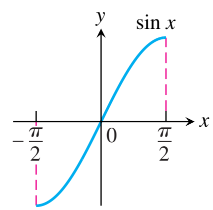
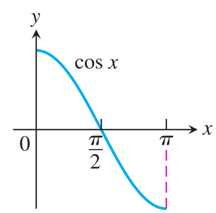
7.7 Inverse Trigonometric Functions

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations. This section shows how these functions are defined, graphed, and evaluated, how their derivatives are computed, and why they appear as important antiderivatives.

Defining the Inverses

The six basic trigonometric functions are not one-to-one (their values repeat periodically). However we can restrict their domains to intervals on which they are one-to-one. The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse $\sin^{-1} x$ (Figure 7.16). Similar domain restrictions can be applied to all six trigonometric functions.

Domain restrictions that make the trigonometric functions one-to-one

Function	Domain	Range	
$\sin x$	$[-\pi/2, \pi/2]$	$[-1, 1]$	
$\cos x$	$[0, \pi]$	$[-1, 1]$	

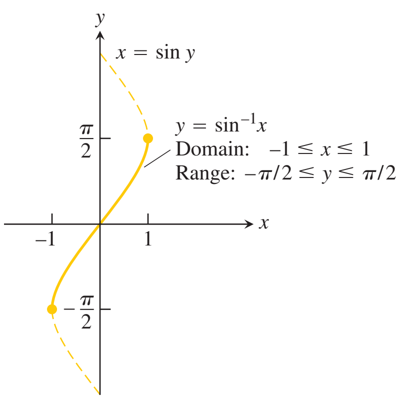
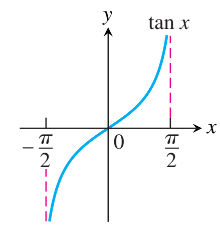
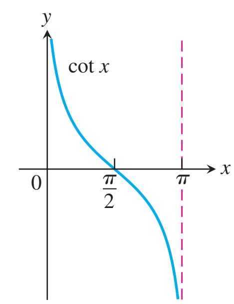
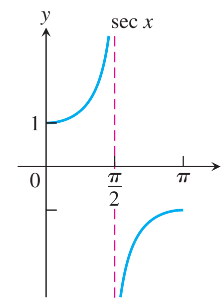
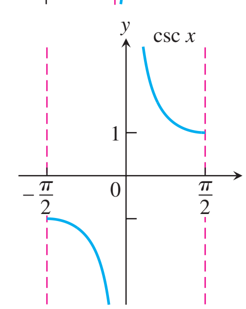


FIGURE 7.16 The graph of $y = \sin^{-1} x$.

$\tan x$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$	
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$	
$\sec x$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$	
$\csc x$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$	

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x$$
$$y = \cos^{-1} x \quad \text{or} \quad y = \arccos x$$
$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x$$
$$y = \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x$$
$$y = \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x$$
$$y = \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x$$

These equations are read "y equals the arcsine of x" or "y equals arcsin x" and so on.

CAUTION The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The graphs of the six inverse trigonometric functions are shown in Figure 7.17. We can obtain these graphs by reflecting the graphs of the restricted trigonometric functions through the line $y = x$, as in Section 7.1. We now take a closer look at these functions and their derivatives.

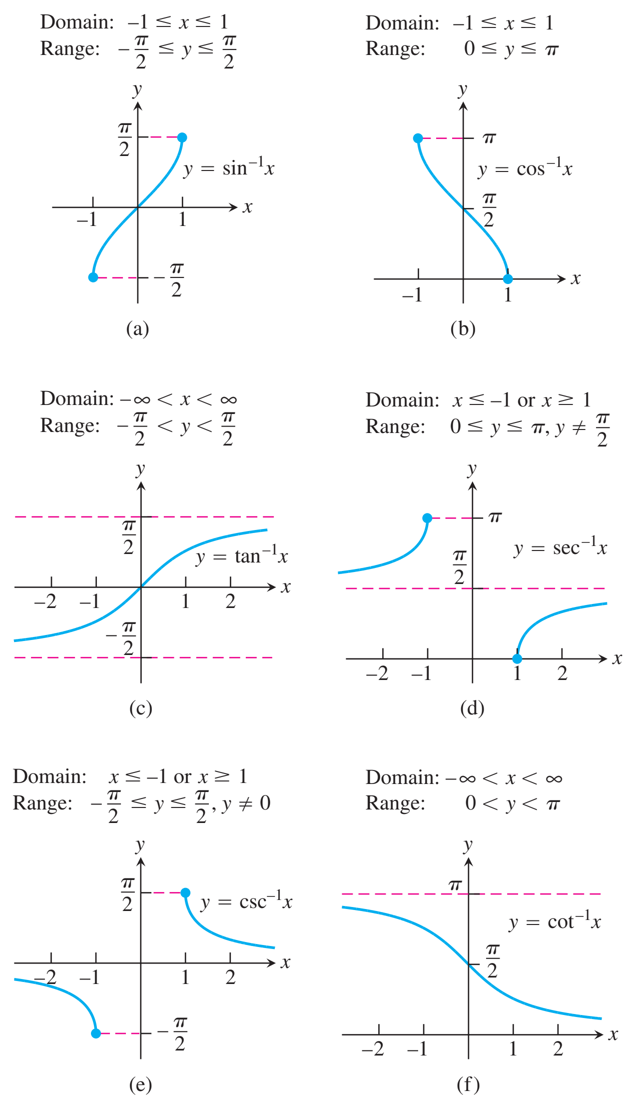


FIGURE 7.17 Graphs of the six basic inverse trigonometric functions.

The Arcsine and Arccosine Functions

The arcsine of x is the angle in $[-\pi/2, \pi/2]$ whose sine is x. The arccosine is an angle in $[0, \pi]$ whose cosine is x.

DEFINITION Arcsine and Arccosine Functions

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

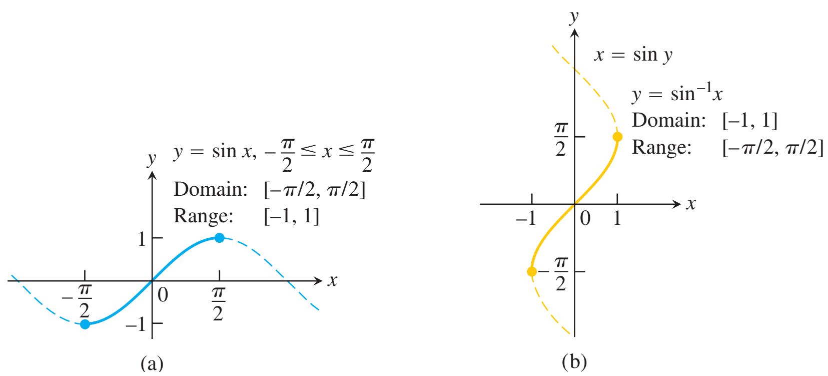
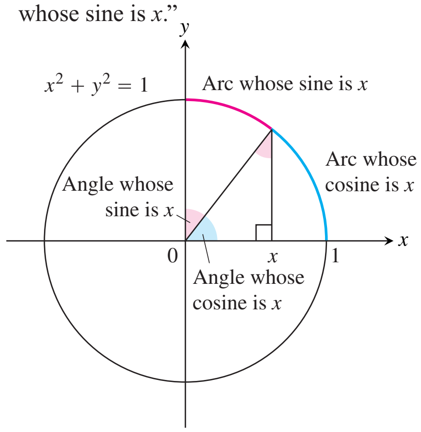


FIGURE 7.18 The graphs of (a) $y = \sin x$, $-\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

The "Arc" in Arc Sine and Arc Cosine

The accompanying figure gives a geometric interpretation of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for radian angles in the first quadrant. For a unit circle, the equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



The graph of $y = \sin^{-1} x$ (Figure 7.18) is symmetric about the origin (it lies along the graph of $x = \sin y$). The arcsine is therefore an odd function:

$$\sin^{-1}(-x) = -\sin^{-1} x. \tag{1}$$

The graph of $y = \cos^{-1} x$ (Figure 7.19) has no such symmetry.

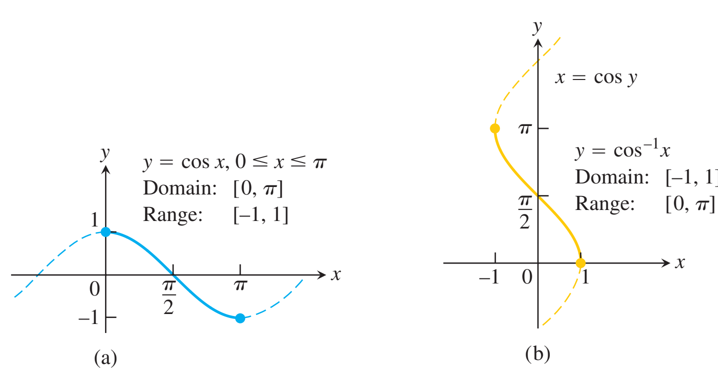
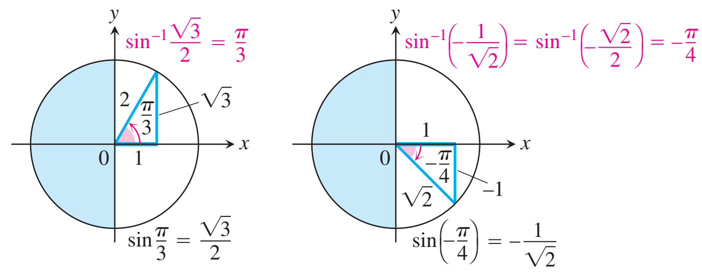


FIGURE 7.19 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Known values of $\sin x$ and $\cos x$ can be inverted to find values of $\sin^{-1} x$ and $\cos^{-1} x$.

x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

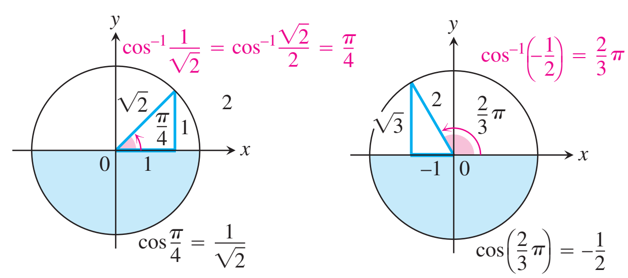
EXAMPLE 1 Common Values of $\sin^{-1} x$



The angles come from the first and fourth quadrants because the range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$. ■

x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$

EXAMPLE 2 Common Values of $\cos^{-1} x$



The angles come from the first and second quadrants because the range of $\cos^{-1} x$ is $[0, \pi]$. ■

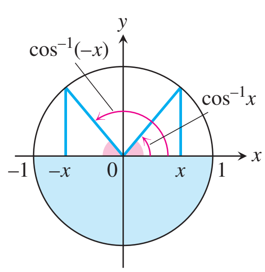


FIGURE 7.20 $\cos^{-1} x$ and $\cos^{-1}(-x)$ are supplementary angles (so their sum is π).

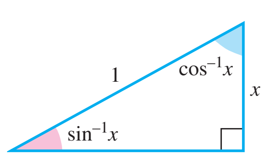


FIGURE 7.21 $\sin^{-1} x$ and $\cos^{-1} x$ are complementary angles (so their sum is $\pi/2$).

