frac{\sin x \cos^4 x}{5} + \frac{\cos^2 x \sin x}{15} + \frac{2}{15}\sin x + C'.$$

**Solution 2** Equation (3) corresponds to Formula 68 in the table, but there is another formula we might use, namely Formula 69. With $a = 1$, Formula 69 gives

$$\int \sin^n x \cos^m x\,dx = \frac{\sin^{n+1} x \cos^{m-1} x}{m+n} + \frac{m-1}{m+n}\int \sin^n x \cos^{m-2} x\,dx.$$

In our case, $n = 2$ and $m = 3$, so that

$$\int \sin^2 x \cos^3 x\,dx = \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5}\int \sin^2 x \cos x\,dx$$
$$= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5}\left(\frac{\sin^3 x}{3}\right) + C$$
$$= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{15}\sin^3 x + C.$$

As you can see, it is faster to use Formula 69, but we often cannot tell beforehand how things will work out. Do not spend a lot of time looking for the "best" formula. Just find one that will work and forge ahead.

Notice also that Formulas 68 (Solution 1) and 69 (Solution 2) lead to different-looking answers. That is often the case with trigonometric integrals and is no cause for concern. The results are equivalent, and we may use whichever one we please. ■

## Nonelementary Integrals

The development of computers and calculators that find antiderivatives by symbolic manipulation has led to a renewed interest in determining which antiderivatives can be expressed as finite combinations of elementary functions (the functions we have been studying) and which cannot. Integrals of functions that do not have elementary antiderivatives are called **nonelementary** integrals. They require infinite series (Chapter 11) or numerical methods for their evaluation. Examples of the latter include the error function (which measures the probability of random errors)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

and integrals such as

$$\int \sin x^2\,dx \qquad \text{and} \qquad \int \sqrt{1 + x^4}\,dx$$

that arise in engineering and physics. These and a number of others, such as

$$\int \frac{e^x}{x}\,dx, \qquad \int e^{(e^x)}\,dx, \qquad \int \frac{1}{\ln x}\,dx, \qquad \int \ln(\ln x)\,dx, \qquad \int \frac{\sin x}{x}\,dx,$$

$$\int \sqrt{1 - k^2 \sin^2 x}\,dx, \qquad 0 < k < 1,$$

look so easy they tempt us to try them just to see how they turn out. It can be proved, however, that there is no way to express these integrals as finite combinations of elementary functions. The same applies to integrals that can be changed into these by substitution. The integrands all have antiderivatives, as a consequence of the Fundamental Theorem of the Calculus, Part 1, because they are continuous. However, none of the antiderivatives is elementary.

None of the integrals you are asked to evaluate in the present chapter falls into this category, but you may encounter nonelementary integrals in your other work.

## Integration with a CAS

A powerful capability of computer algebra systems is their ability to integrate symbolically. This is performed with the **integrate command** specified by the particular system (for example, **int** in Maple, **Integrate** in Mathematica).

### EXAMPLE 9 Using a CAS with a Named Function

Suppose that you want to evaluate the indefinite integral of the function

$$f(x) = x^2\sqrt{a^2 + x^2}.$$

Using Maple, you first define or name the function:

$$> f := x\text{^}2 * \text{sqrt}\,(a\text{^}2 + x\text{^}2);$$

Then you use the integrate command on $f$, identifying the variable of integration:

$$> \text{int}(f, x);$$

Maple returns the answer

$$\frac{1}{4}x(a^2 + x^2)^{3/2} - \frac{1}{8}a^2x\sqrt{a^2 + x^2} - \frac{1}{8}a^4 \ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want to see if the answer can be simplified, enter

$$> \text{simplify}(\%);$$

Maple returns

$$\frac{1}{8}a^2x\sqrt{a^2 + x^2} + \frac{1}{4}x^3\sqrt{a^2 + x^2} - \frac{1}{8}a^4 \ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want the definite integral for $0 \le x \le \pi/2$, you can use the format

$$> \text{int}(f, x = 0..\text{Pi}/2);$$

Maple (Version 5.1) will return the expression

$$\frac{1}{64}\pi(4a^2 + \pi^2)^{(3/2)} - \frac{1}{32}a^2\pi\sqrt{4a^2 + \pi^2} + \frac{1}{8}a^4 \ln(2)$$
$$- \frac{1}{8}a^4 \ln\left(\pi + \sqrt{4a^2 + \pi^2}\right) + \frac{1}{16}a^4 \ln(a^2).$$

You can also find the definite integral for a particular value of the constant $a$:

$$> a := 1;$$

$$> \mathrm{int}(f, x = 0..1);$$

Maple returns the numerical answer

$$\frac{3}{8}\sqrt{2} + \frac{1}{8}\ln\left(\sqrt{2} - 1\right).$$

**EXAMPLE 10** Using a CAS Without Naming the Function

Use a CAS to find

$$\int \sin^2 x \cos^3 x \, dx.$$

**Solution** With Maple, we have the entry

$$> \mathrm{int}\,((\sin\hat{\ }2)(x) * (\cos\hat{\ }3)(x), x);$$

with the immediate return

$$-\frac{1}{5}\sin(x)\cos(x)^4 + \frac{1}{15}\cos(x)^2\sin(x) + \frac{2}{15}\sin(x).$$

**EXAMPLE 11** A CAS May Not Return a Closed Form Solution

Use a CAS to find

$$\int (\cos^{-1} ax)^2 \, dx.$$

**Solution** Using Maple, we enter

$$> \mathrm{int}((\arccos(a * x))\hat{\ }2, x);$$

and Maple returns the expression

$$\int \arccos(ax)^2 \, dx,$$

indicating that it does not have a closed form solution known by Maple. In Chapter 11, you will see how series expansion may help to evaluate such an integral.

Computer algebra systems vary in how they process integrations. We used Maple 5.1 in Examples 9–11. Mathematica 4 would have returned somewhat different results:

**1.** In Example 9, given

$$\mathit{In\ [1]} := \text{Integrate}\,[x\hat{\ }2 * \text{Sqrt}\,[a\hat{\ }2 + x\hat{\ }2], x]$$

Mathematica returns

$$\mathit{Out\ [1]} = \sqrt{a^2 + x^2}\left(\frac{a^2 x}{8} + \frac{x^3}{4}\right) - \frac{1}{8}a^4\,\text{Log}\left[x + \sqrt{a^2 + x^2}\right]$$

without having to simplify an intermediate result. The answer is close to Formula 22 in the integral tables.

**2.** The Mathematica answer to the integral

$$\textit{In [2]:=}\ \text{Integrate [Sin [}x\text{]^2 * Cos [}x\text{]^3, }x\text{]}$$

in Example 10 is

$$\textit{Out [2]=}\ \frac{\text{Sin }[x]}{8} - \frac{1}{48}\text{Sin }[3\,x] - \frac{1}{80}\text{Sin }[5\,x]$$

differing from both the Maple answer and the answers in Example 8.

**3.** Mathematica does give a result for the integration

$$\textit{In [3]:=}\ \text{Integrate [ArcCos [}a * x\text{]^2, }x\text{]}$$

in Example 11, provided $a \neq 0$:

$$\textit{Out [3]=}\ -2x - \frac{2\sqrt{1 - a^2x^2}\,\text{ArcCos }[a\,x]}{a} + x\,\text{ArcCos }[a\,x]^2$$

Although a CAS is very powerful and can aid us in solving difficult problems, each CAS has its own limitations. There are even situations where a CAS may further complicate a problem (in the sense of producing an answer that is extremely difficult to use or interpret). Note, too, that neither Maple nor Mathematica return an arbitrary constant $+C$. On the other hand, a little mathematical thinking on your part may reduce the problem to one that is quite easy to handle. We provide an example in Exercise 111.

# EXERCISES 8.6

## Using Integral Tables

Use the table of integrals at the back of the book to evaluate the integrals in Exercises 1–38.

1. $\displaystyle\int \frac{dx}{x\sqrt{x-3}}$

2. $\displaystyle\int \frac{dx}{x\sqrt{x+4}}$

3. $\displaystyle\int \frac{x\,dx}{\sqrt{x-2}}$

4. $\displaystyle\int \frac{x\,dx}{(2x+3)^{3/2}}$

5. $\displaystyle\int x\sqrt{2x-3}\,dx$

6. $\displaystyle\int x(7x+5)^{3/2}\,dx$

7. $\displaystyle\int \frac{\sqrt{9-4x}}{x^2}\,dx$

8. $\displaystyle\int \frac{dx}{x^2\sqrt{4x-9}}$

9. $\displaystyle\int x\sqrt{4x-x^2}\,dx$

10. $\displaystyle\int \frac{\sqrt{x-x^2}}{x}\,dx$

11. $\displaystyle\int \frac{dx}{x\sqrt{7+x^2}}$

12. $\displaystyle\int \frac{dx}{x\sqrt{7-x^2}}$

13. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$

14. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

15. $\displaystyle\int \sqrt{25-p^2}\,dp$

16. $\displaystyle\int q^2\sqrt{25-q^2}\,dq$

17. $\displaystyle\int \frac{r^2}{\sqrt{4-r^2}}\,dr$

18. $\displaystyle\int \frac{ds}{\sqrt{s^2-2}}$

19. $\displaystyle\int \frac{d\theta}{5+4\sin 2\theta}$

20. $\displaystyle\int \frac{d\theta}{4+5\sin 2\theta}$

21. $\displaystyle\int e^{2t}\cos 3t\,dt$

22. $\displaystyle\int e^{-3t}\sin 4t\,dt$

23. $\displaystyle\int x\cos^{-1}x\,dx$

24. $\displaystyle\int x\tan^{-1}x\,dx$

25. $\displaystyle\int \frac{ds}{(9-s^2)^2}$

26. $\displaystyle\int \frac{d\theta}{(2-\theta^2)^2}$

27. $\displaystyle\int \frac{\sqrt{4x+9}}{x^2}\,dx$

28. $\displaystyle\int \frac{\sqrt{9x-4}}{x^2}\,dx$

29. $\displaystyle\int \frac{\sqrt{3t-4}}{t}\,dt$

30. $\displaystyle\int \frac{\sqrt{3t+9}}{t}\,dt$

31. $\displaystyle\int x^2\tan^{-1}x\,dx$

32. $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx$

33. $\displaystyle\int \sin 3x\cos 2x\,dx$

34. $\displaystyle\int \sin 2x\cos 3x\,dx$

**35.** $\displaystyle\int 8 \sin 4t \sin \frac{t}{2}\, dt$ **36.** $\displaystyle\int \sin \frac{t}{3} \sin \frac{t}{6}\, dt$

**37.** $\displaystyle\int \cos \frac{\theta}{3} \cos \frac{\theta}{4}\, d\theta$ **38.** $\displaystyle\int \cos \frac{\theta}{2} \cos 7\theta\, d\theta$

## Substitution and Integral Tables

In Exercises 39–52, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.

**39.** $\displaystyle\int \frac{x^3 + x + 1}{(x^2 + 1)^2}\, dx$ **40.** $\displaystyle\int \frac{x^2 + 6x}{(x^2 + 3)^2}\, dx$

**41.** $\displaystyle\int \sin^{-1} \sqrt{x}\, dx$ **42.** $\displaystyle\int \frac{\cos^{-1} \sqrt{x}}{\sqrt{x}}\, dx$

**43.** $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1 - x}}\, dx$ **44.** $\displaystyle\int \frac{\sqrt{2 - x}}{\sqrt{x}}\, dx$

**45.** $\displaystyle\int \cot t \sqrt{1 - \sin^2 t}\, dt, \quad 0 < t < \pi/2$

**46.** $\displaystyle\int \frac{dt}{\tan t \sqrt{4 - \sin^2 t}}$ **47.** $\displaystyle\int \frac{dy}{y\sqrt{3 + (\ln y)^2}}$

**48.** $\displaystyle\int \frac{\cos \theta\, d\theta}{\sqrt{5 + \sin^2 \theta}}$ **49.** $\displaystyle\int \frac{3\, dr}{\sqrt{9r^2 - 1}}$

**50.** $\displaystyle\int \frac{3\, dy}{\sqrt{1 + 9y^2}}$ **51.** $\displaystyle\int \cos^{-1} \sqrt{x}\, dx$

**52.** $\displaystyle\int \tan^{-1} \sqrt{y}\, dy$

## Using Reduction Formulas

Use reduction formulas to evaluate the integrals in Exercises 53–72.

