8.5 Trigonometric Substitutions

Trigonometric substitutions can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

Three Basic Substitutions

The most common substitutions are $x = a\tan\theta$, $x = a\sin\theta$, and $x = a\sec\theta$. They come from the reference right triangles in Figure 8.2.

With $x = a\tan\theta$,

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

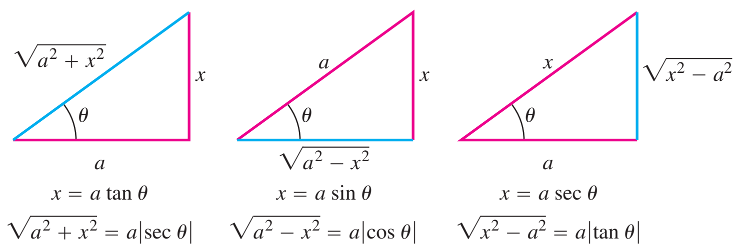


FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

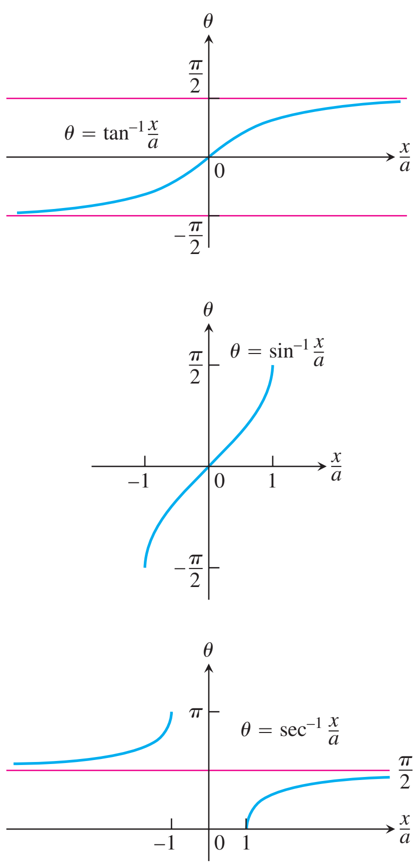


FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a, graphed as functions of x/a.

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a\tan\theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a\sin\theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a\sec\theta$.

As we know from Section 7.7, the functions in these substitutions have inverses only for selected values of θ (Figure 8.3). For reversibility,

$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \ge 1, \\ \frac{\pi}{2} < \theta \le \pi & \text{if } \frac{x}{a} \le -1. \end{cases}$$

To simplify calculations with the substitution $x = a\sec\theta$, we will restrict its use to integrals in which $x/a \ge 1$. This will place θ in $[0, \pi/2)$ and make $\tan\theta \ge 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, free of absolute values, provided $a > 0$.

EXAMPLE 1 Using the Substitution $x = a\tan\theta$

You Try It

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$

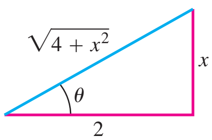


FIGURE 8.4 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} && \sqrt{\sec^2\theta} = |\sec\theta| \\
&= \int \sec\theta\, d\theta && \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C && \text{From Fig. 8.4} \\
&= \ln\left|\sqrt{4+x^2} + x\right| + C'. && \text{Taking } C' = C - \ln 2
\end{aligned}$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Figure 8.4) and read the ratios from the triangle. ■

EXAMPLE 2 Using the Substitution $x = a\sin\theta$

Evaluate

$$\int \frac{x^2\, dx}{\sqrt{9 - x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1 - \sin^2\theta) = 9\cos^2\theta.$$

Then

$$\begin{aligned}
\int \frac{x^2\, dx}{\sqrt{9-x^2}} &= \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{|3\cos\theta|} \\
&= 9\int \sin^2\theta\, d\theta && \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= 9\int \frac{1 - \cos 2\theta}{2}\, d\theta \\
&= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C \\
&= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C && \sin 2\theta = 2\sin\theta\cos\theta \\
&= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C && \text{Fig. 8.5} \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.
\end{aligned}$$

■

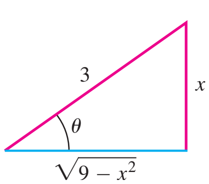


FIGURE 8.5 Reference triangle for $x = 3\sin\theta$ (Example 2):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

EXAMPLE 3 Using the Substitution $x = a\sec\theta$

Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\sqrt{25x^2 - 4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$$
$$= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$
$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$
$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$
$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta. \qquad \text{tan } \theta > 0 \text{ for } 0 < \theta < \pi/2$$

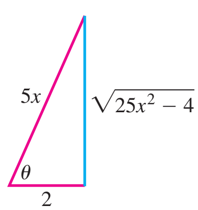


FIGURE 8.6 If $x = (2/5)\sec\theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 3).