Identities Involving Arcsine and Arccosine

As we can see from Figure 7.20, the arccosine of x satisfies the identity

$$\cos^{-1} x + \cos^{-1}(-x) = \pi, \tag{2}$$

or

$$\cos^{-1}(-x) = \pi - \cos^{-1} x. \tag{3}$$

Also, we can see from the triangle in Figure 7.21 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2. \tag{4}$$

Equation (4) holds for the other values of x in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Figure 7.21. It is, however, a consequence of Equations (1) and (3) (Exercise 131).

Inverses of tan x, cot x, sec x, and csc x

The arctangent of x is an angle whose tangent is x. The arccotangent of x is an angle whose cotangent is x.

DEFINITION Arctangent and Arccotangent Functions

$\boldsymbol{y = \tan^{-1} x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$\boldsymbol{y = \cot^{-1} x}$ is the number in $(0, \pi)$ for which $\cot y = x$.

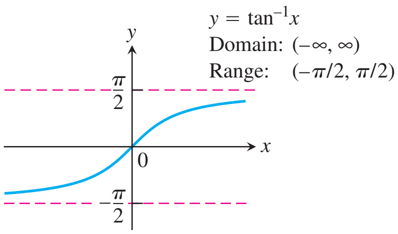


FIGURE 7.22 The graph of $y = \tan^{-1} x$.

We use open intervals to avoid values where the tangent and cotangent are undefined.

The graph of $y = \tan^{-1} x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Figure 7.22). Algebraically this means that

$$\tan^{-1}(-x) = -\tan^{-1} x;$$

the arctangent is an odd function. The graph of $y = \cot^{-1} x$ has no such symmetry (Figure 7.23).

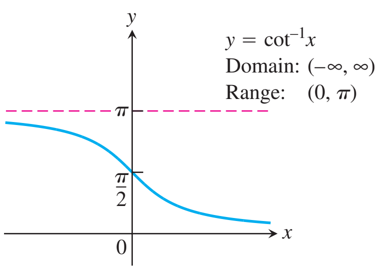


FIGURE 7.23 The graph of $y = \cot^{-1} x$.

The inverses of the restricted forms of $\sec x$ and $\csc x$ are chosen to be the functions graphed in Figures 7.24 and 7.25.

CAUTION There is no general agreement about how to define $\sec^{-1} x$ for negative values of x. We chose angles in the second quadrant between $\pi/2$ and π. This choice makes $\sec^{-1} x = \cos^{-1}(1/x)$. It also makes $\sec^{-1} x$ an increasing function on each interval of its domain. Some tables choose $\sec^{-1} x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Figure 7.26). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1} x = \cos^{-1}(1/x)$. From this, we can derive the identity

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right) \tag{5}$$

by applying Equation (4).

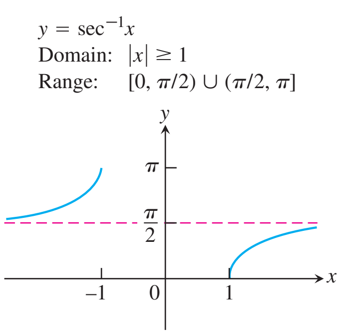


FIGURE 7.24 The graph of $y = \sec^{-1} x$.

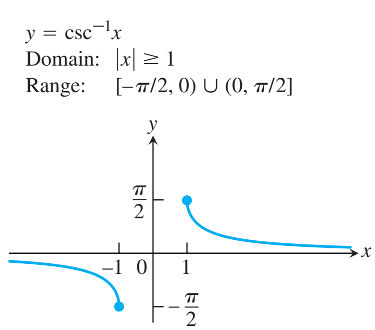


FIGURE 7.25 The graph of $y = \csc^{-1} x$.

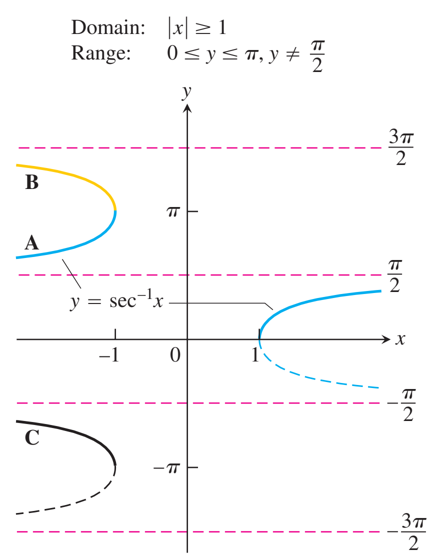
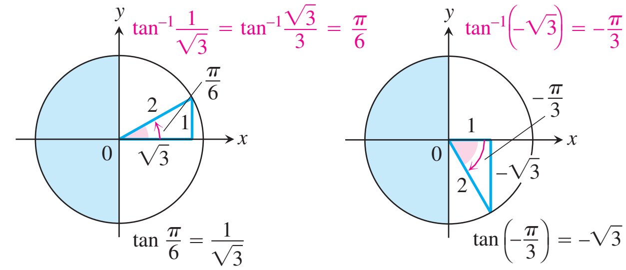


FIGURE 7.26 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, $\sec^{-1} x = \cos^{-1}(1/x)$, a useful identity employed by many calculators.

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

EXAMPLE 3 Common Values of $\tan^{-1} x$



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$. ■

EXAMPLE 4 Find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$ if

$$\alpha = \sin^{-1}\frac{2}{3}.$$

Solution This equation says that $\sin\alpha = 2/3$. We picture α as an angle in a right triangle with opposite side 2 and hypotenuse 3 (Figure 7.27). The length of the remaining side is

$$\sqrt{(3)^2 - (2)^2} = \sqrt{9-4} = \sqrt{5}. \qquad \text{Pythagorean theorem}$$

We add this information to the figure and then read the values we want from the completed triangle:

$$\cos\alpha = \frac{\sqrt{5}}{3}, \quad \tan\alpha = \frac{2}{\sqrt{5}}, \quad \sec\alpha = \frac{3}{\sqrt{5}}, \quad \csc\alpha = \frac{3}{2}, \quad \cot\alpha = \frac{\sqrt{5}}{2}.$$ ■

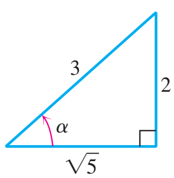


FIGURE 7.27 If $\alpha = \sin^{-1}(2/3)$, then the values of the other basic trigonometric functions of α can be read from this triangle (Example 4).

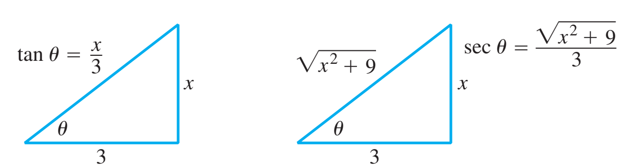
EXAMPLE 5 Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite/adjacent} = x/3.$$

The length of the triangle's hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$



Thus,

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta$$

$$= \frac{\sqrt{x^2+9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$ ■

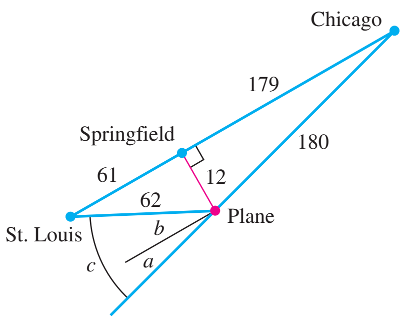


FIGURE 7.28 Diagram for drift correction (Example 6), with distances rounded to the nearest mile (drawing not to scale).

EXAMPLE 6 Drift Correction

During an airplane flight from Chicago to St. Louis the navigator determines that the plane is 12 mi off course, as shown in Figure 7.28. Find the angle a for a course parallel to the original, correct course, the angle b, and the correction angle $c = a + b$.

Solution

$$a = \sin^{-1}\frac{12}{180} \approx 0.067 \text{ radian} \approx 3.8°$$

$$b = \sin^{-1}\frac{12}{62} \approx 0.195 \text{ radian} \approx 11.2°$$

$$c = a + b \approx 15°.$$

The Derivative of $y = \sin^{-1} u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 1 in Section 7.1 therefore assures us that the inverse function $y = \sin^{-1} x$ is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points (see Figure 7.29).

We find the derivative of $y = \sin^{-1} x$ by applying Theorem 1 with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$.

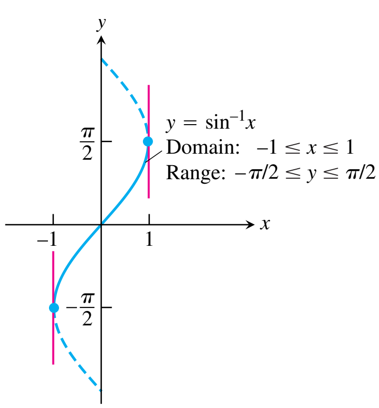


FIGURE 7.29 The graph of $y = \sin^{-1} x$.

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u} \\
&= \frac{1}{\sqrt{1 - x^2}} && \sin(\sin^{-1} x) = x
\end{aligned}$$

Alternate Derivation: Instead of applying Theorem 1 directly, we can find the derivative of $y = \sin^{-1} x$ using implicit differentiation as follows:

$$\begin{aligned}
\sin y &= x && y = \sin^{-1} x \Leftrightarrow \sin y = x \\
\frac{d}{dx}(\sin y) &= 1 && \text{Derivative of both sides with respect to } x \\
\cos y \frac{dy}{dx} &= 1 && \text{Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{\cos y} && \text{We can divide because } \cos y > 0 \text{ for } -\pi/2 < y < \pi/2. \\
&= \frac{1}{\sqrt{1 - x^2}} && \cos y = \sqrt{1 - \sin^2 y}
\end{aligned}$$

No matter which derivation we use, we have that the derivative of $y = \sin^{-1} x$ with respect to x is

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

EXAMPLE 7 Applying the Derivative Formula

You Try It

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1 - x^4}}$$ ■

The Derivative of $y = \tan^{-1} u$

We find the derivative of $y = \tan^{-1} x$ by applying Theorem 1 with $f(x) = \tan x$ and $f^{-1}(x) = \tan^{-1} x$. Theorem 1 can be applied because the derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$.

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u \\
&= \frac{1}{1 + \tan^2(\tan^{-1} x)} && \sec^2 u = 1 + \tan^2 u \\
&= \frac{1}{1 + x^2} && \tan(\tan^{-1} x) = x
\end{aligned}$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2}\frac{du}{dx}.$$

EXAMPLE 8 A Moving Particle

A particle moves along the x-axis so that its position at any time $t \geq 0$ is $x(t) = \tan^{-1}\sqrt{t}$. What is the velocity of the particle when $t = 16$?