**53.** $\displaystyle\int \sin^5 2x\, dx$ **54.** $\displaystyle\int \sin^5 \frac{\theta}{2}\, d\theta$

**55.** $\displaystyle\int 8 \cos^4 2\pi t\, dt$ **56.** $\displaystyle\int 3 \cos^5 3y\, dy$

**57.** $\displaystyle\int \sin^2 2\theta \cos^3 2\theta\, d\theta$ **58.** $\displaystyle\int 9 \sin^3 \theta \cos^{3/2} \theta\, d\theta$

**59.** $\displaystyle\int 2 \sin^2 t \sec^4 t\, dt$ **60.** $\displaystyle\int \csc^2 y \cos^5 y\, dy$

**61.** $\displaystyle\int 4 \tan^3 2x\, dx$ **62.** $\displaystyle\int \tan^4 \left(\frac{x}{2}\right) dx$

**63.** $\displaystyle\int 8 \cot^4 t\, dt$ **64.** $\displaystyle\int 4 \cot^3 2t\, dt$

**65.** $\displaystyle\int 2 \sec^3 \pi x\, dx$ **66.** $\displaystyle\int \frac{1}{2} \csc^3 \frac{x}{2}\, dx$

**67.** $\displaystyle\int 3 \sec^4 3x\, dx$ **68.** $\displaystyle\int \csc^4 \frac{\theta}{3}\, d\theta$

**69.** $\displaystyle\int \csc^5 x\, dx$ **70.** $\displaystyle\int \sec^5 x\, dx$

**71.** $\displaystyle\int 16x^3 (\ln x)^2\, dx$ **72.** $\displaystyle\int (\ln x)^3\, dx$

## Powers of $x$ Times Exponentials

Evaluate the integrals in Exercises 73–80 using table Formulas 103–106. These integrals can also be evaluated using tabular integration (Section 8.2).

**73.** $\displaystyle\int xe^{3x}\, dx$ **74.** $\displaystyle\int xe^{-2x}\, dx$

**75.** $\displaystyle\int x^3 e^{x/2}\, dx$ **76.** $\displaystyle\int x^2 e^{\pi x}\, dx$

**77.** $\displaystyle\int x^2\, 2^x\, dx$ **78.** $\displaystyle\int x^2\, 2^{-x}\, dx$

**79.** $\displaystyle\int x\pi^x\, dx$ **80.** $\displaystyle\int x 2^{\sqrt{2}x}\, dx$

## Substitutions with Reduction Formulas

Evaluate the integrals in Exercises 81–86 by making a substitution (possibly trigonometric) and then applying a reduction formula.

**81.** $\displaystyle\int e^t \sec^3 (e^t - 1)\, dt$ **82.** $\displaystyle\int \frac{\csc^3 \sqrt{\theta}}{\sqrt{\theta}}\, d\theta$

**83.** $\displaystyle\int_0^1 2\sqrt{x^2 + 1}\, dx$ **84.** $\displaystyle\int_0^{\sqrt{3}/2} \frac{dy}{(1 - y^2)^{5/2}}$

**85.** $\displaystyle\int_1^2 \frac{(r^2 - 1)^{3/2}}{r}\, dr$ **86.** $\displaystyle\int_0^{1/\sqrt{3}} \frac{dt}{(t^2 + 1)^{7/2}}$

## Hyperbolic Functions

Use the integral tables to evaluate the integrals in Exercises 87–92.

**87.** $\displaystyle\int \frac{1}{8} \sinh^5 3x\, dx$ **88.** $\displaystyle\int \frac{\cosh^4 \sqrt{x}}{\sqrt{x}}\, dx$

**89.** $\displaystyle\int x^2 \cosh 3x\, dx$ **90.** $\displaystyle\int x \sinh 5x\, dx$

**91.** $\displaystyle\int \operatorname{sech}^7 x \tanh x\, dx$ **92.** $\displaystyle\int \operatorname{csch}^3 2x \coth 2x\, dx$

## Theory and Examples

Exercises 93–100 refer to formulas in the table of integrals at the back of the book.

**93.** Derive Formula 9 by using the substitution $u = ax + b$ to evaluate

$$\int \frac{x}{(ax + b)^2}\, dx.$$

**94.** Derive Formula 17 by using a trigonometric substitution to evaluate

$$\int \frac{dx}{(a^2 + x^2)^2}.$$

**95.** Derive Formula 29 by using a trigonometric substitution to evaluate

$$\int \sqrt{a^2 - x^2}\, dx.$$

**96.** Derive Formula 46 by using a trigonometric substitution to evaluate

$$\int \frac{dx}{x^2\sqrt{x^2 - a^2}}.$$

**97.** Derive Formula 80 by evaluating

$$\int x^n \sin ax\, dx$$

by integration by parts.

**98.** Derive Formula 110 by evaluating

$$\int x^n (\ln ax)^m\, dx$$

by integration by parts.

**99.** Derive Formula 99 by evaluating

$$\int x^n \sin^{-1} ax\, dx$$

by integration by parts.

**100.** Derive Formula 101 by evaluating

$$\int x^n \tan^{-1} ax\, dx$$

by integration by parts.

**101. Surface area** Find the area of the surface generated by revolving the curve $y = \sqrt{x^2 + 2}$, $0 \le x \le \sqrt{2}$, about the $x$-axis.

**102. Arc length** Find the length of the curve $y = x^2$, $0 \le x \le \sqrt{3}/2$.

**103. Centroid** Find the centroid of the region cut from the first quadrant by the curve $y = 1/\sqrt{x + 1}$ and the line $x = 3$.

**104. Moment about *y*-axis** A thin plate of constant density $\delta = 1$ occupies the region enclosed by the curve $y = 36/(2x + 3)$ and the line $x = 3$ in the first quadrant. Find the moment of the plate about the $y$-axis.

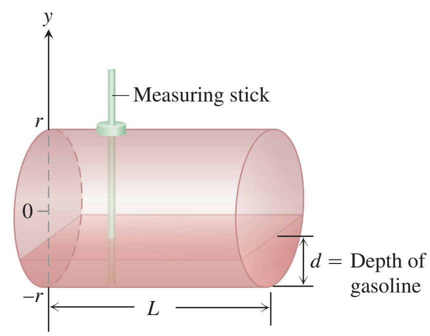
T **105.** Use the integral table and a calculator to find to two decimal places the area of the surface generated by revolving the curve $y = x^2$, $-1 \le x \le 1$, about the $x$-axis.

**106. Volume** The head of your firm's accounting department has asked you to find a formula she can use in a computer program to calculate the year-end inventory of gasoline in the company's tanks. A typical tank is shaped like a right circular cylinder of radius $r$ and length $L$, mounted horizontally, as shown here. The data come to the accounting office as depth measurements taken with a vertical measuring stick marked in centimeters.

**a.** Show, in the notation of the figure here, that the volume of gasoline that fills the tank to a depth $d$ is

$$V = 2L\int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy.$$

**b.** Evaluate the integral.




**107.** What is the largest value

$$\int_a^b \sqrt{x - x^2}\, dx$$

can have for any $a$ and $b$? Give reasons for your answer.

**108.** What is the largest value

$$\int_a^b x\sqrt{2x - x^2}\, dx$$

can have for any $a$ and $b$? Give reasons for your answer.

## COMPUTER EXPLORATIONS

In Exercises 109 and 110, use a CAS to perform the integrations.

**109.** Evaluate the integrals

**a.** $\int x \ln x\, dx$ **b.** $\int x^2 \ln x\, dx$ **c.** $\int x^3 \ln x\, dx$.

**d.** What pattern do you see? Predict the formula for $\int x^4 \ln x\, dx$ and then see if you are correct by evaluating it with a CAS.

**e.** What is the formula for $\int x^n \ln x\, dx$, $n \ge 1$? Check your answer using a CAS.

**110.** Evaluate the integrals

**a.** $\int \frac{\ln x}{x^2}\, dx$ **b.** $\int \frac{\ln x}{x^3}\, dx$ **c.** $\int \frac{\ln x}{x^4}\, dx$.

**d.** What pattern do you see? Predict the formula for

$$\int \frac{\ln x}{x^5}\, dx$$

and then see if you are correct by evaluating it with a CAS.

**e.** What is the formula for

$$\int \frac{\ln x}{x^n}\, dx, \quad n \ge 2?$$

Check your answer using a CAS.

**111. a.** Use a CAS to evaluate

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx$$

where $n$ is an arbitrary positive integer. Does your CAS find the result?

**b.** In succession, find the integral when $n = 1, 2, 3, 5, 7$. Comment on the complexity of the results.

**c.** Now substitute $x = (\pi/2) - u$ and add the new and old integrals. What is the value of

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx ?$$

This exercise illustrates how a little mathematical ingenuity solves a problem not immediately amenable to solution by a CAS.

# 8.7 Numerical Integration

As we have seen, the ideal way to evaluate a definite integral $\int_a^b f(x)\,dx$ is to find a formula $F(x)$ for one of the antiderivatives of $f(x)$ and calculate the number $F(b) - F(a)$. But some antiderivatives require considerable work to find, and still others, like the antiderivatives of $\sin(x^2)$, $1/\ln x$, and $\sqrt{1 + x^4}$, have no elementary formulas.

Another situation arises when a function is defined by a table whose entries were obtained experimentally through instrument readings. In this case a formula for the function may not even exist.

Whatever the reason, when we cannot evaluate a definite integral with an antiderivative, we turn to numerical methods such as the *Trapezoidal Rule* and *Simpson's Rule* developed in this section. These rules usually require far fewer subdivisions of the integration interval to get accurate results compared to the various rectangle rules presented in Sections 5.1 and 5.2. We also estimate the error obtained when using these approximation methods.

## Trapezoidal Approximations

When we cannot find a workable antiderivative for a function $f$ that we have to integrate, we partition the interval of integration, replace $f$ by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the integral of $f$. In our presentation we assume that $f$ is positive, but the only requirement is for $f$ to be continuous over the interval of integration $[a, b]$.

The Trapezoidal Rule for the value of a definite integral is based on approximating the region between a curve and the $x$-axis with trapezoids instead of rectangles, as in Figure 8.10. It is not necessary for the subdivision points $x_0, x_1, x_2, \ldots, x_n$ in the figure to be evenly spaced, but the resulting formula is simpler if they are. We therefore assume that the length of each subinterval is

$$\Delta x = \frac{b - a}{n}.$$

The length $\Delta x = (b - a)/n$ is called the **step size** or **mesh size**. The area of the trapezoid that lies above the $i$th subinterval is

$$\Delta x\left(\frac{y_{i-1} + y_i}{2}\right) = \frac{\Delta x}{2}(y_{i-1} + y_i),$$

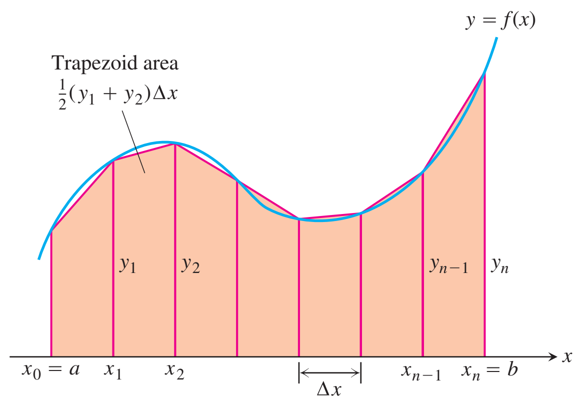



**FIGURE 8.10** The Trapezoidal Rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of $f$ from $a$ to $b$, we add the areas of the trapezoids made by joining the ends of the segments to the $x$-axis.

where $y_{i-1} = f(x_{i-1})$ and $y_i = f(x_i)$. This area is the length $\Delta x$ of the trapezoid's horizontal "altitude" times the average of its two vertical "bases." (See Figure 8.10.) The area below the curve $y = f(x)$ and above the $x$-axis is then approximated by adding the areas of all the trapezoids:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots \\ &\quad + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\ &= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\ &= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n), \end{aligned}$$

where

$$y_0 = f(a), \qquad y_1 = f(x_1), \qquad \ldots, \qquad y_{n-1} = f(x_{n-1}), \qquad y_n = f(b).$$

The Trapezoidal Rule says: Use $T$ to estimate the integral of $f$ from $a$ to $b$.