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta}$$
$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$
$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C. \qquad \text{Fig. 8.6} \quad ■$$

A trigonometric substitution can sometimes help us to evaluate an integral containing an integer power of a quadratic binomial, as in the next example.

EXAMPLE 4 Finding the Volume of a Solid of Revolution

Find the volume of the solid generated by revolving about the x-axis the region bounded by the curve $y = 4/(x^2 + 4)$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution We sketch the region (Figure 8.7) and use the disk method:

$$V = \int_0^2 \pi[R(x)]^2\, dx = 16\pi\int_0^2 \frac{dx}{(x^2 + 4)^2}. \qquad R(x) = \frac{4}{x^2 + 4}$$

To evaluate the integral, we set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad \theta = \tan^{-1}\frac{x}{2},$$
$$x^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$$

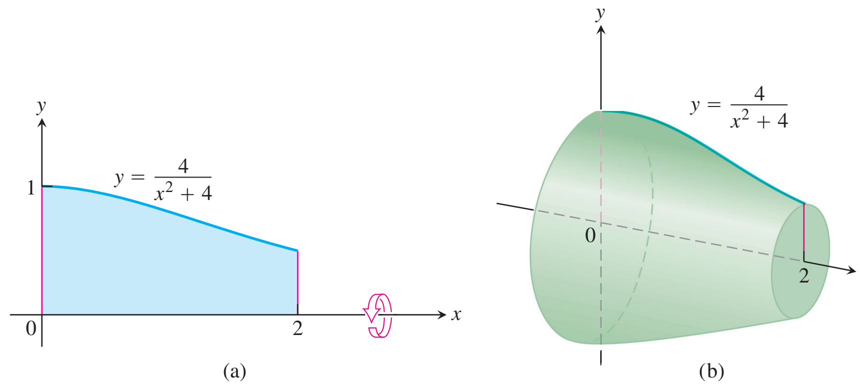


FIGURE 8.7 The region (a) and solid (b) in Example 4.

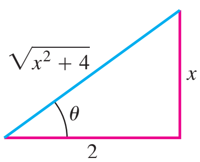


FIGURE 8.8 Reference triangle for $x = 2 \tan \theta$ (Example 4).

(Figure 8.8). With these substitutions,

$$V = 16\pi \int_0^2 \frac{dx}{(x^2 + 4)^2}$$

$$= 16\pi \int_0^{\pi/4} \frac{2 \sec^2 \theta \, d\theta}{(4 \sec^2 \theta)^2} \qquad \theta = 0 \text{ when } x = 0; \; \theta = \pi/4 \text{ when } x = 2$$

$$= 16\pi \int_0^{\pi/4} \frac{2 \sec^2 \theta \, d\theta}{16 \sec^4 \theta} = \pi \int_0^{\pi/4} 2 \cos^2 \theta \, d\theta$$

$$= \pi \int_0^{\pi/4} (1 + \cos 2\theta) \, d\theta = \pi \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/4} \qquad 2 \cos^2 \theta = 1 + \cos 2\theta$$

$$= \pi \left[\frac{\pi}{4} + \frac{1}{2} \right] \approx 4.04.$$

■

EXAMPLE 5 Finding the Area of an Ellipse

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

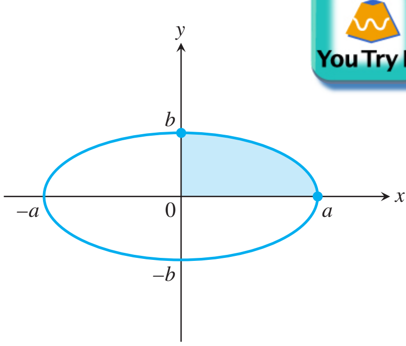


FIGURE 8.9 The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in Example 5.

Solution Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (Figure 8.9). Solving the equation of the ellipse for $y \geq 0$, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2},$$

or

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \leq x \leq a$$

The area of the ellipse is

$$
\begin{aligned}
A &= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx \\
&= 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta \\
&= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta \\
&= 4ab\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta \\
&= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\
&= 2ab\left[\frac{\pi}{2} + 0 - 0\right] = \pi ab.
\end{aligned}
$$

$x = a\sin\theta$, $dx = a\cos\theta\,d\theta$,
$\theta = 0$ when $x = 0$;
$\theta = \pi/2$ when $x = a$

If $a = b = r$ we get that the area of a circle with radius r is πr^2. ■

EXERCISES 8.5

Basic Trigonometric Substitutions

Evaluate the integrals in Exercises 1–28.