Solution

$$v(t) = \frac{d}{dt}\tan^{-1}\sqrt{t} = \frac{1}{1 + (\sqrt{t})^2} \cdot \frac{d}{dt}\sqrt{t} = \frac{1}{1 + t} \cdot \frac{1}{2\sqrt{t}}$$

When $t = 16$, the velocity is

$$v(16) = \frac{1}{1 + 16} \cdot \frac{1}{2\sqrt{16}} = \frac{1}{136}.$$

The Derivative of $y = \sec^{-1} u$

Since the derivative of sec x is positive for $0 < x < \pi/2$ and $\pi/2 < x < \pi$, Theorem 1 says that the inverse function $y = \sec^{-1} x$ is differentiable. Instead of applying the formula in Theorem 1 directly, we find the derivative of $y = \sec^{-1} x$, $|x| > 1$, using implicit differentiation and the Chain Rule as follows:

$$y = \sec^{-1} x$$

$$\sec y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\sec y) = \frac{d}{dx} x \qquad \text{Differentiate both sides.}$$

$$\sec y \tan y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \qquad \text{Since } |x| > 1, y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \neq 0.$$

To express the result in terms of x, we use the relationships

$$\sec y = x \qquad \text{and} \qquad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}}.$$

Can we do anything about the $\pm$ sign? A glance at Figure 7.30 shows that the slope of the graph $y = \sec^{-1} x$ is always positive. Thus,

$$\frac{d}{dx}\sec^{-1} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

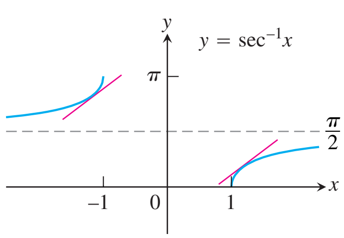


FIGURE 7.30 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

With the absolute value symbol, we can write a single expression that eliminates the "$\pm$" ambiguity:

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

EXAMPLE 9 Using the Formula

$$\frac{d}{dx}\sec^{-1}(5x^4) = \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}}\frac{d}{dx}(5x^4)$$

$$= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) \qquad 5x^4 > 0$$

$$= \frac{4}{x\sqrt{25x^8 - 1}}$$

■

Derivatives of the Other Three

We could use the same techniques to find the derivatives of the other three inverse trigonometric functions—arccosine, arccotangent, and arccosecant—but there is a much easier way, thanks to the following identities.

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1}x = \pi/2 - \sin^{-1}x$$
$$\cot^{-1}x = \pi/2 - \tan^{-1}x$$
$$\csc^{-1}x = \pi/2 - \sec^{-1}x$$

We saw the first of these identities in Equation (4). The others are derived in a similar way. It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions. For example, the derivative of $\cos^{-1}x$ is calculated as follows:

$$\frac{d}{dx}(\cos^{-1}x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1}x\right) \qquad \text{Identity}$$

$$= -\frac{d}{dx}(\sin^{-1}x)$$

$$= -\frac{1}{\sqrt{1 - x^2}} \qquad \text{Derivative of arcsine}$$

EXAMPLE 10 A Tangent Line to the Arccotangent Curve

Find an equation for the line tangent to the graph of $y = \cot^{-1}x$ at $x = -1$.

Solution First we note that

$$\cot^{-1}(-1) = \pi/2 - \tan^{-1}(-1) = \pi/2 - (-\pi/4) = 3\pi/4.$$

The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=-1} = -\left.\frac{1}{1 + x^2}\right|_{x=-1} = -\frac{1}{1 + (-1)^2} = -\frac{1}{2},$$

so the tangent line has equation $y - 3\pi/4 = (-1/2)(x + 1)$. ■

The derivatives of the inverse trigonometric functions are summarized in Table 7.3.

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1+u^2}$

4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1+u^2}$

5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

Integration Formulas

The derivative formulas in Table 7.3 yield three useful integration formulas in Table 7.4. The formulas are readily verified by differentiating the functions on the right-hand sides.

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

The derivative formulas in Table 7.3 have $a = 1$, but in most integrations $a \neq 1$, and the formulas in Table 7.4 are more useful.

EXAMPLE 11 Using the Integral Formulas

(a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big]_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(b) $\displaystyle\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

(c) $\displaystyle\int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x\Big]_{2/\sqrt{3}}^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$ ■

EXAMPLE 12 Using Substitution and Table 7.4

You Try It

(a) $\displaystyle\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{(3)^2 - x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$ Table 7.4 Formula 1, with $a = 3$, $u = x$

You Try It

(b) $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}}$ $a = \sqrt{3}$, $u = 2x$, and $du/2 = dx$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula 1}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$ ■

EXAMPLE 13 Completing the Square

Evaluate

$$\int \frac{dx}{\sqrt{4x - x^2}}.$$

Solution The expression $\sqrt{4x - x^2}$ does not match any of the formulas in Table 7.4, so we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x-2)^2.$$

Then we substitute $a = 2$, $u = x - 2$, and $du = dx$ to get

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x-2)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}} \qquad a = 2,\ u = x - 2, \text{ and } du = dx$$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 7.4, Formula 1}$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$ ■

EXAMPLE 14 Completing the Square

You Try It

Evaluate

$$\int \frac{dx}{4x^2 + 4x + 2}.$$

Solution We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.$$

Then,

$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x + 1)^2 + 1} = \frac{1}{2}\int \frac{du}{u^2 + a^2} \qquad a = 1, u = 2x + 1, \text{ and } du/2 = dx$$

$$= \frac{1}{2} \cdot \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 7.4, Formula 2}$$

$$= \frac{1}{2} \tan^{-1}(2x + 1) + C \qquad a = 1, u = 2x + 1$$

■

EXAMPLE 15 Using Substitution

You Try It

Evaluate

$$\int \frac{dx}{\sqrt{e^{2x} - 6}}.$$

Solution

$$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{du/u}{\sqrt{u^2 - a^2}} \qquad u = e^x, du = e^x\,dx, dx = du/e^x = du/u, a = \sqrt{6}$$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C \qquad \text{Table 7.4, Formula 3}$$

$$= \frac{1}{\sqrt{6}} \sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

■

EXERCISES 7.7

Common Values of Inverse Trignonometric Functions

Use reference triangles like those in Examples 1–3 to find the angles in Exercises 1–12.

1. **a.** $\tan^{-1} 1$ **b.** $\tan^{-1}(-\sqrt{3})$ **c.** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

2. **a.** $\tan^{-1}(-1)$ **b.** $\tan^{-1}\sqrt{3}$ **c.** $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

3. **a.** $\sin^{-1}\left(\frac{-1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

4. **a.** $\sin^{-1}\left(\frac{1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

5. **a.** $\cos^{-1}\left(\frac{1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

6. **a.** $\cos^{-1}\left(\frac{-1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

7. **a.** $\sec^{-1}(-\sqrt{2})$ **b.** $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **c.** $\sec^{-1}(-2)$

8. **a.** $\sec^{-1}\sqrt{2}$ **b.** $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ **c.** $\sec^{-1}2$

9. **a.** $\csc^{-1}\sqrt{2}$ **b.** $\csc^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ **c.** $\csc^{-1}2$

10. **a.** $\csc^{-1}(-\sqrt{2})$ **b.** $\csc^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **c.** $\csc^{-1}(-2)$

11. **a.** $\cot^{-1}(-1)$ **b.** $\cot^{-1}(\sqrt{3})$ **c.** $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

12. **a.** $\cot^{-1}(1)$ **b.** $\cot^{-1}(-\sqrt{3})$ **c.** $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Trigonometric Function Values

13. Given that $\alpha = \sin^{-1}(5/13)$, find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

14. Given that $\alpha = \tan^{-1}(4/3)$, find $\sin\alpha$, $\cos\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

15. Given that $\alpha = \sec^{-1}(-\sqrt{5})$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

16. Given that $\alpha = \sec^{-1}(-\sqrt{13}/2)$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

Evaluating Trigonometric and Inverse Trigonometric Terms

Find the values in Exercises 17–28.

17. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ **18.** $\sec\left(\cos^{-1}\frac{1}{2}\right)$

19. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ **20.** $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

21. $\csc(\sec^{-1}2) + \cos(\tan^{-1}(-\sqrt{3}))$

22. $\tan(\sec^{-1}1) + \sin(\csc^{-1}(-2))$

23. $\sin\left(\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)\right)$

24. $\cot\left(\sin^{-1}\left(-\frac{1}{2}\right) - \sec^{-1}2\right)$

25. $\sec(\tan^{-1}1 + \csc^{-1}1)$ **26.** $\sec(\cot^{-1}\sqrt{3} + \csc^{-1}(-1))$

27. $\sec^{-1}\left(\sec\left(-\frac{\pi}{6}\right)\right)$ (The answer is *not* $-\pi/6$.)

28. $\cot^{-1}\left(\cot\left(-\frac{\pi}{4}\right)\right)$ (The answer is *not* $-\pi/4$.)

Finding Trigonometric Expressions

Evaluate the expressions in Exercises 29–40.

29. $\sec\left(\tan^{-1}\frac{x}{2}\right)$ **30.** $\sec(\tan^{-1}2x)$

31. $\tan(\sec^{-1}3y)$ **32.** $\tan\left(\sec^{-1}\frac{y}{5}\right)$

33. $\cos(\sin^{-1}x)$ **34.** $\tan(\cos^{-1}x)$

35. $\sin(\tan^{-1}\sqrt{x^2 - 2x})$, $x \geq 2$

36. $\sin\left(\tan^{-1}\frac{x}{\sqrt{x^2+1}}\right)$ **37.** $\cos\left(\sin^{-1}\frac{2y}{3}\right)$

38. $\cos\left(\sin^{-1}\frac{y}{5}\right)$ **39.** $\sin\left(\sec^{-1}\frac{x}{4}\right)$

40. $\sin\sec^{-1}\left(\frac{\sqrt{x^2+4}}{x}\right)$

Limits

Find the limits in Exercises 41–48. (If in doubt, look at the function's graph.)

41. $\lim_{x\to 1^-}\sin^{-1}x$ **42.** $\lim_{x\to -1^+}\cos^{-1}x$

43. $\lim_{x\to\infty}\tan^{-1}x$ **44.** $\lim_{x\to-\infty}\tan^{-1}x$

45. $\lim_{x\to\infty}\sec^{-1}x$ **46.** $\lim_{x\to-\infty}\sec^{-1}x$

47. $\lim_{x\to\infty}\csc^{-1}x$ **48.** $\lim_{x\to-\infty}\csc^{-1}x$

Finding Derivatives

In Exercises 49–70, find the derivative of y with respect to the appropriate variable.

49. $y = \cos^{-1}(x^2)$ **50.** $y = \cos^{-1}(1/x)$

51. $y = \sin^{-1}\sqrt{2}\,t$ **52.** $y = \sin^{-1}(1 - t)$

53. $y = \sec^{-1}(2s + 1)$ **54.** $y = \sec^{-1}5s$

55. $y = \csc^{-1}(x^2 + 1)$, $x > 0$

56. $y = \csc^{-1}\frac{x}{2}$

57. $y = \sec^{-1}\frac{1}{t}$, $0 < t < 1$ **58.** $y = \sin^{-1}\frac{3}{t^2}$

59. $y = \cot^{-1}\sqrt{t}$ **60.** $y = \cot^{-1}\sqrt{t - 1}$

61. $y = \ln(\tan^{-1}x)$ **62.** $y = \tan^{-1}(\ln x)$

63. $y = \csc^{-1}(e^t)$ **64.** $y = \cos^{-1}(e^{-t})$

65. $y = s\sqrt{1 - s^2} + \cos^{-1}s$ **66.** $y = \sqrt{s^2 - 1} - \sec^{-1}s$

67. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1}x$, $x > 1$

68. $y = \cot^{-1}\frac{1}{x} - \tan^{-1}x$ **69.** $y = x\sin^{-1}x + \sqrt{1 - x^2}$

70. $y = \ln(x^2 + 4) - x\tan^{-1}\left(\frac{x}{2}\right)$

Evaluating Integrals

Evaluate the integrals in Exercises 71–94.