### The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The $y$'s are the values of $f$ at the partition points

$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n - 1)\Delta x, x_n = b,$
where $\Delta x = (b - a)/n$.

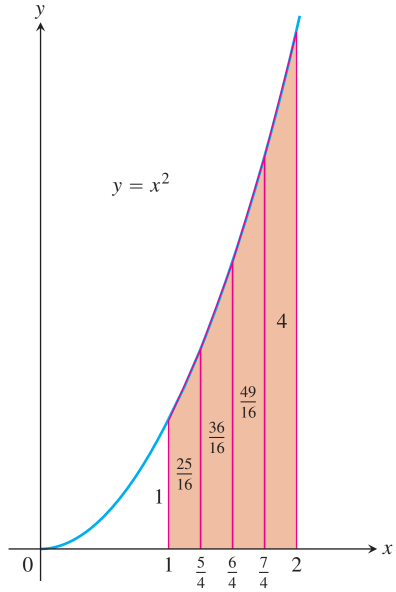



**FIGURE 8.11** The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate (Example 1).

**TABLE 8.3**

| $x$ | $y = x^2$ |
|---|---|
| 1 | 1 |
| $\frac{5}{4}$ | $\frac{25}{16}$ |
| $\frac{6}{4}$ | $\frac{36}{16}$ |
| $\frac{7}{4}$ | $\frac{49}{16}$ |
| 2 | 4 |

## EXAMPLE 1 Applying the Trapezoidal Rule

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

**Solution** Partition [1, 2] into four subintervals of equal length (Figure 8.11). Then evaluate $y = x^2$ at each partition point (Table 8.3).

Using these $y$ values, $n = 4$, and $\Delta x = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \\ &= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) \\ &= \frac{75}{32} = 2.34375. \end{aligned}$$

The exact value of the integral is

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\bigg]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The $T$ approximation overestimates the integral by about half a percent of its true value of $7/3$. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.00446$, or 0.446%. ■

We could have predicted that the Trapezoidal Rule would overestimate the integral in Example 1 by considering the geometry of the graph in Figure 8.11. Since the parabola is concave *up*, the approximating segments lie above the curve, giving each trapezoid slightly more area than the corresponding strip under the curve. In Figure 8.10, we see that the straight segments lie *under* the curve on those intervals where the curve is concave *down*, causing the Trapezoidal Rule to *underestimate* the integral on those intervals.

## EXAMPLE 2 Averaging Temperatures

An observer measures the outside temperature every hour from noon until midnight, recording the temperatures in the following table.

| Time | N | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Temp | 63 | 65 | 66 | 68 | 70 | 69 | 68 | 68 | 65 | 64 | 62 | 58 | 55 |

What was the average temperature for the 12-hour period?

**Solution** We are looking for the average value of a continuous function (temperature) for which we know values at discrete times that are one unit apart. We need to find

$$\operatorname{av}(f) = \frac{1}{b - a}\int_a^b f(x)\,dx,$$

without having a formula for $f(x)$. The integral, however, can be approximated by the Trapezoidal Rule, using the temperatures in the table as function values at the points of a 12-subinterval partition of the 12-hour interval (making $\Delta x = 1$).

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{11} + y_{12}\right) \\ &= \frac{1}{2}\left(63 + 2\cdot 65 + 2\cdot 66 + \cdots + 2\cdot 58 + 55\right) \\ &= 782 \end{aligned}$$

Using $T$ to approximate $\int_a^b f(x)\,dx$, we have

$$\text{av}(f) \approx \frac{1}{b-a} \cdot T = \frac{1}{12} \cdot 782 \approx 65.17.$$

Rounding to the accuracy of the given data, we estimate the average temperature as 65 degrees. ■

## Error Estimates for the Trapezoidal Rule

As $n$ increases and the step size $\Delta x = (b-a)/n$ approaches zero, $T$ approaches the exact value of $\int_a^b f(x)\,dx$. To see why, write

$$\begin{aligned} T &= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\ &= (y_1 + y_2 + \cdots + y_n)\Delta x + \frac{1}{2}(y_0 - y_n)\,\Delta x \\ &= \sum_{k=1}^{n} f(x_k)\Delta x + \frac{1}{2}[f(a) - f(b)]\,\Delta x. \end{aligned}$$

As $n \to \infty$ and $\Delta x \to 0$,

$$\sum_{k=1}^{n} f(x_k)\Delta x \to \int_a^b f(x)\,dx \qquad \text{and} \qquad \frac{1}{2}[f(a) - f(b)]\Delta x \to 0.$$

Therefore,

$$\lim_{n\to\infty} T = \int_a^b f(x)\,dx + 0 = \int_a^b f(x)\,dx.$$

This means that in theory we can make the difference between $T$ and the integral as small as we want by taking $n$ large enough, assuming only that $f$ is integrable. In practice, though, how do we tell how large $n$ should be for a given tolerance?

We answer this question with a result from advanced calculus, which says that if $f''$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = T - \frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

for some number $c$ between $a$ and $b$. Thus, as $\Delta x$ approaches zero, the error defined by

$$E_T = -\frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

approaches zero as the *square* of $\Delta x$.

The inequality

$$|E_T| \le \frac{b-a}{12} \max|f''(x)|(\Delta x)^2,$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. In practice, we usually cannot find the exact value of $\max|f''(x)|$ and have to estimate an upper bound or "worst case" value for it instead. If $M$ is any upper bound for the values of $|f''(x)|$ on $[a, b]$, so that $|f''(x)| \le M$ on $[a, b]$, then

$$|E_T| \le \frac{b-a}{12} M(\Delta x)^2.$$

If we substitute $(b - a)/n$ for $\Delta x$, we get

$$|E_T| \le \frac{M(b-a)^3}{12n^2}.$$

This is the inequality we normally use in estimating $|E_T|$. We find the best $M$ we can and go on to estimate $|E_T|$ from there. This may sound careless, but it works. To make $|E_T|$ small for a given $M$, we just make $n$ large.

**The Error Estimate for the Trapezoidal Rule**
If $f''$ is continuous and $M$ is any upper bound for the values of $|f''|$ on $[a, b]$, then the error $E_T$ in the trapezoidal approximation of the integral of $f$ from $a$ to $b$ for $n$ steps satisfies the inequality

$$|E_T| \le \frac{M(b-a)^3}{12n^2}.$$

### EXAMPLE 3 Bounding the Trapezoidal Rule Error

Find an upper bound for the error incurred in estimating

$$\int_0^{\pi} x \sin x \, dx$$

with the Trapezoidal Rule with $n = 10$ steps (Figure 8.12).

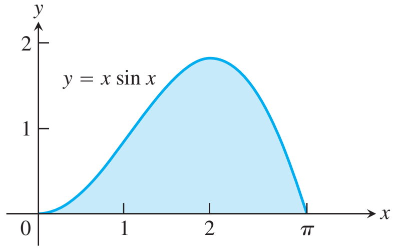



**FIGURE 8.12** Graph of the integrand in Example 3.

**Solution** With $a = 0$, $b = \pi$, and $n = 10$, the error estimate gives

$$|E_T| \le \frac{M(b-a)^3}{12n^2} = \frac{\pi^3}{1200}M.$$

The number $M$ can be any upper bound for the magnitude of the second derivative of $f(x) = x \sin x$ on $[0, \pi]$. A routine calculation gives

$$f''(x) = 2\cos x - x \sin x,$$

so

$$\begin{aligned} |f''(x)| &= |2\cos x - x\sin x| \\ &\le 2|\cos x| + |x||\sin x| \\ &\le 2 \cdot 1 + \pi \cdot 1 = 2 + \pi. \end{aligned}$$

$|\cos x|$ and $|\sin x|$ never exceed 1, and $0 \le x \le \pi$.

We can safely take $M = 2 + \pi$. Therefore,

$$|E_T| \le \frac{\pi^3}{1200}M = \frac{\pi^3(2+\pi)}{1200} < 0.133.$$

Rounded up to be safe

The absolute error is no greater than 0.133.

For greater accuracy, we would not try to improve $M$ but would take more steps. With $n = 100$ steps, for example, we get

$$|E_T| \le \frac{(2+\pi)\pi^3}{120{,}000} < 0.00133 = 1.33 \times 10^{-3}.$$ ■

### EXAMPLE 4 Finding How Many Steps Are Needed for a Specific Accuracy

How many subdivisions should be used in the Trapezoidal Rule to approximate

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

with an error whose absolute value is less than $10^{-4}$?

**Solution** With $a = 1$ and $b = 2$, the error estimate is

$$|E_T| \le \frac{M(2-1)^3}{12n^2} = \frac{M}{12n^2}.$$

This is one of the rare cases in which we actually can find $\max|f''|$ rather than having to settle for an upper bound $M$. With $f(x) = 1/x$, we find

$$f''(x) = \frac{d^2}{dx^2}(x^{-1}) = 2x^{-3} = \frac{2}{x^3}.$$

On $[1, 2]$, $y = 2/x^3$ decreases steadily from a maximum of $y = 2$ to a minimum of $y = 1/4$ (Figure 8.13). Therefore, $M = 2$ and

$$|E_T| \le \frac{2}{12n^2} = \frac{1}{6n^2}.$$

The error's absolute value will therefore be less than $10^{-4}$ if

$$\frac{1}{6n^2} < 10^{-4}, \qquad \frac{10^4}{6} < n^2, \qquad \frac{100}{\sqrt{6}} < n, \qquad \text{or} \qquad 40.83 < n.$$

The first integer beyond 40.83 is $n = 41$. With $n = 41$ subdivisions we can guarantee calculating ln 2 with an error of magnitude less than $10^{-4}$. Any larger $n$ will work, too. ■

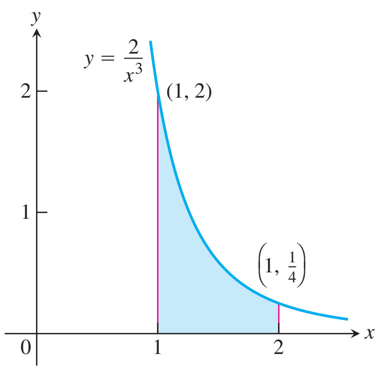



**FIGURE 8.13** The continuous function $y = 2/x^3$ has its maximum value on $[1, 2]$ at $x = 1$.

## Simpson's Rule: Approximations Using Parabolas

Riemann sums and the Trapezoidal Rule both give reasonable approximations to the integral of a continuous function over a closed interval. The Trapezoidal Rule is more efficient, giving a better approximation for small values of $n$, which makes it a faster algorithm for numerical integration.

Another rule for approximating the definite integral of a continuous function results from using parabolas instead of the straight line segments which produced trapezoids. As before, we partition the interval $[a, b]$ into $n$ subintervals of equal length $h = \Delta x = (b - a)/n$, but this time we require that $n$ be an *even* number. On each consecutive pair of intervals we approximate the curve $y = f(x) \ge 0$ by a parabola, as shown in Figure 8.14. A typical parabola passes through three consecutive points $(x_{i-1}, y_{i-1})$, $(x_i, y_i)$, and $(x_{i+1}, y_{i+1})$ on the curve.