1. $\displaystyle\int \frac{dy}{\sqrt{9+y^2}}$

2. $\displaystyle\int \frac{3\,dy}{\sqrt{1+9y^2}}$

3. $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\displaystyle\int_{0}^{2} \frac{dx}{8+2x^2}$

5. $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\displaystyle\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\displaystyle\int \sqrt{25-t^2}\,dt$

8. $\displaystyle\int \sqrt{1-9t^2}\,dt$

9. $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

10. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

11. $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

12. $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

14. $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

15. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

16. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

17. $\displaystyle\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

18. $\displaystyle\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

19. $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

20. $\displaystyle\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$

21. $\displaystyle\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

22. $\displaystyle\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

23. $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

24. $\displaystyle\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

25. $\displaystyle\int \frac{8\,dx}{(4x^2+1)^2}$

26. $\displaystyle\int \frac{6\,dt}{(9t^2+1)^2}$

27. $\displaystyle\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

28. $\displaystyle\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 29–36, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

29. $\displaystyle\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

30. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

31. $\displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

32. $\displaystyle\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

33. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

34. $\displaystyle\int \frac{dx}{1+x^2}$

35. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$

36. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$

Initial Value Problems

Solve the initial value problems in Exercises 37–40 for y as a function of x.

37. $\displaystyle x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \ge 2, \quad y(2) = 0$

38. $\displaystyle \sqrt{x^2-9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

39. $\displaystyle (x^2+4)\frac{dy}{dx} = 3, \quad y(2) = 0$

40. $\displaystyle (x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

Applications

Exercises

41. Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9 - x^2/3}$.

42. Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = 2/(1 + x^2)$, and the line $x = 1$.

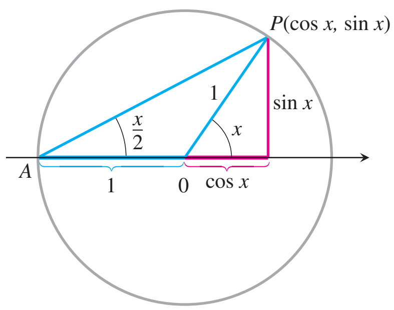
The Substitution $z = \tan(x/2)$

The substitution

$$z = \tan\frac{x}{2} \tag{1}$$

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions.

From the accompanying figure



we can read the relation

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

To see the effect of the substitution, we calculate

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1$$

$$= \frac{2}{1 + \tan^2(x/2)} - 1 = \frac{2}{1 + z^2} - 1$$

$$\cos x = \frac{1 - z^2}{1 + z^2}, \tag{2}$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin(x/2)}{\cos(x/2)} \cdot \cos^2\left(\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2} \cdot \frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2z}{1 + z^2}. \tag{3}$$

Finally, $x = 2\tan^{-1} z$, so

$$dx = \frac{2\,dz}{1 + z^2}. \tag{4}$$

Examples

a. $$\int \frac{1}{1 + \cos x}\,dx = \int \frac{1 + z^2}{2}\,\frac{2\,dz}{1 + z^2}$$

$$= \int dz = z + C$$

$$= \tan\left(\frac{x}{2}\right) + C$$

b. $$\int \frac{1}{2 + \sin x}\,dx = \int \frac{1 + z^2}{2 + 2z + 2z^2}\,\frac{2\,dz}{1 + z^2}$$

$$= \int \frac{dz}{z^2 + z + 1} = \int \frac{dz}{(z + (1/2))^2 + 3/4}$$

$$= \int \frac{du}{u^2 + a^2}$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z + 1}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1 + 2\tan(x/2)}{\sqrt{3}} + C$$

Use the substitutions in Equations (1)–(4) to evaluate the integrals in Exercises 43–50. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

43. $\displaystyle\int \frac{dx}{1 - \sin x}$

44. $\displaystyle\int \frac{dx}{1 + \sin x + \cos x}$

45. $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sin x}$

46. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{1 - \cos x}$

47. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{2 + \cos\theta}$

48. $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta + \sin\theta}$

49. $\displaystyle\int \frac{dt}{\sin t - \cos t}$

50. $\displaystyle\int \frac{\cos t\,dt}{1 - \cos t}$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 51 and 52.

51. $\displaystyle\int \sec\theta\,d\theta$

52. $\displaystyle\int \csc\theta\,d\theta$