71. $\int \frac{dx}{\sqrt{9 - x^2}}$ **72.** $\int \frac{dx}{\sqrt{1 - 4x^2}}$

73. $\displaystyle\int \frac{dx}{17 + x^2}$

74. $\displaystyle\int \frac{dx}{9 + 3x^2}$

75. $\displaystyle\int \frac{dx}{x\sqrt{25x^2 - 2}}$

76. $\displaystyle\int \frac{dx}{x\sqrt{5x^2 - 4}}$

77. $\displaystyle\int_0^1 \frac{4\,ds}{\sqrt{4 - s^2}}$

78. $\displaystyle\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9 - 4s^2}}$

79. $\displaystyle\int_0^2 \frac{dt}{8 + 2t^2}$

80. $\displaystyle\int_{-2}^2 \frac{dt}{4 + 3t^2}$

81. $\displaystyle\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$

82. $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2 - 1}}$

83. $\displaystyle\int \frac{3\,dr}{\sqrt{1 - 4(r - 1)^2}}$

84. $\displaystyle\int \frac{6\,dr}{\sqrt{4 - (r + 1)^2}}$

85. $\displaystyle\int \frac{dx}{2 + (x - 1)^2}$

86. $\displaystyle\int \frac{dx}{1 + (3x + 1)^2}$

87. $\displaystyle\int \frac{dx}{(2x - 1)\sqrt{(2x - 1)^2 - 4}}$

88. $\displaystyle\int \frac{dx}{(x + 3)\sqrt{(x + 3)^2 - 25}}$

89. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{1 + (\sin\theta)^2}$

90. $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\,dx}{1 + (\cot x)^2}$

91. $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{1 + e^{2x}}$

92. $\displaystyle\int_1^{e^{\pi/4}} \frac{4\,dt}{t(1 + \ln^2 t)}$

93. $\displaystyle\int \frac{y\,dy}{\sqrt{1 - y^4}}$

94. $\displaystyle\int \frac{\sec^2 y\,dy}{\sqrt{1 - \tan^2 y}}$

Evaluate the integrals in Exercises 95–104.

95. $\displaystyle\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$

96. $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$

97. $\displaystyle\int_{-1}^0 \frac{6\,dt}{\sqrt{3 - 2t - t^2}}$

98. $\displaystyle\int_{1/2}^1 \frac{6\,dt}{\sqrt{3 + 4t - 4t^2}}$

99. $\displaystyle\int \frac{dy}{y^2 - 2y + 5}$

100. $\displaystyle\int \frac{dy}{y^2 + 6y + 10}$

101. $\displaystyle\int_1^2 \frac{8\,dx}{x^2 - 2x + 2}$

102. $\displaystyle\int_2^4 \frac{2\,dx}{x^2 - 6x + 10}$

103. $\displaystyle\int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}$

104. $\displaystyle\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$

Evaluate the integrals in Exercises 105–112.

105. $\displaystyle\int \frac{e^{\sin^{-1}x}\,dx}{\sqrt{1 - x^2}}$

106. $\displaystyle\int \frac{e^{\cos^{-1}x}\,dx}{\sqrt{1 - x^2}}$

107. $\displaystyle\int \frac{(\sin^{-1}x)^2\,dx}{\sqrt{1 - x^2}}$

108. $\displaystyle\int \frac{\sqrt{\tan^{-1}x}\,dx}{1 + x^2}$

109. $\displaystyle\int \frac{dy}{(\tan^{-1}y)(1 + y^2)}$

110. $\displaystyle\int \frac{dy}{(\sin^{-1}y)\sqrt{1 - y^2}}$

111. $\displaystyle\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1}x)\,dx}{x\sqrt{x^2 - 1}}$

112. $\displaystyle\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1}x)\,dx}{x\sqrt{x^2 - 1}}$

Limits

Find the limits in Exercises 113–116.

113. $\displaystyle\lim_{x\to 0} \frac{\sin^{-1}5x}{x}$

114. $\displaystyle\lim_{x\to 1^+} \frac{\sqrt{x^2 - 1}}{\sec^{-1}x}$

115. $\displaystyle\lim_{x\to\infty} x\tan^{-1}\frac{2}{x}$

116. $\displaystyle\lim_{x\to 0} \frac{2\tan^{-1}3x^2}{7x^2}$

Integration Formulas

Verify the integration formulas in Exercises 117–120.

117. $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1}x}{x} + C$

118. $\displaystyle\int x^3\cos^{-1}5x\,dx = \frac{x^4}{4}\cos^{-1}5x + \frac{5}{4}\int \frac{x^4\,dx}{\sqrt{1 - 25x^2}}$

119. $\displaystyle\int (\sin^{-1}x)^2\,dx = x(\sin^{-1}x)^2 - 2x + 2\sqrt{1 - x^2}\sin^{-1}x + C$

120. $\displaystyle\int \ln(a^2 + x^2)\,dx = x\ln(a^2 + x^2) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

Initial Value Problems

Solve the initial value problems in Exercises 121–124.

121. $\displaystyle\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}, \quad y(0) = 0$

122. $\displaystyle\frac{dy}{dx} = \frac{1}{x^2 + 1} - 1, \quad y(0) = 1$

123. $\displaystyle\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}, \quad x > 1; \quad y(2) = \pi$

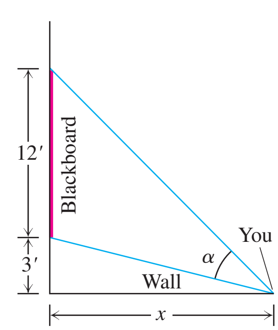
124. $\displaystyle\frac{dy}{dx} = \frac{1}{1 + x^2} - \frac{2}{\sqrt{1 - x^2}}, \quad y(0) = 2$

Applications and Theory

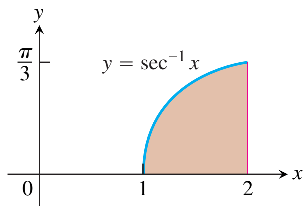
125. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$$

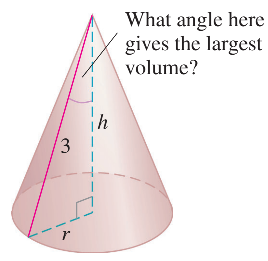
if you are x ft from the front wall.



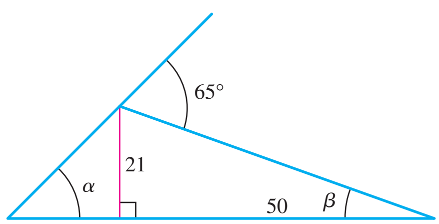
126. The region between the curve $y = \sec^{-1} x$ and the x-axis from $x = 1$ to $x = 2$ (shown here) is revolved about the y-axis to generate a solid. Find the volume of the solid.



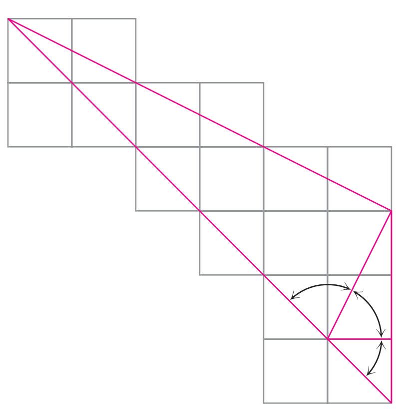
127. The slant height of the cone shown here is 3 m. How large should the indicated angle be to maximize the cone's volume?



128. Find the angle α.

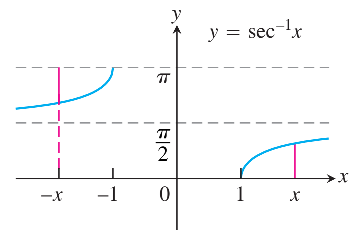


129. Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



130. Two derivations of the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$

a. (*Geometric*) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1} x$. See if you can tell what is going on.



b. (*Algebraic*) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad \text{Eq. (3)}$$

$$\sec^{-1} x = \cos^{-1}(1/x) \qquad \text{Eq. (5)}$$

131. The identity $\sin^{-1} x + \cos^{-1} x = \pi/2$ Figure 7.21 establishes the identity for $0 < x < 1$. To establish it for the rest of $[-1, 1]$, verify by direct calculation that it holds for $x = 1$, 0, and -1. Then, for values of x in $(-1, 0)$, let $x = -a$, $a > 0$, and apply Eqs. (1) and (3) to the sum $\sin^{-1}(-a) + \cos^{-1}(-a)$.

132. Show that the sum $\tan^{-1} x + \tan^{-1}(1/x)$ is constant.

Which of the expressions in Exercises 133–136 are defined, and which are not? Give reasons for your answers.

133. a. $\tan^{-1} 2$ **b.** $\cos^{-1} 2$

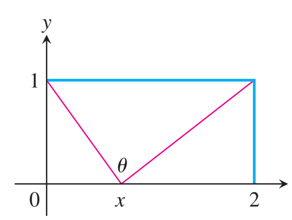
134. a. $\csc^{-1}(1/2)$ **b.** $\csc^{-1} 2$

135. a. $\sec^{-1} 0$ **b.** $\sin^{-1}\sqrt{2}$

136. a. $\cot^{-1}(-1/2)$ **b.** $\cos^{-1}(-5)$

137. (*Continuation of Exercise 125.*) You want to position your chair along the wall to maximize your viewing angle α. How far from the front of the room should you sit?

138. What value of x maximizes the angle θ shown here? How large is θ at that point? Begin by showing that $\theta = \pi - \cot^{-1} x - \cot^{-1}(2 - x)$.



139. Can the integrations in (a) and (b) both be correct? Explain.

a. $$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$$

b. $$\int \frac{dx}{\sqrt{1 - x^2}} = -\int -\frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1} x + C$$

140. Can the integrations in (a) and (b) both be correct? Explain.

a. $$\int \frac{dx}{\sqrt{1 - x^2}} = -\int -\frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1} x + C$$

b. $$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{-du}{\sqrt{1-(-u)^2}} \qquad \begin{aligned} x &= -u, \\ dx &= -du \end{aligned}$$
$$= \int \frac{-du}{\sqrt{1-u^2}}$$
$$= \cos^{-1} u + C$$
$$= \cos^{-1}(-x) + C \qquad u = -x$$

141. Use the identity

$$\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u$$

to derive the formula for the derivative of $\csc^{-1} u$ in Table 7.3 from the formula for the derivative of $\sec^{-1} u$.

142. Derive the formula

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

for the derivative of $y = \tan^{-1} x$ by differentiating both sides of the equivalent equation $\tan y = x$.

143. Use the Derivative Rule in Section 7.1, Theorem 1, to derive

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

144. Use the identity

$$\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u$$

to derive the formula for the derivative of $\cot^{-1} u$ in Table 7.3 from the formula for the derivative of $\tan^{-1} u$.