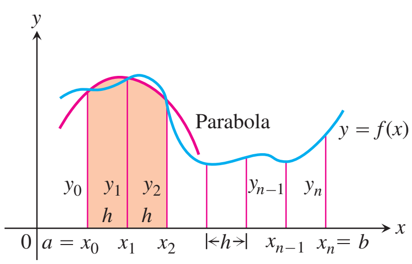



**FIGURE 8.14** Simpson's Rule approximates short stretches of the curve with parabolas.

Let's calculate the shaded area beneath a parabola passing through three consecutive points. To simplify our calculations, we first take the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$ (Figure 8.15), where $h = \Delta x = (b - a)/n$. The area under the parabola will be the same if we shift the $y$-axis to the left or right. The parabola has an equation of the form

$$y = Ax^2 + Bx + C,$$

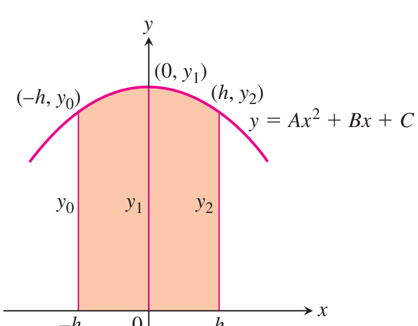



**FIGURE 8.15** By integrating from $-h$ to $h$, we find the shaded area to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

so the area under it from $x = -h$ to $x = h$ is

$$A_p = \int_{-h}^{h} (Ax^2 + Bx + C)\,dx$$

$$= \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\Big]_{-h}^{h}$$

$$= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C).$$

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, and $(h, y_2)$, we also have

$$y_0 = Ah^2 - Bh + C, \qquad y_1 = C, \qquad y_2 = Ah^2 + Bh + C,$$

from which we obtain

$$C = y_1,$$

$$Ah^2 - Bh = y_0 - y_1,$$

$$Ah^2 + Bh = y_2 - y_1,$$

$$2Ah^2 = y_0 + y_2 - 2y_1.$$

Hence, expressing the area $A_p$ in terms of the ordinates $y_0$, $y_1$, and $y_2$, we have

$$A_p = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}((y_0 + y_2 - 2y_1) + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now shifting the parabola horizontally to its shaded position in Figure 8.14 does not change the area under it. Thus the area under the parabola through $(x_0, y_0)$, $(x_1, y_1)$, and $(x_2, y_2)$ in Figure 8.14 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly, the area under the parabola through the points $(x_2, y_2)$, $(x_3, y_3)$, and $(x_4, y_4)$ is

$$\frac{h}{3}(y_2 + 4y_3 + y_4).$$

Computing the areas under all the parabolas and adding the results gives the approximation

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots$$

$$+ \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The result is known as Simpson's Rule, and it is again valid for any continuous function $y = f(x)$ (Exercise 38). The function need not be positive, as in our derivation. The number $n$ of subintervals must be even to apply the rule because each parabolic arc uses two subintervals.

Biography

HISTORICAL BIOGRAPHY

Thomas Simpson
(1720–1761)

**Simpson's Rule**

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The $y$'s are the values of $f$ at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number $n$ is even, and $\Delta x = (b - a)/n$.

Note the pattern of the coefficients in the above rule: 1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 1.

**TABLE 8.4**

| $x$ | $y = 5x^4$ |
|---|---|
| 0 | 0 |
| $\frac{1}{2}$ | $\frac{5}{16}$ |
| 1 | 5 |
| $\frac{3}{2}$ | $\frac{405}{16}$ |
| 2 | 80 |

### EXAMPLE 5 Applying Simpson's Rule

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

**Solution** Partition [0, 2] into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.4). Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

$$\begin{aligned} S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right) \\ &= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right) \\ &= 32\frac{1}{12}. \end{aligned}$$

This estimate differs from the exact value (32) by only 1/12, a percentage error of less than three-tenths of one percent, and this was with just four subintervals. ■

## Error Estimates for Simpson's Rule

To estimate the error in Simpson's rule, we start with a result from advanced calculus that says that if the fourth derivative $f^{(4)}$ is continuous, then

$$\int_a^b f(x)\,dx = S - \frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4$$

for some point $c$ between $a$ and $b$. Thus, as $\Delta x$ approaches zero, the error,

$$E_S = -\frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4,$$

approaches zero as the *fourth power* of $\Delta x$ (This helps to explain why Simpson's Rule is likely to give better results than the Trapezoidal Rule.)

The inequality

$$|E_S| \le \frac{b-a}{180} \max|f^{(4)}(x)|\,(\Delta x)^4$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. As with $\max|f''|$ in the error formula for the Trapezoidal Rule, we usually cannot find the exact value of $\max|f^{(4)}(x)|$ and have to replace it with an upper bound. If $M$ is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then

$$|E_S| \le \frac{b-a}{180} M(\Delta x)^4.$$

Substituting $(b-a)/n$ for $\Delta x$ in this last expression gives

$$|E_S| \le \frac{M(b-a)^5}{180n^4}.$$

This is the formula we usually use in estimating the error in Simpson's Rule. We find a reasonable value for $M$ and go on to estimate $|E_S|$ from there.

**The Error Estimate for Simpson's Rule**

If $f^{(4)}$ is continuous and $M$ is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error $E_S$ in the Simpson's Rule approximation of the integral of $f$ from $a$ to $b$ for $n$ steps satisfies the inequality

$$|E_S| \le \frac{M(b-a)^5}{180n^4}.$$

As with the Trapezoidal Rule, we often cannot find the smallest possible value of $M$. We just find the best value we can and go on from there.

### EXAMPLE 6 Bounding the Error in Simpson's Rule

Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$ (Example 5).

**Solution** To estimate the error, we first find an upper bound $M$ for the magnitude of the fourth derivative of $f(x) = 5x^4$ on the interval $0 \le x \le 2$. Since the fourth derivative has the constant value $f^{(4)}(x) = 120$, we take $M = 120$. With $b - a = 2$ and $n = 4$, the error estimate for Simpson's Rule gives

$$|E_S| \le \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{180 \cdot 4^4} = \frac{1}{12}.$$ ■

### EXAMPLE 7 Comparing the Trapezoidal Rule and Simpson's Rule Approximations

As we saw in Chapter 7, the value of ln 2 can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx.$$

Table 8.5 shows $T$ and $S$ values for approximations of $\int_1^2 (1/x)\,dx$ using various values of $n$. Notice how Simpson's Rule dramatically improves over the Trapezoidal Rule. In particular, notice that when we double the value of $n$ (thereby halving the value of $h = \Delta x$), the $T$ error is divided by 2 *squared*, whereas the $S$ error is divided by 2 *to the fourth*.

**TABLE 8.5** Trapezoidal Rule approximations ($T_n$) and Simpson's Rule approximations ($S_n$) of $\ln 2 = \int_1^2 (1/x)\,dx$

| $n$ | $T_n$ | \|Error\| less than . . . | $S_n$ | \|Error\| less than . . . |
|---|---|---|---|---|
| 10 | 0.6937714032 | 0.0006242227 | 0.6931502307 | 0.0000030502 |
| 20 | 0.6933033818 | 0.0001562013 | 0.6931473747 | 0.0000001942 |
| 30 | 0.6932166154 | 0.0000694349 | 0.6931472190 | 0.0000000385 |
| 40 | 0.6931862400 | 0.0000390595 | 0.6931471927 | 0.0000000122 |
| 50 | 0.6931721793 | 0.0000249988 | 0.6931471856 | 0.0000000050 |
| 100 | 0.6931534305 | 0.0000062500 | 0.6931471809 | 0.0000000004 |

This has a dramatic effect as $\Delta x = (2 - 1)/n$ gets very small. The Simpson approximation for $n = 50$ rounds accurately to seven places and for $n = 100$ agrees to nine decimal places (billionths)! ■

If $f(x)$ is a polynomial of degree less than four, then its fourth derivative is zero, and

$$E_S = -\frac{b-a}{180} f^{(4)}(c)(\Delta x)^4 = -\frac{b-a}{180}(0)(\Delta x)^4 = 0.$$

Thus, there will be no error in the Simpson approximation of any integral of $f$. In other words, if $f$ is a constant, a linear function, or a quadratic or cubic polynomial, Simpson's Rule will give the value of any integral of $f$ exactly, whatever the number of subdivisions. Similarly, if $f$ is a constant or a linear function, then its second derivative is zero and

$$E_T = -\frac{b-a}{12} f''(c)(\Delta x)^2 = -\frac{b-a}{12}(0)(\Delta x)^2 = 0.$$

The Trapezoidal Rule will therefore give the exact value of any integral of $f$. This is no surprise, for the trapezoids fit the graph perfectly. Although decreasing the step size $\Delta x$ reduces the error in the Simpson and Trapezoidal approximations in theory, it may fail to do so in practice.

When $\Delta x$ is very small, say $\Delta x = 10^{-5}$, computer or calculator round-off errors in the arithmetic required to evaluate $S$ and $T$ may accumulate to such an extent that the error formulas no longer describe what is going on. Shrinking $\Delta x$ below a certain size can actually make things worse. Although this is not an issue in this book, you should consult a text on numerical analysis for alternative methods if you are having problems with round-off.

**EXAMPLE 8** Estimate

$$\int_0^2 x^3\,dx$$

with Simpson's Rule.

**Solution** The fourth derivative of $f(x) = x^3$ is zero, so we expect Simpson's Rule to give the integral's exact value with any (even) number of steps. Indeed, with $n = 2$ and $\Delta x = (2 - 0)/2 = 1$,

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$
$$= \frac{1}{3}((0)^3 + 4(1)^3 + (2)^3) = \frac{12}{3} = 4,$$

while

$$\int_0^2 x^3\,dx = \frac{x^4}{4}\bigg]_0^2 = \frac{16}{4} - 0 = 4.$$

## EXAMPLE 9 Draining a Swamp

A town wants to drain and fill a small polluted swamp (Figure 8.16). The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?

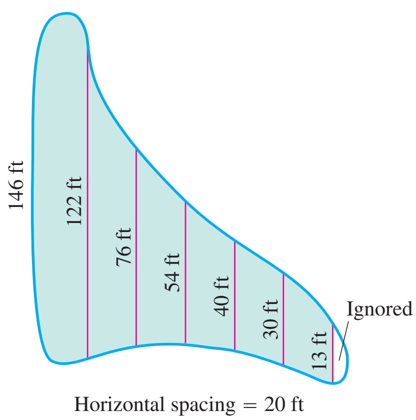



**FIGURE 8.16** The dimensions of the swamp in Example 9.

**Solution** To calculate the volume of the swamp, we estimate the surface area and multiply by 5. To estimate the area, we use Simpson's Rule with $\Delta x = 20$ ft and the $y$'s equal to the distances measured across the swamp, as shown in Figure 8.16.

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$
$$= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100$$

The volume is about $(8100)(5) = 40{,}500$ ft$^3$ or 1500 yd$^3$.

# EXERCISES 8.7

## Estimating Integrals

The instructions for the integrals in Exercises 1–10 have two parts, one for the Trapezoidal Rule and one for Simpson's Rule.

**I. Using the Trapezoidal Rule**

**a.** Estimate the integral with $n = 4$ steps and find an upper bound for $|E_T|$.

**b.** Evaluate the integral directly and find $|E_T|$.

**c.** Use the formula $(|E_T|/(\text{true value})) \times 100$ to express $|E_T|$ as a percentage of the integral's true value.

**II. Using Simpson's Rule**

**a.** Estimate the integral with $n = 4$ steps and find an upper bound for $|E_S|$.

**b.** Evaluate the integral directly and find $|E_S|$.

**c.** Use the formula $(|E_S|/(\text{true value})) \times 100$ to express $|E_S|$ as a percentage of the integral's true value.

**1.** $\displaystyle\int_1^2 x\,dx$

**2.** $\displaystyle\int_1^3 (2x - 1)\,dx$

**3.** $\displaystyle\int_{-1}^1 (x^2 + 1)\,dx$

**4.** $\displaystyle\int_{-2}^0 (x^2 - 1)\,dx$

**5.** $\displaystyle\int_0^2 (t^3 + t)\,dt$

**6.** $\displaystyle\int_{-1}^1 (t^3 + 1)\,dt$

**7.** $\displaystyle\int_1^2 \frac{1}{s^2}\,ds$

**8.** $\displaystyle\int_2^4 \frac{1}{(s - 1)^2}\,ds$

**9.** $\displaystyle\int_0^\pi \sin t\,dt$

**10.** $\displaystyle\int_0^1 \sin \pi t\,dt$

In Exercises 11–14, use the tabulated values of the integrand to estimate the integral with **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule with $n = 8$ steps. Round your answers to five decimal places. Then **(c)** find the integral's exact value and the approximation error $E_T$ or $E_S$, as appropriate.