145. What is special about the functions

$$f(x) = \sin^{-1} \frac{x-1}{x+1}, \quad x \geq 0, \quad \text{and} \quad g(x) = 2\tan^{-1}\sqrt{x}?$$

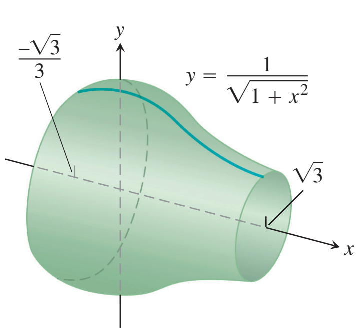
Explain.

146. What is special about the functions

$$f(x) = \sin^{-1} \frac{1}{\sqrt{x^2+1}} \quad \text{and} \quad g(x) = \tan^{-1} \frac{1}{x}?$$

Explain.

147. Find the volume of the solid of revolution shown here.



148. Arc length Find the length of the curve $y = \sqrt{1-x^2}$, $-1/2 \leq x \leq 1/2$.

Volumes by Slicing

Find the volumes of the solids in Exercises 149 and 150.

149. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are

a. circles whose diameters stretch from the curve $y = -1/\sqrt{1+x^2}$ to the curve $y = 1/\sqrt{1+x^2}$.

b. vertical squares whose base edges run from the curve $y = -1/\sqrt{1+x^2}$ to the curve $y = 1/\sqrt{1+x^2}$.

150. The solid lies between planes perpendicular to the x-axis at $x = -\sqrt{2}/2$ and $x = \sqrt{2}/2$. The cross-sections perpendicular to the x-axis are

a. circles whose diameters stretch from the x-axis to the curve $y = 2/\sqrt[4]{1-x^2}$.

b. squares whose diagonals stretch from the x-axis to the curve $y = 2/\sqrt[4]{1-x^2}$.

T Calculator and Grapher Explorations

151. Find the values of

a. $\sec^{-1} 1.5$ **b.** $\csc^{-1}(-1.5)$ **c.** $\cot^{-1} 2$

152. Find the values of

a. $\sec^{-1}(-3)$ **b.** $\csc^{-1} 1.7$ **c.** $\cot^{-1}(-2)$

In Exercises 153–155, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

153. a. $y = \tan^{-1}(\tan x)$ **b.** $y = \tan(\tan^{-1} x)$

154. a. $y = \sin^{-1}(\sin x)$ **b.** $y = \sin(\sin^{-1} x)$

155. a. $y = \cos^{-1}(\cos x)$ **b.** $y = \cos(\cos^{-1} x)$

156. Graph $y = \sec(\sec^{-1} x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

157. Newton's serpentine Graph Newton's serpentine, $y = 4x/(x^2+1)$. Then graph $y = 2\sin(2\tan^{-1} x)$ in the same graphing window. What do you see? Explain.

158. Graph the rational function $y = (2-x^2)/x^2$. Then graph $y = \cos(2\sec^{-1} x)$ in the same graphing window. What do you see? Explain.

159. Graph $f(x) = \sin^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

160. Graph $f(x) = \tan^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

7.8 Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and they are important in applications. For instance, they are used in problems such as computing the tension in a cable suspended by its two ends, as in an electric transmission line. They also play an important role in finding solutions to differential equations. In this section, we give a brief introduction to hyperbolic functions, their graphs, how their derivatives are calculated, and why they appear as important antiderivatives.

Even and Odd Parts of the Exponential Function

Recall the definitions of even and odd functions from Section 1.4, and the symmetries of their graphs. An even function f satisfies $f(-x) = f(x)$, while an odd function satisfies $f(-x) = -f(x)$. Every function f that is defined on an interval centered at the origin can be written in a unique way as the sum of one even function and one odd function. The decomposition is

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd part}}.$$

If we write e^x this way, we get

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even part}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd part}}.$$

The even and odd parts of e^x, called the hyperbolic cosine and hyperbolic sine of x, respectively, are useful in their own right. They describe the motions of waves in elastic solids and the temperature distributions in metal cooling fins. The centerline of the Gateway Arch to the West in St. Louis is a weighted hyperbolic cosine curve.

Definitions and Identities

The hyperbolic cosine and hyperbolic sine functions are defined by the first two equations in Table 7.5. The table also lists the definitions of the hyperbolic tangent, cotangent, secant, and cosecant. As we will see, the hyperbolic functions bear a number of similarities to the trigonometric functions after which they are named. (See Exercise 84 as well.)

The notation $\cosh x$ is often read "kosh x," rhyming with "gosh x," and $\sinh x$ is pronounced as if spelled "cinch x," rhyming with "pinch x."

Hyperbolic functions satisfy the identities in Table 7.6. Except for differences in sign, these resemble identities we already know for trigonometric functions.

The second equation is obtained as follows:

$$\begin{aligned} 2\sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{e^{2x} - e^{-2x}}{2} \\ &= \sinh 2x. \end{aligned}$$

TABLE 7.5 The six basic hyperbolic functions

Hyperbolic sine of x:	$\sinh x = \dfrac{e^x - e^{-x}}{2}$
Hyperbolic cosine of x:	$\cosh x = \dfrac{e^x + e^{-x}}{2}$
Hyperbolic tangent:	$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
Hyperbolic cotangent:	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
Hyperbolic secant:	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
Hyperbolic cosecant:	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

FIGURE 7.31

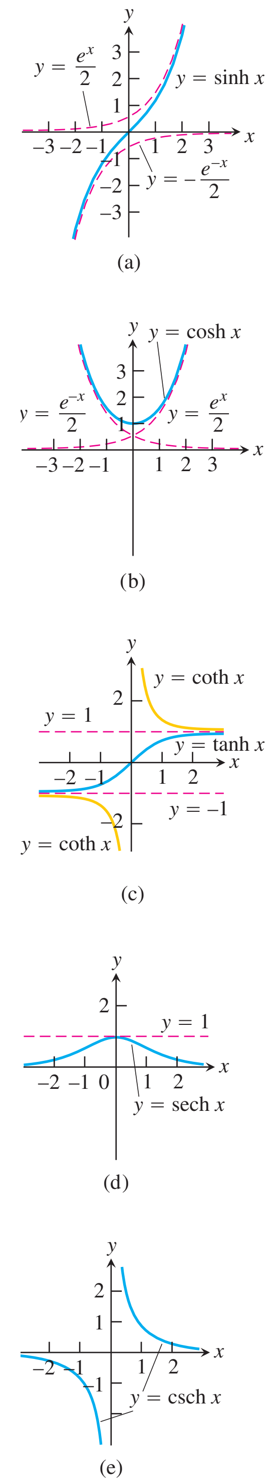


TABLE 7.6 Identities for hyperbolic functions

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

$\coth^2 x = 1 + \operatorname{csch}^2 x$

The other identities are obtained similarly, by substituting in the definitions of the hyperbolic functions and using algebra. Like many standard functions, hyperbolic functions and their inverses are easily evaluated with calculators, which have special keys or keystroke sequences for that purpose.

Derivatives and Integrals

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined (Table 7.7). Again, there are similarities with trigonometric functions. The derivative formulas in Table 7.7 lead to the integral formulas in Table 7.8.

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

The derivative formulas are derived from the derivative of e^u:

$$\begin{aligned}
\frac{d}{dx}(\sinh u) &= \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) && \text{Definition of } \sinh u \\
&= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2} && \text{Derivative of } e^u \\
&= \cosh u \frac{du}{dx} && \text{Definition of } \cosh u
\end{aligned}$$

This gives the first derivative formula. The calculation

$$\begin{aligned}
\frac{d}{dx}(\operatorname{csch} u) &= \frac{d}{dx}\left(\frac{1}{\sinh u}\right) && \text{Definition of } \operatorname{csch} u \\
&= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} && \text{Quotient Rule} \\
&= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} && \text{Rearrange terms.} \\
&= -\operatorname{csch} u \coth u \frac{du}{dx} && \text{Definitions of } \operatorname{csch} u \text{ and } \coth u
\end{aligned}$$

gives the last formula. The others are obtained similarly.

EXAMPLE 1 Finding Derivatives and Integrals

(a)
$$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1 = t^2}\cdot\frac{d}{dt}\left(\sqrt{1+t^2}\right)$$
$$= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

(b)
$$\int \coth 5x\,dx = \int \frac{\cosh 5x}{\sinh 5x}\,dx = \frac{1}{5}\int\frac{du}{u} \qquad \begin{matrix} u = \sinh 5x, \\ du = 5\cosh 5x\,dx \end{matrix}$$
$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

(c)
$$\int_0^1 \sinh^2 x\,dx = \int_0^1 \frac{\cosh 2x - 1}{2}\,dx \qquad \text{Table 7.6}$$
$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1)\,dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$
$$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672 \qquad \text{Evaluate with a calculator}$$

(d)
$$\int_0^{\ln 2} 4e^x \sinh x\,dx = \int_0^{\ln 2} 4e^x\,\frac{e^x - e^{-x}}{2}\,dx = \int_0^{\ln 2}(2e^{2x} - 2)\,dx$$
$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0)$$
$$= 4 - 2\ln 2 - 1$$
$$\approx 1.6137$$

Inverse Hyperbolic Functions

The inverses of the six basic hyperbolic functions are very useful in integration. Since $d(\sinh x)/dx = \cosh x > 0$, the hyperbolic sine is an increasing function of x. We denote its inverse by

$$y = \sinh^{-1} x.$$

For every value of x in the interval $-\infty < x < \infty$, the value of $y = \sinh^{-1} x$ is the number whose hyperbolic sine is x. The graphs of $y = \sinh x$ and $y = \sinh^{-1} x$ are shown in Figure 7.32a.

The function $y = \cosh x$ is not one-to-one, as we can see from the graph in Figure 7.31b. The restricted function $y = \cosh x, x \geq 0$, however, is one-to-one and therefore has an inverse, denoted by

$$y = \cosh^{-1} x.$$

For every value of $x \geq 1, y = \cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x. The graphs of $y = \cosh x, x \geq 0$, and $y = \cosh^{-1} x$ are shown in Figure 7.32b.

Like $y = \cosh x$, the function $y = \operatorname{sech} x = 1/\cosh x$ fails to be one-to-one, but its restriction to nonnegative values of x does have an inverse, denoted by

$$y = \operatorname{sech}^{-1} x.$$

For every value of x in the interval $(0, 1], y = \operatorname{sech}^{-1} x$ is the nonnegative number whose hyperbolic secant is x. The graphs of $y = \operatorname{sech} x, x \geq 0$, and $y = \operatorname{sech}^{-1} x$ are shown in Figure 7.32c.