**11.** $\displaystyle\int_0^1 x\sqrt{1 - x^2}\,dx$

| $x$ | $x\sqrt{1-x^2}$ |
|---|---|
| 0 | 0.0 |
| 0.125 | 0.12402 |
| 0.25 | 0.24206 |
| 0.375 | 0.34763 |
| 0.5 | 0.43301 |
| 0.625 | 0.48789 |
| 0.75 | 0.49608 |
| 0.875 | 0.42361 |
| 1.0 | 0 |

**12.** $\displaystyle\int_0^3 \frac{\theta}{\sqrt{16 + \theta^2}}\,d\theta$

| $\theta$ | $\theta/\sqrt{16+\theta^2}$ |
|---|---|
| 0 | 0.0 |
| 0.375 | 0.09334 |
| 0.75 | 0.18429 |
| 1.125 | 0.27075 |
| 1.5 | 0.35112 |
| 1.875 | 0.42443 |
| 2.25 | 0.49026 |
| 2.625 | 0.58466 |
| 3.0 | 0.6 |

**13.** $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2 + \sin t)^2}\,dt$

| $t$ | $(3\cos t)/(2+\sin t)^2$ |
|---|---|
| −1.57080 | 0.0 |
| −1.17810 | 0.99138 |
| −0.78540 | 1.26906 |
| −0.39270 | 1.05961 |
| 0 | 0.75 |
| 0.39270 | 0.48821 |
| 0.78540 | 0.28946 |
| 1.17810 | 0.13429 |
| 1.57080 | 0 |

**14.** $\displaystyle\int_{\pi/4}^{\pi/2} (\csc^2 y)\sqrt{\cot y}\,dy$

| $y$ | $(\csc^2 y)\sqrt{\cot y}$ |
|---|---|
| 0.78540 | 2.0 |
| 0.88357 | 1.51606 |
| 0.98175 | 1.18237 |
| 1.07992 | 0.93998 |
| 1.17810 | 0.75402 |
| 1.27627 | 0.60145 |
| 1.37445 | 0.46364 |
| 1.47262 | 0.31688 |
| 1.57080 | 0 |

## The Minimum Number of Subintervals

In Exercises 15–26, estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than $10^{-4}$ by **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule. (The integrals in Exercises 15–22 are the integrals from Exercises 1–8.)

**15.** $\displaystyle\int_1^2 x\,dx$

**16.** $\displaystyle\int_1^3 (2x - 1)\,dx$

**17.** $\displaystyle\int_{-1}^1 (x^2 + 1)\,dx$

**18.** $\displaystyle\int_{-2}^0 (x^2 - 1)\,dx$

**19.** $\displaystyle\int_0^2 (t^3 + t)\,dt$

**20.** $\displaystyle\int_{-1}^1 (t^3 + 1)\,dt$

**21.** $\displaystyle\int_1^2 \frac{1}{s^2}\,ds$

**22.** $\displaystyle\int_2^4 \frac{1}{(s-1)^2}\,ds$

**23.** $\displaystyle\int_0^3 \sqrt{x+1}\,dx$

**24.** $\displaystyle\int_0^3 \frac{1}{\sqrt{x+1}}\,dx$

**25.** $\displaystyle\int_0^2 \sin(x + 1)\,dx$

**26.** $\displaystyle\int_{-1}^1 \cos(x + \pi)\,dx$

## Applications

**27. Volume of water in a swimming pool** A rectangular swimming pool is 30 ft wide and 50 ft long. The table shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$, applied to the integral

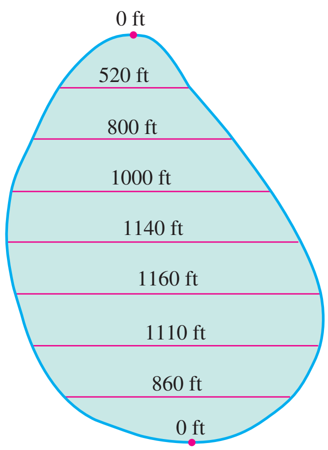
$$V = \int_0^{50} 30 \cdot h(x)\,dx.$$

| Position (ft) $x$ | Depth (ft) $h(x)$ | Position (ft) $x$ | Depth (ft) $h(x)$ |
|---|---|---|---|
| 0 | 6.0 | 30 | 11.5 |
| 5 | 8.2 | 35 | 11.9 |
| 10 | 9.1 | 40 | 12.3 |
| 15 | 9.9 | 45 | 12.7 |
| 20 | 10.5 | 50 | 13.0 |
| 25 | 11.0 | | |

**28. Stocking a fish pond** As the fish and game warden of your township, you are responsible for stocking the town pond with fish before the fishing season. The average depth of the pond is 20 ft. Using a scaled map, you measure distances across the pond at 200-ft intervals, as shown in the accompanying diagram.

**a.** Use the Trapezoidal Rule to estimate the volume of the pond.

**b.** You plan to start the season with one fish per 1000 cubic feet. You intend to have at least 25% of the opening day's fish population left at the end of the season. What is the maximum number of licenses the town can sell if the average seasonal catch is 20 fish per license?



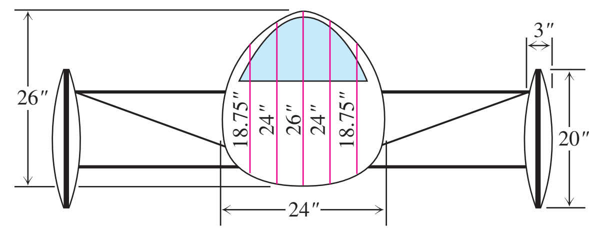

Vertical spacing = 200 ft

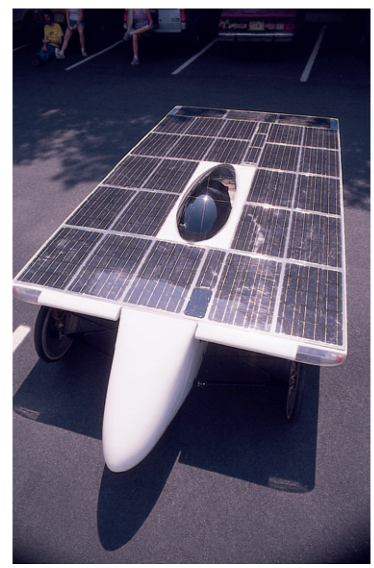
**29. Ford® Mustang Cobra™** The accompanying table shows time-to-speed data for a 1994 Ford Mustang Cobra accelerating from rest to 130 mph. How far had the Mustang traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: The time intervals vary in length.)

| Speed change | Time (sec) |
|---|---|
| Zero to 30 mph | 2.2 |
| 40 mph | 3.2 |
| 50 mph | 4.5 |
| 60 mph | 5.9 |
| 70 mph | 7.8 |
| 80 mph | 10.2 |
| 90 mph | 12.7 |
| 100 mph | 16.0 |
| 110 mph | 20.6 |
| 120 mph | 26.2 |
| 130 mph | 37.1 |

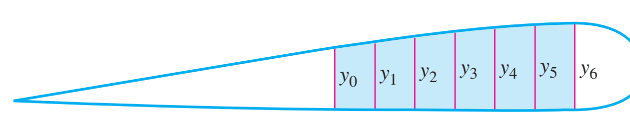
*Source: Car and Driver*, April 1994.

**30. Aerodynamic drag** A vehicle's aerodynamic drag is determined in part by its cross-sectional area, so, all other things being equal, engineers try to make this area as small as possible. Use Simpson's Rule to estimate the cross-sectional area of the body of James Worden's solar-powered Solectria® automobile at MIT from the diagram.



**31. Wing design** The design of a new airplane requires a gasoline tank of constant cross-sectional area in each wing. A scale drawing of a cross-section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft$^3$. Estimate the length of the tank.




$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

**32. Oil consumption on Pathfinder Island** A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced.

Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

| Day | Oil consumption rate (liters/h) |
|---|---|
| Sun | 0.019 |
| Mon | 0.020 |
| Tue | 0.021 |
| Wed | 0.023 |
| Thu | 0.025 |
| Fri | 0.028 |
| Sat | 0.031 |
| Sun | 0.035 |

## Theory and Examples

**33. Usable values of the sine-integral function** *The sine-integral function,*

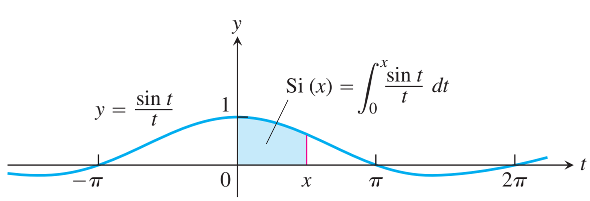
$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \text{"Sine integral of } x\text{"}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of $\operatorname{Si}(x)$, however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth, and you can expect good results from Simpson's Rule.




**a.** Use the fact that $|f^{(4)}| \leq 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\operatorname{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t}\,dt$$

is estimated by Simpson's Rule with $n = 4$.

**b.** Estimate $\operatorname{Si}(\pi/2)$ by Simpson's Rule with $n = 4$.

**c.** Express the error bound you found in part (a) as a percentage of the value you found in part (b).

**34. The error function** *The error function,*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt,$$

important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of $e^{-t^2}$.

**a.** Use Simpson's Rule with $n = 10$ to estimate $\operatorname{erf}(1)$.

**b.** In $[0, 1]$,

$$\left|\frac{d^4}{dt^4}\left(e^{-t^2}\right)\right| \leq 12.$$

Give an upper bound for the magnitude of the error of the estimate in part (a).

**35.** (*Continuation of Example 3.*) The error bounds for $E_T$ and $E_S$ are "worst case" estimates, and the Trapezoidal and Simpson Rules are often more accurate than the bounds suggest. The Trapezoidal Rule estimate of

$$\int_0^{\pi} x \sin x\,dx$$

in Example 3 is a case in point.

**a.** Use the Trapezoidal Rule with $n = 10$ to approximate the value of the integral. The table to the right gives the necessary $y$-values.

| $x$ | $x \sin x$ |
|---|---|
| 0 | 0 |
| $(0.1)\pi$ | 0.09708 |
| $(0.2)\pi$ | 0.36932 |
| $(0.3)\pi$ | 0.76248 |
| $(0.4)\pi$ | 1.19513 |
| $(0.5)\pi$ | 1.57080 |
| $(0.6)\pi$ | 1.79270 |
| $(0.7)\pi$ | 1.77912 |
| $(0.8)\pi$ | 1.47727 |
| $(0.9)\pi$ | 0.87372 |
| $\pi$ | 0 |

**b.** Find the magnitude of the difference between $\pi$, the integral's value, and your approximation in part (a). You will find the difference to be considerably less than the upper bound of 0.133 calculated with $n = 10$ in Example 3.

T **c.** The upper bound of 0.133 for $|E_T|$ in Example 3 could have been improved somewhat by having a better bound for

$$|f''(x)| = |2\cos x - x \sin x|$$

on $[0, \pi]$. The upper bound we used was $2 + \pi$. Graph $f''$ over $[0, \pi]$ and use Trace or Zoom to improve this upper bound.

Use the improved upper bound as $M$ to make an improved estimate of $|E_T|$. Notice that the Trapezoidal Rule approximation in part (a) is also better than this improved estimate would suggest.

T **36.** (*Continuation of Exercise 35.*)

**a.** Show that the fourth derivative of $f(x) = x \sin x$ is

$$f^{(4)}(x) = -4\cos x + x \sin x.$$

Use Trace or Zoom to find an upper bound $M$ for the values of $|f^{(4)}|$ on $[0, \pi]$.

**b.** Use the value of $M$ from part (a) to obtain an upper bound for the magnitude of the error in estimating the value of

$$\int_0^{\pi} x \sin x \, dx$$

with Simpson's Rule with $n = 10$ steps.