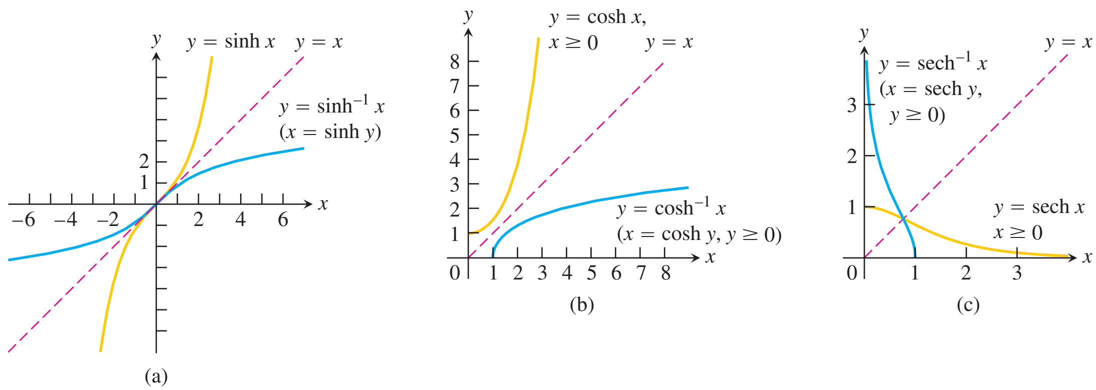


FIGURE 7.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \qquad y = \coth^{-1} x, \qquad y = \operatorname{csch}^{-1} x.$$

These functions are graphed in Figure 7.33.

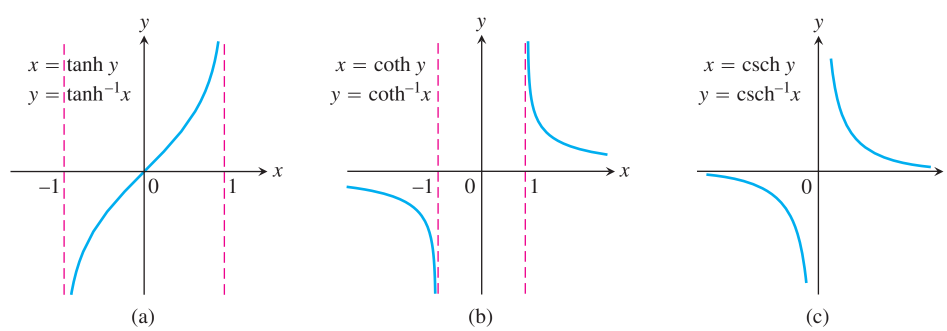


FIGURE 7.33 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

TABLE 7.9 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

Useful Identities

We use the identities in Table 7.9 to calculate the values of $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$, and $\coth^{-1} x$ on calculators that give only $\cosh^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$. These identities are direct consequences of the definitions. For example, if $0 < x \leq 1$, then

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x$$

so

$$\cosh^{-1}\left(\frac{1}{x}\right) = \operatorname{sech}^{-1} x$$

since the hyperbolic secant is one-to-one on $(0, 1]$.

Derivatives and Integrals

The chief use of inverse hyperbolic functions lies in integrations that reverse the derivative formulas in Table 7.10.

TABLE 7.10 Derivatives of inverse hyperbolic functions

$\dfrac{d(\sinh^{-1} u)}{dx} = \dfrac{1}{\sqrt{1 + u^2}} \dfrac{du}{dx}$	
$\dfrac{d(\cosh^{-1} u)}{dx} = \dfrac{1}{\sqrt{u^2 - 1}} \dfrac{du}{dx},$	$u > 1$
$\dfrac{d(\tanh^{-1} u)}{dx} = \dfrac{1}{1 - u^2} \dfrac{du}{dx},$	$\lvert u \rvert < 1$
$\dfrac{d(\coth^{-1} u)}{dx} = \dfrac{1}{1 - u^2} \dfrac{du}{dx},$	$\lvert u \rvert > 1$
$\dfrac{d(\operatorname{sech}^{-1} u)}{dx} = \dfrac{-du/dx}{u\sqrt{1 - u^2}},$	$0 < u < 1$
$\dfrac{d(\operatorname{csch}^{-1} u)}{dx} = \dfrac{-du/dx}{\lvert u \rvert\sqrt{1 + u^2}},$	$u \neq 0$

The restrictions $|u| < 1$ and $|u| > 1$ on the derivative formulas for $\tanh^{-1} u$ and $\coth^{-1} u$ come from the natural restrictions on the values of these functions. (See Figure 7.33a and b.) The distinction between $|u| < 1$ and $|u| > 1$ becomes important when we convert the derivative formulas into integral formulas. If $|u| < 1$, the integral of $1/(1 - u^2)$ is $\tanh^{-1} u + C$. If $|u| > 1$, the integral is $\coth^{-1} u + C$.

We illustrate how the derivatives of the inverse hyperbolic functions are found in Example 2, where we calculate $d(\cosh^{-1} u)/dx$. The other derivatives are obtained by similar calculations.

HISTORICAL BIOGRAPHY

Sonya Kovalevsky
(1850–1891)

EXAMPLE 2 Derivative of the Inverse Hyperbolic Cosine

Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

You Try It

Solution First we find the derivative of $y = \cosh^{-1} x$ for $x > 1$ by applying Theorem 1 with $f(x) = \cosh x$ and $f^{-1}(x) = \cosh^{-1} x$. Theorem 1 can be applied because the derivative of $\cosh x$ is positive for $0 < x$.

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\sinh(\cosh^{-1} x)} && f'(u) = \sinh u \\
&= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} && \cosh^2 u - \sinh^2 u = 1, \\
& && \sinh u = \sqrt{\cosh^2 u - 1} \\
&= \frac{1}{\sqrt{x^2 - 1}} && \cosh(\cosh^{-1} x) = x
\end{aligned}$$

In short,

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}.$$

The Chain Rule gives the final result:

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

■

Instead of applying Theorem 1 directly, as in Example 2, we could also find the derivative of $y = \cosh^{-1} x$, $x > 1$, using implicit differentiation and the Chain Rule:

$$\begin{aligned}
y &= \cosh^{-1} x \\
x &= \cosh y && \text{Equivalent equation} \\
1 &= \sinh y \frac{dy}{dx} && \text{Implicit differentiation with respect to } x\text{, and the Chain Rule} \\
\frac{dy}{dx} &= \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} && \text{Since } x > 1, y > 0 \text{ and } \sinh y > 0 \\
&= \frac{1}{\sqrt{x^2 - 1}}. && \cosh y = x
\end{aligned}$$

With appropriate substitutions, the derivative formulas in Table 7.10 lead to the integration formulas in Table 7.11. Each of the formulas in Table 7.11 can be verified by differentiating the expression on the right-hand side.

EXAMPLE 3 Using Table 7.11

You Try It

Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}.$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2 - u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

Solution The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3 + 4x^2}} = \int \frac{du}{\sqrt{a^2 + u^2}} \qquad u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.11}$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Big]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$ ■

EXERCISES 7.8

Hyperbolic Function Values and Identities

Each of Exercises 1–4 gives a value of sinh x or cosh x. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\dfrac{3}{4}$

2. $\sinh x = \dfrac{4}{3}$

3. $\cosh x = \dfrac{17}{15}, \quad x > 0$

4. $\cosh x = \dfrac{13}{5}, \quad x > 0$

Rewrite the expressions in Exercises 5–10 in terms of exponentials and simplify the results as much as you can.

5. $2 \cosh (\ln x)$

6. $\sinh (2 \ln x)$

7. $\cosh 5x + \sinh 5x$

8. $\cosh 3x - \sinh 3x$

9. $(\sinh x + \cosh x)^4$

10. $\ln (\cosh x + \sinh x) + \ln (\cosh x - \sinh x)$

11. Use the identities

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

to show that

a. $\sinh 2x = 2\sinh x \cosh x$

b. $\cosh 2x = \cosh^2 x + \sinh^2 x$.

12. Use the definitions of $\cosh x$ and $\sinh x$ to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

Derivatives

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

Exercises

13. $y = 6\sinh\frac{x}{3}$

14. $y = \frac{1}{2}\sinh(2x+1)$

15. $y = 2\sqrt{t}\tanh\sqrt{t}$

16. $y = t^2\tanh\frac{1}{t}$

17. $y = \ln(\sinh z)$

18. $y = \ln(\cosh z)$

19. $y = \operatorname{sech}\theta(1 - \ln\operatorname{sech}\theta)$

20. $y = \operatorname{csch}\theta(1 - \ln\operatorname{csch}\theta)$

21. $y = \ln\cosh v - \frac{1}{2}\tanh^2 v$

22. $y = \ln\sinh v - \frac{1}{2}\coth^2 v$

23. $y = (x^2+1)\operatorname{sech}(\ln x)$

(*Hint:* Before differentiating, express in terms of exponentials and simplify.)

24. $y = (4x^2-1)\operatorname{csch}(\ln 2x)$

In Exercises 25–36, find the derivative of y with respect to the appropriate variable.

Exercises

25. $y = \sinh^{-1}\sqrt{x}$

26. $y = \cosh^{-1} 2\sqrt{x+1}$

27. $y = (1-\theta)\tanh^{-1}\theta$

28. $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta+1)$

29. $y = (1-t)\coth^{-1}\sqrt{t}$

30. $y = (1-t^2)\coth^{-1} t$

31. $y = \cos^{-1} x - x\operatorname{sech}^{-1} x$

32. $y = \ln x + \sqrt{1-x^2}\operatorname{sech}^{-1} x$

33. $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$

34. $y = \operatorname{csch}^{-1} 2^{\theta}$

35. $y = \sinh^{-1}(\tan x)$

36. $y = \cosh^{-1}(\sec x),\quad 0 < x < \pi/2$

Integration Formulas

Verify the integration formulas in Exercises 37–40.

37. a. $\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x) + C$

b. $\int \operatorname{sech} x\,dx = \sin^{-1}(\tanh x) + C$

38. $\int x\operatorname{sech}^{-1} x\,dx = \frac{x^2}{2}\operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$

39. $\int x\coth^{-1} x\,dx = \frac{x^2-1}{2}\coth^{-1} x + \frac{x}{2} + C$

40. $\int \tanh^{-1} x\,dx = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + C$

Indefinite Integrals

Evaluate the integrals in Exercises 41–50.

Exercises

41. $\int \sinh 2x\,dx$

42. $\int \sinh\frac{x}{5}\,dx$

43. $\int 6\cosh\left(\frac{x}{2} - \ln 3\right)dx$

44. $\int 4\cosh(3x - \ln 2)\,dx$

45. $\int \tanh\frac{x}{7}\,dx$

46. $\int \coth\frac{\theta}{\sqrt{3}}\,d\theta$

47. $\int \operatorname{sech}^2\left(x - \frac{1}{2}\right)dx$

48. $\int \operatorname{csch}^2(5-x)\,dx$

49. $\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}\,dt}{\sqrt{t}}$

50. $\int \frac{\operatorname{csch}(\ln t)\coth(\ln t)\,dt}{t}$

Definite Integrals

Evaluate the integrals in Exercises 51–60.

Exercises

51. $\int_{\ln 2}^{\ln 4} \coth x\,dx$

52. $\int_0^{\ln 2} \tanh 2x\,dx$

53. $\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\,d\theta$

54. $\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\,d\theta$

55. $\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\,d\theta$

56. $\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\,d\theta$

57. $\int_1^2 \frac{\cosh(\ln t)}{t}\,dt$

58. $\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\,dx$

59. $\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right)dx$

60. $\int_0^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right)dx$

Evaluating Inverse Hyperbolic Functions and Related Integrals

When hyperbolic function keys are not available on a calculator, it is still possible to evaluate the inverse hyperbolic functions by expressing them as logarithms, as shown here.