**c.** Use the data in the table in Exercise 35 to estimate $\int_0^{\pi} x \sin x \, dx$ with Simpson's Rule with $n = 10$ steps.

**d.** To six decimal places, find the magnitude of the difference between your estimate in part (c) and the integral's true value, $\pi$. You will find the error estimate obtained in part (b) to be quite good.

**37.** Prove that the sum $T$ in the Trapezoidal Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for $f$ continuous on $[a, b]$. (*Hint:* Use the Intermediate Value Theorem to show the existence of $c_k$ in the subinterval $[x_{k-1}, x_k]$ satisfying $f(c_k) = (f(x_{k-1}) + f(x_k))/2$.)

**38.** Prove that the sum $S$ in Simpson's Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for $f$ continuous on $[a, b]$. (See Exercise 37.)

## T Numerical Integration

As we mentioned at the beginning of the section, the definite integrals of many continuous functions cannot be evaluated with the Fundamental Theorem of Calculus because their antiderivatives lack elementary formulas. Numerical integration offers a practical way to estimate the values of these so-called *nonelementary integrals*. If your calculator or computer has a numerical integration routine, try it on the integrals in Exercises 39–42.

**39.** $\displaystyle\int_0^1 \sqrt{1 + x^4}\, dx$ — A nonelementary integral that came up in Newton's research

**40.** $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\, dx$ — The integral from Exercise 33. To avoid division by zero, you may have to start the integration at a small positive number like $10^{-6}$ instead of 0.

**41.** $\displaystyle\int_0^{\pi/2} \sin(x^2)\, dx$ — An integral associated with the diffraction of light

**42.** $\displaystyle\int_0^{\pi/2} 40\sqrt{1 - 0.64\cos^2 t}\, dt$ — The length of the ellipse $(x^2/25) + (y^2/9) = 1$

**T 43.** Consider the integral $\int_0^{\pi} \sin x\, dx$.

**a.** Find the Trapezoidal Rule approximations for $n = 10, 100$, and 1000.

**b.** Record the errors with as many decimal places of accuracy as you can.

**c.** What pattern do you see?

**d.** Explain how the error bound for $E_T$ accounts for the pattern.

**T 44.** (*Continuation of Exercise 43.*) Repeat Exercise 43 with Simpson's Rule and $E_S$.

**45.** Consider the integral $\int_{-1}^{1} \sin(x^2)\, dx$.

**a.** Find $f''$ for $f(x) = \sin(x^2)$.

**b.** Graph $y = f''(x)$ in the viewing window $[-1, 1]$ by $[-3, 3]$.

**c.** Explain why the graph in part (b) suggests that $|f''(x)| \le 3$ for $-1 \le x \le 1$.

**d.** Show that the error estimate for the Trapezoidal Rule in this case becomes

$$|E_T| \le \frac{(\Delta x)^2}{2}.$$

**e.** Show that the Trapezoidal Rule error will be less than or equal to 0.01 in magnitude if $\Delta x \le 0.1$.

**f.** How large must $n$ be for $\Delta x \le 0.1$?

**46.** Consider the integral $\int_{-1}^{1} \sin(x^2)\, dx$.

**a.** Find $f^{(4)}$ for $f(x) = \sin(x^2)$. (You may want to check your work with a CAS if you have one available.)

**b.** Graph $y = f^{(4)}(x)$ in the viewing window $[-1, 1]$ by $[-30, 10]$.

**c.** Explain why the graph in part (b) suggests that $|f^{(4)}(x)| \le 30$ for $-1 \le x \le 1$.

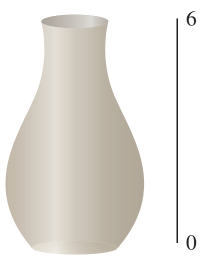
**d.** Show that the error estimate for Simpson's Rule in this case becomes

$$|E_S| \le \frac{(\Delta x)^4}{3}.$$

**e.** Show that the Simpson's Rule error will be less than or equal to 0.01 in magnitude if $\Delta x \le 0.4$.

**f.** How large must $n$ be for $\Delta x \le 0.4$?

**T 47. A vase** We wish to estimate the volume of a flower vase using only a calculator, a string, and a ruler. We measure the height of the vase to be 6 in. We then use the string and the ruler to find circumferences of the vase (in inches) at half-inch intervals. (We list them from the top down to correspond with the picture of the vase.)




| **Circumferences** | |
|---|---|
| 5.4 | 10.8 |
| 4.5 | 11.6 |
| 4.4 | 11.6 |
| 5.1 | 10.8 |
| 6.3 | 9.0 |
| 7.8 | 6.3 |
| 9.4 | |

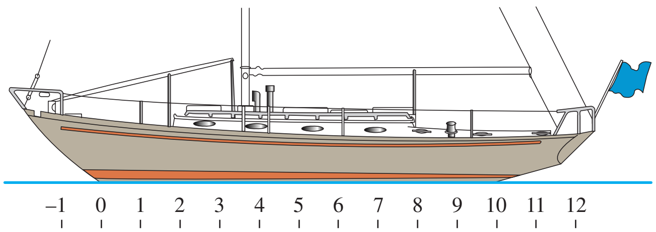
**a.** Find the areas of the cross-sections that correspond to the given circumferences.

**b.** Express the volume of the vase as an integral with respect to $y$ over the interval $[0, 6]$.

**c.** Approximate the integral using the Trapezoidal Rule with $n = 12$.

**d.** Approximate the integral using Simpson's Rule with $n = 12$. Which result do you think is more accurate? Give reasons for your answer.

T **48. A sailboat's displacement** To find the volume of water displaced by a sailboat, the common practice is to partition the waterline into 10 subintervals of equal length, measure the cross-sectional area $A(x)$ of the submerged portion of the hull at each partition point, and then use Simpson's Rule to estimate the integral of $A(x)$ from one end of the waterline to the other. The table here lists the area measurements at "Stations" 0 through 10, as the partition points are called, for the cruising sloop *Pipedream*, shown here. The common subinterval length (distance between consecutive stations) is $\Delta x = 2.54$ ft (about 2 ft 6-1/2 in., chosen for the convenience of the builder).




**a.** Estimate *Pipedream*'s displacement volume to the nearest cubic foot.

| Station | Submerged area (ft$^2$) |
|---|---|
| 0 | 0 |
| 1 | 1.07 |
| 2 | 3.84 |
| 3 | 7.82 |
| 4 | 12.20 |
| 5 | 15.18 |
| 6 | 16.14 |
| 7 | 14.00 |
| 8 | 9.21 |
| 9 | 3.24 |
| 10 | 0 |

**b.** The figures in the table are for seawater, which weighs 64 lb/ft$^3$. How many pounds of water does *Pipedream* displace? (Displacement is given in pounds for small craft and in long tons (1 long ton = 2240 lb) for larger vessels.) (Data from *Skene's Elements of Yacht Design* by Francis S. Kinney (Dodd, Mead, 1962.)

**c. Prismatic coefficients** A boat's prismatic coefficient is the ratio of the displacement volume to the volume of a prism whose height equals the boat's waterline length and whose base equals the area of the boat's largest submerged cross-section. The best sailboats have prismatic coefficients between 0.51 and 0.54. Find *Pipedream*'s prismatic coefficient, given a waterline length of 25.4 ft and a largest submerged cross-sectional area of 16.14 ft$^2$ (at Station 6).

T **49. Elliptic integrals** The length of the ellipse

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi$$

turns out to be

$$\text{Length} = 4a\int_0^{\pi/2}\sqrt{1 - e^2\cos^2 t}\,dt,$$

where $e$ is the ellipse's eccentricity. The integral in this formula, called an *elliptic integral*, is nonelementary except when $e = 0$ or 1.

**a.** Use the Trapezoidal Rule with $n = 10$ to estimate the length of the ellipse when $a = 1$ and $e = 1/2$.

**b.** Use the fact that the absolute value of the second derivative of $f(t) = \sqrt{1 - e^2\cos^2 t}$ is less than 1 to find an upper bound for the error in the estimate you obtained in part (a).

T **50.** The length of one arch of the curve $y = \sin x$ is given by

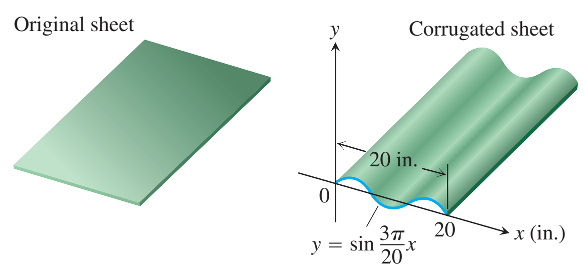
$$L = \int_0^{\pi}\sqrt{1 + \cos^2 x}\,dx.$$

Estimate $L$ by Simpson's Rule with $n = 8$.

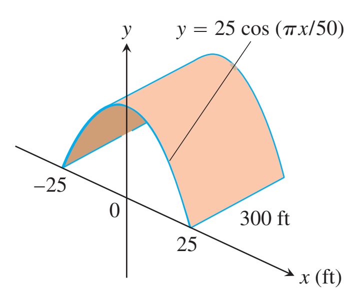
T **51.** Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross-sections of the corrugated sheets are to conform to the curve

$$y = \sin\frac{3\pi}{20}x, \quad 0 \le x \le 20 \text{ in.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to two decimal places.




T **52.** Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 300 ft long and 50 ft wide at the base. The cross-section is shaped like one arch of the curve $y = 25\cos(\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$1.75 per square foot to apply. How much will it cost to apply the sealer? (*Hint:* Use numerical integration to find the length of the cosine curve.)



## Surface Area

Find, to two decimal places, the areas of the surfaces generated by revolving the curves in Exercises 53–56 about the $x$-axis.

**53.** $y = \sin x, \quad 0 \le x \le \pi$

**54.** $y = x^2/4, \quad 0 \le x \le 2$

**55.** $y = x + \sin 2x, \quad -2\pi/3 \le x \le 2\pi/3$ (the curve in Section 4.4, Exercise 5)

**56.** $y = \frac{x}{12}\sqrt{36 - x^2}$, $0 \le x \le 6$ (the surface of the plumb bob in Section 6.1, Exercise 56)

## Estimating Function Values

**57.** Use numerical integration to estimate the value of

$$\sin^{-1} 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1 - x^2}}.$$

For reference, $\sin^{-1} 0.6 = 0.64350$ to five decimal places.

**58.** Use numerical integration to estimate the value of

$$\pi = 4\int_0^1 \frac{1}{1 + x^2}\,dx.$$

# 8.8 Improper Integrals

Up to now, definite integrals have been required to have two properties. First, that the domain of integration $[a, b]$ be finite. Second, that the range of the integrand be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite (Figure 8.17a). The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the range of the integrand is infinite (Figure 8.17b). In either case, the integrals are said to be *improper* and are calculated as limits. We will see that improper integrals play an important role when investigating the convergence of certain infinite series in Chapter 11.

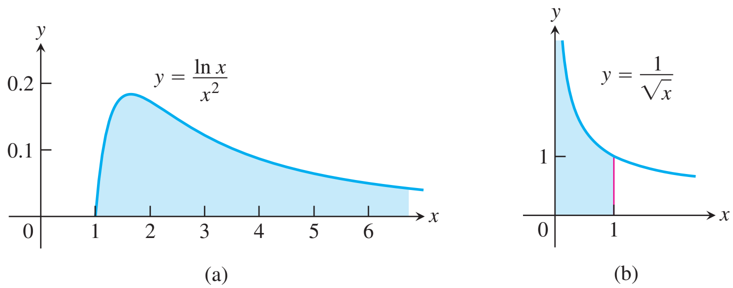



**FIGURE 8.17** Are the areas under these infinite curves finite?

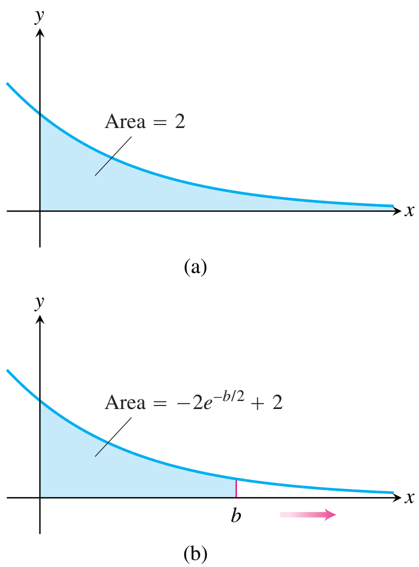



**FIGURE 8.18** (a) The area in the first quadrant under the curve $y = e^{-x/2}$ is (b) an improper integral of the first type.