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right),$	$-\infty < x < \infty$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right),$	$x \geq 1$
$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x},$	$\lvert x\rvert < 1$
$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),$	$0 < x \leq 1$
$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right),$	$x \neq 0$
$\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1},$	$\lvert x\rvert > 1$

Use the formulas in the box here to express the numbers in Exercises 61–66 in terms of natural logarithms.

61. $\sinh^{-1}(-5/12)$

62. $\cosh^{-1}(5/3)$

63. $\tanh^{-1}(-1/2)$

64. $\coth^{-1}(5/4)$

65. $\operatorname{sech}^{-1}(3/5)$

66. $\operatorname{csch}^{-1}(-1/\sqrt{3})$

Evaluate the integrals in Exercises 67–74 in terms of

a. inverse hyperbolic functions.

b. natural logarithms.

67. $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}}$

68. $\displaystyle\int_0^{1/3} \frac{6\,dx}{\sqrt{1+9x^2}}$

69. $\displaystyle\int_{5/4}^{2} \frac{dx}{1-x^2}$

70. $\displaystyle\int_0^{1/2} \frac{dx}{1-x^2}$

71. $\displaystyle\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}}$

72. $\displaystyle\int_1^{2} \frac{dx}{x\sqrt{4+x^2}}$

73. $\displaystyle\int_0^{\pi} \frac{\cos x\,dx}{\sqrt{1+\sin^2 x}}$

74. $\displaystyle\int_1^{e} \frac{dx}{x\sqrt{1+(\ln x)^2}}$

Applications and Theory

75. a. Show that if a function f is defined on an interval symmetric about the origin (so that f is defined at $-x$ whenever it is defined at x), then

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}. \qquad (1)$$

Then show that $(f(x)+f(-x))/2$ is even and that $(f(x)-f(-x))/2$ is odd.

b. Equation (1) simplifies considerably if f itself is (i) even or (ii) odd. What are the new equations? Give reasons for your answers.

76. Derive the formula $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$, $-\infty < x < \infty$. Explain in your derivation why the plus sign is used with the square root instead of the minus sign.

77. Skydiving If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity t sec into the fall satisfies the differential equation

$$m\frac{dv}{dt} = mg - kv^2,$$

where k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is short enough so that the variation in the air's density will not affect the outcome significantly.)

a. Show that

$$v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}\,t\right)$$

satisfies the differential equation and the initial condition that $v = 0$ when $t = 0$.

b. Find the body's *limiting velocity*, $\lim_{t\to\infty} v$.

c. For a 160-lb skydiver ($mg = 160$), with time in seconds and distance in feet, a typical value for k is 0.005. What is the diver's limiting velocity?

78. Accelerations whose magnitudes are proportional to displacement Suppose that the position of a body moving along a coordinate line at time t is

a. $s = a\cos kt + b\sin kt$

b. $s = a\cosh kt + b\sinh kt$.

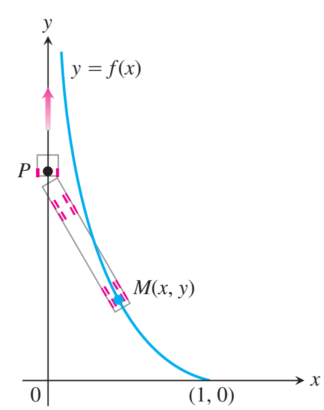
Show in both cases that the acceleration d^2s/dt^2 is proportional to s but that in the first case it is directed toward the origin, whereas in the second case it is directed away from the origin.

79. Tractor trailers and the tractrix When a tractor trailer turns into a cross street or driveway, its rear wheels follow a curve like the one shown here. (This is why the rear wheels sometimes ride up over the curb.) We can find an equation for the curve if we picture the rear wheels as a mass M at the point (1, 0) on the x-axis attached by a rod of unit length to a point P representing the cab at the origin. As the point P moves up the y-axis, it drags M along behind it. The curve traced by M—called a *tractrix* from the Latin word *tractum*, for "drag"—can be shown to be the graph of the function $y = f(x)$ that solves the initial value problem

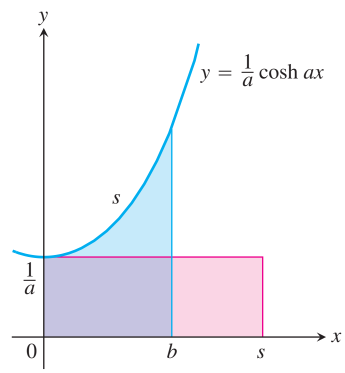
Differential equation: $\displaystyle\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$

Initial condition: $y = 0$ when $x = 1$.

Solve the initial value problem to find an equation for the curve. (You need an inverse hyperbolic function.)



80. Area Show that the area of the region in the first quadrant enclosed by the curve $y = (1/a)\cosh ax$, the coordinate axes, and the line $x = b$ is the same as the area of a rectangle of height $1/a$ and length s, where s is the length of the curve from $x = 0$ to $x = b$. (See accompanying figure.)

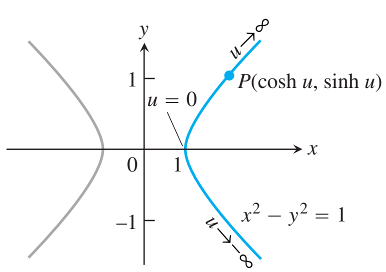


81. Volume A region in the first quadrant is bounded above by the curve $y = \cosh x$, below by the curve $y = \sinh x$, and on the left and right by the y-axis and the line $x = 2$, respectively. Find the volume of the solid generated by revolving the region about the x-axis.

82. Volume The region enclosed by the curve $y = \operatorname{sech} x$, the x-axis, and the lines $x = \pm \ln \sqrt{3}$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

83. Arc length Find the length of the segment of the curve $y = (1/2)\cosh 2x$ from $x = 0$ to $x = \ln \sqrt{5}$.

84. The hyperbolic in hyperbolic functions In case you are wondering where the name *hyperbolic* comes from, here is the answer: Just as $x = \cos u$ and $y = \sin u$ are identified with points (x, y) on the unit circle, the functions $x = \cosh u$ and $y = \sinh u$ are identified with points (x, y) on the right-hand branch of the unit hyperbola, $x^2 - y^2 = 1$.



Since $\cosh^2 u - \sinh^2 u = 1$, the point $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$ for every value of u (Exercise 84).

Another analogy between hyperbolic and circular functions is that the variable u in the coordinates $(\cosh u, \sinh u)$ for the points of the right-hand branch of the hyperbola $x^2 - y^2 = 1$ is twice the area of the sector AOP pictured in the accompanying figure. To see why this is so, carry out the following steps.

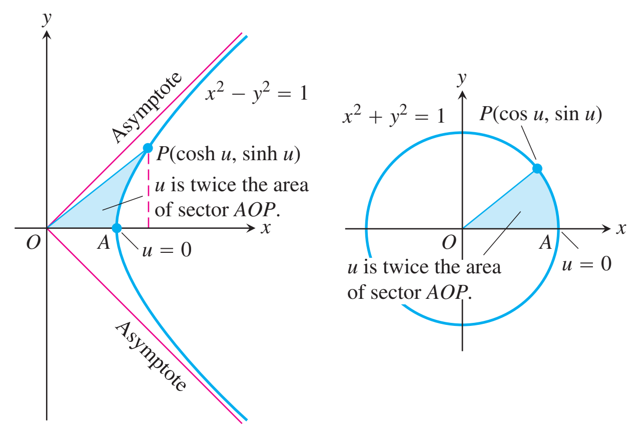
a. Show that the area $A(u)$ of sector AOP is

$$A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\, dx.$$

b. Differentiate both sides of the equation in part (a) with respect to u to show that

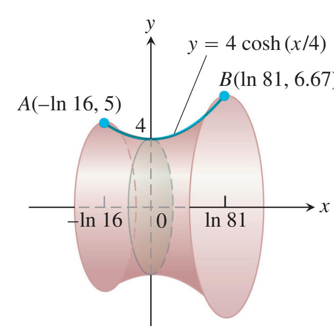
$$A'(u) = \frac{1}{2}.$$

c. Solve this last equation for $A(u)$. What is the value of $A(0)$? What is the value of the constant of integration C in your solution? With C determined, what does your solution say about the relationship of u to $A(u)$?



One of the analogies between hyperbolic and circular functions is revealed by these two diagrams (Exercise 84).

85. A minimal surface Find the area of the surface swept out by revolving about the x-axis the curve $y = 4\cosh(x/4)$, $-\ln 16 \le x \le \ln 81$.



It can be shown that, of all continuously differentiable curves joining points A and B in the figure, the curve $y = 4\cosh(x/4)$ generates the surface of least area. If you made a rigid wire frame of the end-circles through A and B and dipped them in a soap-film solution, the surface spanning the circles would be the one generated by the curve.

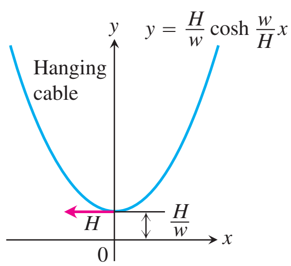
T **86. a.** Find the centroid of the curve $y = \cosh x$, $-\ln 2 \le x \le \ln 2$.

b. Evaluate the coordinates to two decimal places. Then sketch the curve and plot the centroid to show its relation to the curve.

Hanging Cables

87. Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is w and the horizontal tension at its lowest point is a vector of length H. If we choose a coordinate system for the plane of the cable in which the x-axis is horizontal, the force of gravity is straight down, the positive y-axis points straight up, and the lowest point of the cable lies at the point $y = H/w$ on the y-axis (see accompanying figure), then it can be shown that the cable lies along the graph of the hyperbolic cosine

$$y = \frac{H}{w}\cosh\frac{w}{H}x.$$

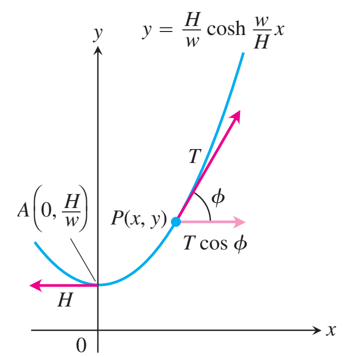


Such a curve is sometimes called a **chain curve** or a **catenary**, the latter deriving from the Latin *catena*, meaning "chain."

a. Let $P(x, y)$ denote an arbitrary point on the cable. The next accompanying figure displays the tension at P as a vector of length (magnitude) T, as well as the tension H at the lowest point A. Show that the cable's slope at P is

$$\tan\phi = \frac{dy}{dx} = \sinh\frac{w}{H}x.$$

b. Using the result from part (a) and the fact that the horizontal tension at P must equal H (the cable is not moving), show that $T = wy$. Hence, the magnitude of the tension at $P(x, y)$ is exactly equal to the weight of y units of cable.