## Infinite Limits of Integration

Consider the infinite region that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure 8.18a). You might think this region has infinite area, but we will see that the natural value to assign is finite. Here is how to assign a value to the area. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Figure 8.18b).

$$A(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (-2e^{-b/2} + 2) = 2.$$

The value we assign to the area under the curve from 0 to $\infty$ is

$$\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx = 2.$$

**DEFINITION Type I Improper Integrals**

Integrals with infinite limits of integration are **improper integrals of Type I**.

**1.** If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

**2.** If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

**3.** If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where $c$ is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

It can be shown that the choice of $c$ in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^{\infty} f(x)\,dx$ with any convenient choice.

Any of the integrals in the above definition can be interpreted as an area if $f \ge 0$ on the interval of integration. For instance, we interpreted the improper integral in Figure 8.18 as an area. In that case, the area has the finite value 2. If $f \ge 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

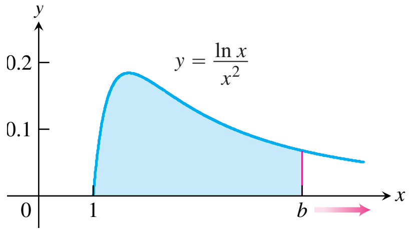



**FIGURE 8.19** The area under this curve is an improper integral (Example 1).

## EXAMPLE 1 Evaluating an Improper Integral on $[1, \infty)$

Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

**Solution** We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve (Figure 8.19). The area from 1 to $b$ is

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

Integration by parts with $u = \ln x$, $dv = dx/x^2$, $du = dx/x$, $v = -1/x$.

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

$$= -\left[\lim_{b\to\infty} \frac{\ln b}{b}\right] - 0 + 1$$

$$= -\left[\lim_{b\to\infty} \frac{1/b}{1}\right] + 1 = 0 + 1 = 1. \qquad \text{l'Hôpital's Rule}$$

Thus, the improper integral converges and the area has finite value 1. ■

## EXAMPLE 2 Evaluating an Integral on $(-\infty, \infty)$

Evaluate

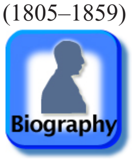
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

**Solution** According to the definition (Part 3), we can write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

HISTORICAL BIOGRAPHY

Lejeune Dirichlet
(1805–1859)




Next we evaluate each improper integral on the right side of the equation above.

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \tan^{-1} x\Big]_a^0$$

$$= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \tan^{-1} x\Big]_0^b$$

$$= \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

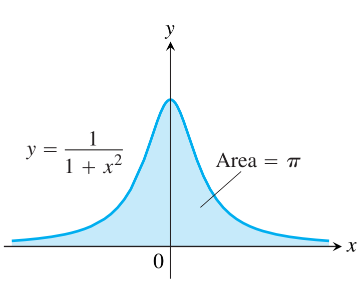



**FIGURE 8.20** The area under this curve is finite (Example 2).

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the $x$-axis (Figure 8.20). ■

## The Integral $\int_1^\infty \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \le 1$.

**EXAMPLE 3** Determining Convergence

For what values of $p$ does the integral $\int_1^\infty dx/x^p$ converge? When the integral does converge, what is its value?

**Solution** If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

because

$$\lim_{b\to\infty}\frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x}$$

$$= \lim_{b\to\infty}\int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \ln x\bigg]_1^b$$

$$= \lim_{b\to\infty}(\ln b - \ln 1) = \infty.$$

■

## Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand $f$ is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of $f$ and above the $x$-axis between the limits of integration.

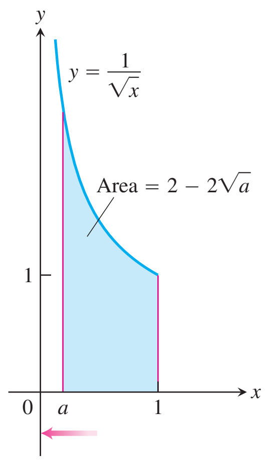



**FIGURE 8.21** The area under this curve is

$$\lim_{a \to 0^+} \int_a^1 \left( \frac{1}{\sqrt{x}} \right) dx = 2,$$

an improper integral of the second kind.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.17b). First we find the area of the portion from $a$ to 1 (Figure 8.21).

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big]_a^1 = 2 - 2\sqrt{a}$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a \to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \left( 2 - 2\sqrt{a} \right) = 2.$$

The area under the curve from 0 to 1 is finite and equals

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

**DEFINITION Type II Improper Integrals**

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at $a$ then

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \int_c^b f(x)\, dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at $b$, then

$$\int_a^b f(x)\, dx = \lim_{c \to b^-} \int_a^c f(x)\, dx.$$

3. If $f(x)$ is discontinuous at $c$, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

In Part 3 of the definition, the integral on the left side of the equation converges if *both* integrals on the right side converge; otherwise it diverges.

## EXAMPLE 4 A Divergent Improper Integral

Investigate the convergence of

$$\int_0^1 \frac{1}{1 - x}\, dx.$$

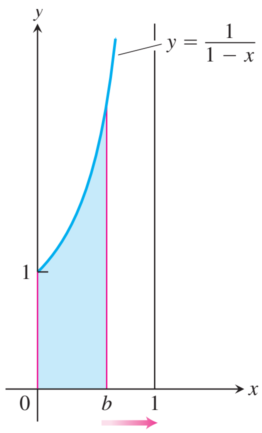



**FIGURE 8.22** The limit does not exist:

$$\int_0^1 \left(\frac{1}{1-x}\right) dx = \lim_{b \to 1^-} \int_0^b \frac{1}{1-x}\, dx = \infty.$$

The area beneath the curve and above the $x$-axis for $[0, 1)$ is not a real number (Example 4).

**Solution** The integrand $f(x) = 1/(1 - x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ (Figure 8.22). We evaluate the integral as

$$\begin{aligned} \lim_{b \to 1^-} \int_0^b \frac{1}{1-x}\, dx &= \lim_{b \to 1^-} \Big[-\ln|1 - x|\Big]_0^b \\ &= \lim_{b \to 1^-} [-\ln(1 - b) + 0] = \infty. \end{aligned}$$

The limit is infinite, so the integral diverges. ■

## EXAMPLE 5 Vertical Asympote at an Interior Point

Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

**Solution** The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.23). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\begin{aligned} \int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{b \to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{b \to 1^-} 3(x-1)^{1/3}\Big]_0^b \\ &= \lim_{b \to 1^-} [3(b-1)^{1/3} + 3] = 3 \end{aligned}$$

$$\begin{aligned} \int_1^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{c \to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{c \to 1^+} 3(x-1)^{1/3}\Big]_c^3 \\ &= \lim_{c \to 1^+} \Big[3(3-1)^{1/3} - 3(c-1)^{1/3}\Big] = 3\sqrt[3]{2} \end{aligned}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$ ■

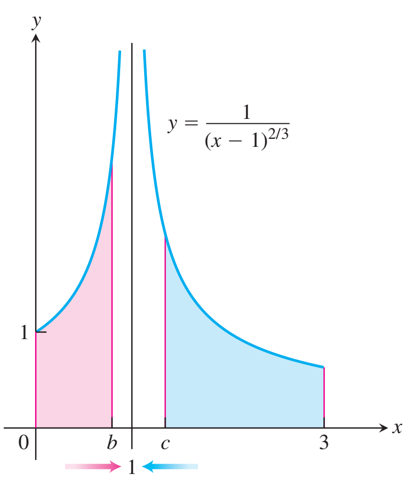



**FIGURE 8.23** Example 5 shows the convergence of

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = 3 + 3\sqrt[3]{2},$$

so the area under the curve exists (so it is a real number).

## EXAMPLE 6 A Convergent Improper Integral

Evaluate

$$\int_2^\infty \frac{x + 3}{(x-1)(x^2+1)}\, dx.$$

**Solution**

$$\begin{aligned}
\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx &= \lim_{b\to\infty}\int_2^b \frac{x+3}{(x-1)(x^2+1)}\,dx \\
&= \lim_{b\to\infty}\int_2^b \left(\frac{2}{x-1} - \frac{2x+1}{x^2+1}\right)dx \qquad \text{Partial fractions} \\
&= \lim_{b\to\infty}\Big[2\ln(x-1) - \ln(x^2+1) - \tan^{-1}x\Big]_2^b \\
&= \lim_{b\to\infty}\left[\ln\frac{(x-1)^2}{x^2+1} - \tan^{-1}x\right]_2^b \qquad \text{Combine the logarithms.} \\
&= \lim_{b\to\infty}\left[\ln\left(\frac{(b-1)^2}{b^2+1}\right) - \tan^{-1}b\right] - \ln\left(\frac{1}{5}\right) + \tan^{-1}2 \\
&= 0 - \frac{\pi}{2} + \ln 5 + \tan^{-1}2 \approx 1.1458
\end{aligned}$$

Notice that we combined the logarithms in the antiderivative *before* we calculated the limit as $b \to \infty$. Had we not done so, we would have encountered the indeterminate form

$$\lim_{b\to\infty}(2\ln(b-1) - \ln(b^2+1)) = \infty - \infty.$$

The way to evaluate the indeterminate form, of course, is to combine the logarithms, so we would have arrived at the same answer in the end. ■

Computer algebra systems can evaluate many convergent improper integrals. To evaluate the integral in Example 6 using Maple, enter

$$> f := (x+3)/((x-1)*(x\verb|^|2+1));$$

Then use the integration command

$$> \text{int}(f, x = 2..\text{infinity});$$

Maple returns the answer

$$-\frac{1}{2}\pi + \ln(5) + \arctan(2).$$

To obtain a numerical result, use the evaluation command **evalf** and specify the number of digits, as follows:

$$> \text{evalf}(\%, 6);$$

The symbol % instructs the computer to evaluate the last expression on the screen, in this case $(-1/2)\pi + \ln(5) + \arctan(2)$. Maple returns 1.14579.

Using Mathematica, entering

$$\textit{In [1]:=}\ \text{Integrate}\,[(x+3)/((x-1)(x\verb|^|2+1)), \{x, 2, \text{Infinity}\}]$$

returns

$$\textit{Out [1]=}\ \frac{-\text{Pi}}{2} + \text{ArcTan}\,[2] + \text{Log}\,[5].$$

To obtain a numerical result with six digits, use the command "N[%, 6]"; it also yields 1.14579.

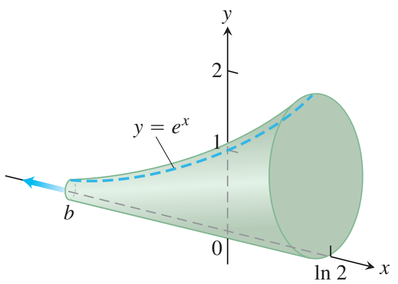



**FIGURE 8.24** The calculation in Example 7 shows that this infinite horn has a finite volume.

## EXAMPLE 7 Finding the Volume of an Infinite Solid

The cross-sections of the solid horn in Figure 8.24 perpendicular to the $x$-axis are circular disks with diameters reaching from the $x$-axis to the curve $y = e^x$, $-\infty < x \le \ln 2$. Find the volume of the horn.

**Solution** The area of a typical cross-section is

$$A(x) = \pi(\text{radius})^2 = \pi\left(\frac{1}{2}y\right)^2 = \frac{\pi}{4}e^{2x}.$$

We define the volume of the horn to be the limit as $b \to -\infty$ of the volume of the portion from $b$ to $\ln 2$. As in Section 6.1 (the method of slicing), the volume of this portion is

$$\begin{aligned} V &= \int_b^{\ln 2} A(x)\,dx = \int_b^{\ln 2} \frac{\pi}{4}e^{2x}\,dx = \frac{\pi}{8}e^{2x}\Big]_b^{\ln 2} \\ &= \frac{\pi}{8}\left(e^{\ln 4} - e^{2b}\right) = \frac{\pi}{8}\left(4 - e^{2b}\right). \end{aligned}$$

As $b \to -\infty$, $e^{2b} \to 0$ and $V \to (\pi/8)(4 - 0) = \pi/2$. The volume of the horn is $\pi/2$. ■

## EXAMPLE 8 An Incorrect Calculation

Evaluate

$$\int_0^3 \frac{dx}{x-1}.$$

**Solution** Suppose we fail to notice the discontinuity of the integrand at $x = 1$, interior to the interval of integration. If we evaluate the integral as an ordinary integral we get

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big]_0^3 = \ln 2 - \ln 1 = \ln 2.$$

This result is *wrong* because the integral is improper. The correct evaluation uses limits:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where

$$\begin{aligned} \int_0^1 \frac{dx}{x-1} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{x-1} = \lim_{b\to 1^-} \ln|x-1|\Big]_0^b \\ &= \lim_{b\to 1^-}\left(\ln|b-1| - \ln|-1|\right) \\ &= \lim_{b\to 1^-} \ln(1-b) = -\infty. \end{aligned}$$

$1 - b \to 0^+$ as $b \to 1^-$

Since $\int_0^1 dx/(x-1)$ is divergent, the original integral $\int_0^3 dx/(x-1)$ is divergent. ■

Example 8 illustrates what can go wrong if you mistake an improper integral for an ordinary integral. Whenever you encounter an integral $\int_a^b f(x)\,dx$ you must examine the function $f$ on $[a, b]$ and then decide if the integral is improper. If $f$ is continuous on $[a, b]$, it will be proper, an ordinary integral.

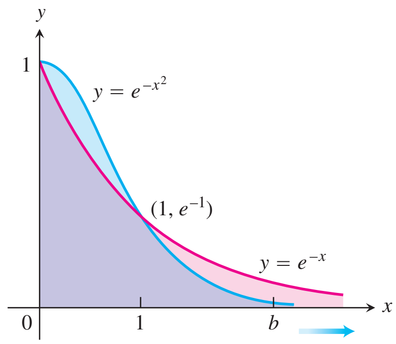



**FIGURE 8.25** The graph of $e^{-x^2}$ lies below the graph of $e^{-x}$ for $x > 1$ (Example 9).

## Tests for Convergence and Divergence

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

**EXAMPLE 9** Investigating Convergence

Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

**Solution** By definition,

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx.$$

We cannot evaluate the latter integral directly because it is nonelementary. But we *can* show that its limit as $b \to \infty$ is finite. We know that $\int_1^b e^{-x^2}\,dx$ is an increasing function of $b$. Therefore either it becomes infinite as $b \to \infty$ or it has a finite limit as $b \to \infty$. It does not become infinite: For every value of $x \ge 1$ we have $e^{-x^2} \le e^{-x}$ (Figure 8.25), so that

$$\int_1^b e^{-x^2}\,dx \le \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788.$$

Hence

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$$

converges to some definite finite value. We do not know exactly what the value is except that it is something positive and less than 0.37. Here we are relying on the completeness property of the real numbers, discussed in Appendix 4. ■

The comparison of $e^{-x^2}$ and $e^{-x}$ in Example 9 is a special case of the following test.

### Historical Biography

Karl Weierstrass
(1815–1897)

**THEOREM 1 Direct Comparison Test**

Let $f$ and $g$ be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges
2. $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

The reasoning behind the argument establishing Theorem 1 is similar to that in Example 9.

If $0 \le f(x) \le g(x)$ for $x \ge a$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx, \qquad b > a.$$

From this it can be argued, as in Example 9, that

$$\int_a^\infty f(x)\,dx \text{ converges if } \int_a^\infty g(x)\,dx \text{ converges}.$$

Turning this around says that

$$\int_a^\infty g(x)\,dx \text{ diverges if } \int_a^\infty f(x)\,dx \text{ diverges}.$$

**EXAMPLE 10** Using the Direct Comparison Test

**(a)** $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\,dx \text{ converges}.$$ Example 3

**(b)** $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x}\,dx \text{ diverges}.$$ Example 3

■

**THEOREM 2 Limit Comparison Test**

If the positive functions $f$ and $g$ are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

A proof of Theorem 2 is given in advanced calculus.

Although the improper integrals of two functions from $a$ to $\infty$ may both converge, this does not mean that their integrals necessarily have the same value, as the next example shows.

### EXAMPLE 11 Using the Limit Comparison Test

Show that

$$\int_1^\infty \frac{dx}{1 + x^2}$$

converges by comparison with $\int_1^\infty (1/x^2)\,dx$. Find and compare the two integral values.

**Solution** The functions $f(x) = 1/x^2$ and $g(x) = 1/(1 + x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^2}{1/(1 + x^2)} = \lim_{x\to\infty} \frac{1 + x^2}{x^2}$$

$$= \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,$$

a positive finite limit (Figure 8.26). Therefore, $\int_1^\infty \frac{dx}{1 + x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

The integrals converge to different values, however.

$$\int_1^\infty \frac{dx}{x^2} = \frac{1}{2 - 1} = 1 \qquad \text{Example 3}$$

and

$$\int_1^\infty \frac{dx}{1 + x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1 + x^2}$$

$$= \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

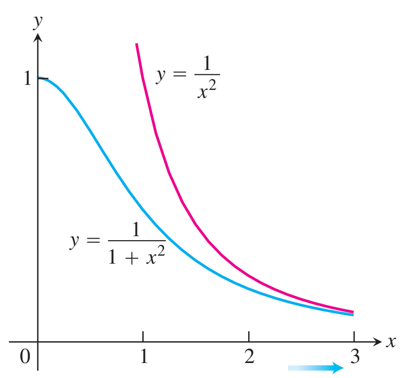



**FIGURE 8.26** The functions in Example 11.

### EXAMPLE 12 Using the Limit Comparison Test

Show that

$$\int_1^\infty \frac{3}{e^x + 5}\,dx$$

converges.

**Solution** From Example 9, it is easy to see that $\int_1^\infty e^{-x}\,dx = \int_1^\infty (1/e^x)\,dx$ converges. Moreover, we have

$$\lim_{x\to\infty} \frac{1/e^x}{3/(e^x + 5)} = \lim_{x\to\infty} \frac{e^x + 5}{3e^x} = \lim_{x\to\infty} \left(\frac{1}{3} + \frac{5}{3e^x}\right) = \frac{1}{3},$$

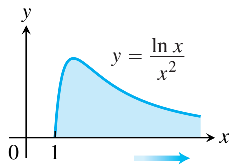
a positive finite limit. As far as the convergence of the improper integral is concerned, $3/(e^x + 5)$ behaves like $1/e^x$.

## Types of Improper Integrals Discussed in This Section

### Infinite Limits of Integration: Type I

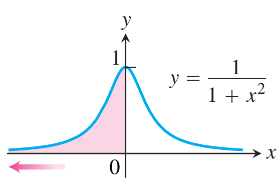
**1.** Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$



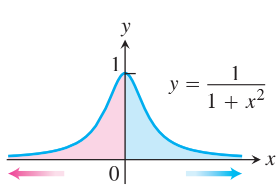

**2.** Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2}$$




**3.** Both limits

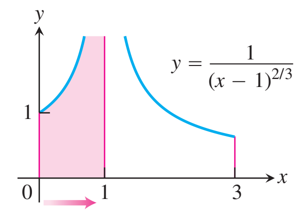
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^0 \frac{dx}{1+x^2} + \lim_{c\to\infty} \int_0^c \frac{dx}{1+x^2}$$




### Integrand Becomes Infinite: Type II

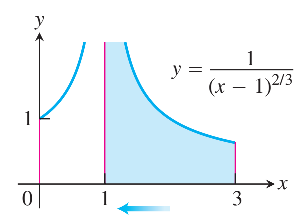
**4.** Upper endpoint

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$



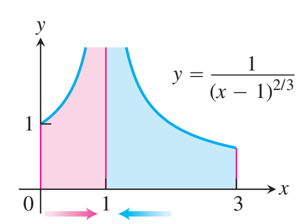

**5.** Lower endpoint

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{d\to1^+} \int_d^3 \frac{dx}{(x-1)^{2/3}}$$




**6.** Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$



# EXERCISES 8.8

## Evaluating Improper Integrals

Evaluate the integrals in Exercises 1–34 without using tables.

1. $\int_0^\infty \frac{dx}{x^2+1}$
2. $\int_1^\infty \frac{dx}{x^{1.001}}$
3. $\int_0^1 \frac{dx}{\sqrt{x}}$
4. $\int_0^4 \frac{dx}{\sqrt{4-x}}$
5. $\int_{-1}^1 \frac{dx}{x^{2/3}}$
6. $\int_{-8}^1 \frac{dx}{x^{1/3}}$
7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$
8. $\int_0^1 \frac{dr}{r^{0.999}}$
9. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$
10. $\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$
11. $\int_2^\infty \frac{2}{v^2-v}\,dv$
12. $\int_2^\infty \frac{2\,dt}{t^2-1}$
13. $\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$
14. $\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{3/2}}$
15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$
16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$
17. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$
18. $\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$
19. $\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$
20. $\int_0^\infty \frac{16\tan^{-1} x}{1+x^2}\,dx$
21. $\int_{-\infty}^0 \theta e^\theta\,d\theta$
22. $\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$
23. $\int_{-\infty}^0 e^{-|x|}\,dx$
24. $\int_{-\infty}^\infty 2xe^{-x^2}\,dx$
25. $\int_0^1 x\ln x\,dx$
26. $\int_0^1 (-\ln x)\,dx$
27. $\int_0^2 \frac{ds}{\sqrt{4-s^2}}$
28. $\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}}$
29. $\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$
30. $\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$
31. $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$
32. $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$
33. $\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6}$
34. $\int_0^\infty \frac{dx}{(x+1)(x^2+1)}$

## Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\int_0^{\pi/2} \tan\theta\,d\theta$
36. $\int_0^{\pi/2} \cot\theta\,d\theta$
37. $\int_0^\pi \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$
38. $\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi-2\theta)^{1/3}}$
39. $\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$
40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$
41. $\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$
42. $\int_0^1 \frac{dt}{t-\sin t}$ (*Hint:* $t \ge \sin t$ for $t \ge 0$)
43. $\int_0^2 \frac{dx}{1-x^2}$
44. $\int_0^2 \frac{dx}{1-x}$
45. $\int_{-1}^1 \ln|x|\,dx$
46. $\int_{-1}^1 -x\ln|x|\,dx$
47. $\int_1^\infty \frac{dx}{x^3+1}$
48. $\int_4^\infty \frac{dx}{\sqrt{x}-1}$
49. $\int_2^\infty \frac{dv}{\sqrt{v-1}}$
50. $\int_0^\infty \frac{d\theta}{1+e^\theta}$
51. $\int_0^\infty \frac{dx}{\sqrt{x^6+1}}$
52. $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$
53. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$
54. $\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$
55. $\int_\pi^\infty \frac{2+\cos x}{x}\,dx$
56. $\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$
57. $\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$
58. $\int_2^\infty \frac{1}{\ln x}\,dx$
59. $\int_1^\infty \frac{e^x}{x}\,dx$
60. $\int_{e^e}^\infty \ln(\ln x)\,dx$
61. $\int_1^\infty \frac{1}{\sqrt{e^x-x}}\,dx$
62. $\int_1^\infty \frac{1}{e^x-2^x}\,dx$
63. $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$
64. $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

## Theory and Examples

65. Find the values of $p$ for which each integral converges.

    **a.** $\int_1^2 \frac{dx}{x(\ln x)^p}$ **b.** $\int_2^\infty \frac{dx}{x(\ln x)^p}$

66. **$\int_{-\infty}^\infty f(x)\,dx$ may not equal $\lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$** Show that

$$\int_0^\infty \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$$