88. (*Continuation of Exercise 87.*) The length of arc AP in the Exercise 87 figure is $s = (1/a)\sinh ax$, where $a = w/H$. Show that the coordinates of P may be expressed in terms of s as

$$x = \frac{1}{a}\sinh^{-1} as, \qquad y = \sqrt{s^2 + \frac{1}{a^2}}.$$

89. The sag and horizontal tension in a cable The ends of a cable 32 ft long and weighing 2 lb/ft are fastened at the same level to posts 30 ft apart.

a. Model the cable with the equation

$$y = \frac{1}{a}\cosh ax, \quad -15 \le x \le 15.$$

Use information from Exercise 88 to show that a satisfies the equation

$$16a = \sinh 15a. \tag{2}$$

T **b.** Solve Equation (2) graphically by estimating the coordinates of the points where the graphs of the equations $y = 16a$ and $y = \sinh 15a$ intersect in the ay-plane.

T **c.** Solve Equation (2) for a numerically. Compare your solution with the value you found in part (b).

d. Estimate the horizontal tension in the cable at the cable's lowest point.

T **e.** Using the value found for a in part (c), graph the catenary

$$y = \frac{1}{a}\cosh ax$$

over the interval $-15 \le x \le 15$. Estimate the sag in the cable at its center.

Chapter 7 Questions to Guide Your Review

1. What functions have inverses? How do you know if two functions f and g are inverses of one another? Give examples of functions that are (are not) inverses of one another.
2. How are the domains, ranges, and graphs of functions and their inverses related? Give an example.
3. How can you sometimes express the inverse of a function of x as a function of x?
4. Under what circumstances can you be sure that the inverse of a function f is differentiable? How are the derivatives of f and f^{-1} related?

5. What is the natural logarithm function? What are its domain, range, and derivative? What arithmetic properties does it have? Comment on its graph.

6. What is logarithmic differentiation? Give an example.

7. What integrals lead to logarithms? Give examples. What are the integrals of $\tan x$ and $\cot x$?

8. How is the exponential function e^x defined? What are its domain, range, and derivative? What laws of exponents does it obey? Comment on its graph.

9. How are the functions a^x and $\log_a x$ defined? Are there any restrictions on a? How is the graph of $\log_a x$ related to the graph of $\ln x$? What truth is there in the statement that there is really only one exponential function and one logarithmic function?

10. Describe some of the applications of base 10 logarithms.

11. What is the law of exponential change? How can it be derived from an initial value problem? What are some of the applications of the law?

12. How do you compare the growth rates of positive functions as $x \to \infty$?

13. What roles do the functions e^x and $\ln x$ play in growth comparisons?

14. Describe big-oh and little-oh notation. Give examples.

15. Which is more efficient—a sequential search or a binary search? Explain.

16. How are the inverse trigonometric functions defined? How can you sometimes use right triangles to find values of these functions? Give examples.

17. What are the derivatives of the inverse trigonometric functions? How do the domains of the derivatives compare with the domains of the functions?

18. What integrals lead to inverse trigonometric functions? How do substitution and completing the square broaden the application of these integrals?

19. What are the six basic hyperbolic functions? Comment on their domains, ranges, and graphs. What are some of the identities relating them?

20. What are the derivatives of the six basic hyperbolic functions? What are the corresponding integral formulas? What similarities do you see here with the six basic trigonometric functions?

21. How are the inverse hyperbolic functions defined? Comment on their domains, ranges, and graphs. How can you find values of $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$, and $\coth^{-1} x$ using a calculator's keys for $\cosh^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$?

22. What integrals lead naturally to inverse hyperbolic functions?

Chapter 7 Practice Exercises

Differentiation

In Exercises 1–24, find the derivative of y with respect to the appropriate variable.

1. $y = 10e^{-x/5}$
2. $y = \sqrt{2}e^{\sqrt{2}x}$
3. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$
4. $y = x^2e^{-2/x}$
5. $y = \ln(\sin^2\theta)$
6. $y = \ln(\sec^2\theta)$
7. $y = \log_2(x^2/2)$
8. $y = \log_5(3x - 7)$
9. $y = 8^{-t}$
10. $y = 9^{2t}$
11. $y = 5x^{3.6}$
12. $y = \sqrt{2}x^{-\sqrt{2}}$
13. $y = (x + 2)^{x+2}$
14. $y = 2(\ln x)^{x/2}$
15. $y = \sin^{-1}\sqrt{1 - u^2}, \quad 0 < u < 1$
16. $y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right), \quad v > 1$
17. $y = \ln\cos^{-1}x$
18. $y = z\cos^{-1}z - \sqrt{1 - z^2}$
19. $y = t\tan^{-1}t - \frac{1}{2}\ln t$
20. $y = (1 + t^2)\cot^{-1}2t$
21. $y = z\sec^{-1}z - \sqrt{z^2 - 1}, \quad z > 1$
22. $y = 2\sqrt{x - 1}\sec^{-1}\sqrt{x}$
23. $y = \csc^{-1}(\sec\theta), \quad 0 < \theta < \pi/2$
24. $y = (1 + x^2)e^{\tan^{-1}x}$

Logarithmic Differentiation

In Exercises 25–30, use logarithmic differentiation to find the derivative of y with respect to the appropriate variable.

25. $y = \dfrac{2(x^2 + 1)}{\sqrt{\cos 2x}}$
26. $y = \sqrt[10]{\dfrac{3x + 4}{2x - 4}}$
27. $y = \left(\dfrac{(t + 1)(t - 1)}{(t - 2)(t + 3)}\right)^5, \quad t > 2$
28. $y = \dfrac{2u2^u}{\sqrt{u^2 + 1}}$
29. $y = (\sin\theta)^{\sqrt{\theta}}$
30. $y = (\ln x)^{1/(\ln x)}$

Integration

Evaluate the integrals in Exercises 31–78.

31. $\int e^x \sin(e^x)\,dx$
32. $\int e^t \cos(3e^t - 2)\,dt$

33. $\int e^x \sec^2(e^x - 7)\,dx$

34. $\int e^y \csc(e^y + 1)\cot(e^y + 1)\,dy$

35. $\int \sec^2(x)e^{\tan x}\,dx$

36. $\int \csc^2 x\, e^{\cot x}\,dx$

37. $\int_{-1}^{1} \frac{dx}{3x - 4}$

38. $\int_{1}^{e} \frac{\sqrt{\ln x}}{x}\,dx$

39. $\int_0^{\pi} \tan\frac{x}{3}\,dx$

40. $\int_{1/6}^{1/4} 2\cot \pi x\,dx$

41. $\int_0^4 \frac{2t}{t^2 - 25}\,dt$

42. $\int_{-\pi/2}^{\pi/6} \frac{\cos t}{1 - \sin t}\,dt$

43. $\int \frac{\tan(\ln v)}{v}\,dv$

44. $\int \frac{dv}{v \ln v}$

45. $\int \frac{(\ln x)^{-3}}{x}\,dx$

46. $\int \frac{\ln(x - 5)}{x - 5}\,dx$

47. $\int \frac{1}{r}\csc^2(1 + \ln r)\,dr$

48. $\int \frac{\cos(1 - \ln v)}{v}\,dv$

49. $\int x3^{x^2}\,dx$

50. $\int 2^{\tan x}\sec^2 x\,dx$

51. $\int_1^7 \frac{3}{x}\,dx$

52. $\int_1^{32} \frac{1}{5x}\,dx$

53. $\int_1^4 \left(\frac{x}{8} + \frac{1}{2x}\right)dx$

54. $\int_1^8 \left(\frac{2}{3x} - \frac{8}{x^2}\right)dx$

55. $\int_{-2}^{-1} e^{-(x+1)}\,dx$

56. $\int_{-\ln 2}^{0} e^{2w}\,dw$

57. $\int_0^{\ln 5} e^r(3e^r + 1)^{-3/2}\,dr$

58. $\int_0^{\ln 9} e^{\theta}(e^{\theta} - 1)^{1/2}\,d\theta$

59. $\int_1^e \frac{1}{x}(1 + 7\ln x)^{-1/3}\,dx$

60. $\int_e^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx$

61. $\int_1^3 \frac{(\ln(v + 1))^2}{v + 1}\,dv$

62. $\int_2^4 (1 + \ln t)t\ln t\,dt$

63. $\int_1^8 \frac{\log_4 \theta}{\theta}\,d\theta$

64. $\int_1^e \frac{8\ln 3\log_3 \theta}{\theta}\,d\theta$

65. $\int_{-3/4}^{3/4} \frac{6\,dx}{\sqrt{9 - 4x^2}}$

66. $\int_{-1/5}^{1/5} \frac{6\,dx}{\sqrt{4 - 25x^2}}$

67. $\int_{-2}^{2} \frac{3\,dt}{4 + 3t^2}$

68. $\int_{\sqrt{3}}^{3} \frac{dt}{3 + t^2}$

69. $\int \frac{dy}{y\sqrt{4y^2 - 1}}$

70. $\int \frac{24\,dy}{y\sqrt{y^2 - 16}}$

71. $\int_{\sqrt{2}/3}^{2/3} \frac{dy}{|y|\sqrt{9y^2 - 1}}$

72. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{dy}{|y|\sqrt{5y^2 - 3}}$

73. $\int \frac{dx}{\sqrt{-2x - x^2}}$

74. $\int \frac{dx}{\sqrt{-x^2 + 4x - 1}}$

75. $\int_{-2}^{-1} \frac{2\,dv}{v^2 + 4v + 5}$

76. $\int_{-1}^{1} \frac{3\,dv}{4v^2 + 4v + 4}$

77. $\int \frac{dt}{(t + 1)\sqrt{t^2 + 2t - 8}}$

78. $\int \frac{dt}{(3t + 1)\sqrt{9t^2 + 6t}}$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 79–84, solve for y.

79. $3^y = 2^{y+1}$

80. $4^{-y} = 3^{y+2}$

81. $9e^{2y} = x^2$

82. $3^y = 3\ln x$

83. $\ln(y - 1) = x + \ln y$

84. $\ln(10\ln y) = \ln 5x$

Evaluating Limits

Find the limits in Exercises 85–96.

85. $\lim_{x\to 0} \frac{10^x - 1}{x}$

86. $\lim_{\theta\to 0} \frac{3^{\theta} - 1}{\theta}$

87. $\lim_{x\to 0} \frac{2^{\sin x} - 1}{e^x - 1}$

88. $\lim_{x\to 0} \frac{2^{-\sin x} - 1}{e^x - 1}$

89. $\lim_{x\to 0} \frac{5 - 5\cos x}{e^x - x - 1}$

90. $\lim_{x\to 0} \frac{4 - 4e^x}{xe^x}$

91. $\lim_{t\to 0^+} \frac{t - \ln(1 + 2t)}{t^2}$

92. $\lim_{x\to 4} \frac{\sin^2(\pi x)}{e^{x-4} + 3 - x}$

93. $\lim_{t\to 0^+} \left(\frac{e^t}{t} - \frac{1}{t}\right)$

94. $\lim_{y\to 0^+} e^{-1/y}\ln y$

95. $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^x$

96. $\lim_{x\to 0^+} \left(1 + \frac{3}{x}\right)^x$

Comparing Growth Rates of Functions

97. Does f grow faster, slower, or at the same rate as g as $x \to \infty$? Give reasons for your answers.

a. $f(x) = \log_2 x$, $g(x) = \log_3 x$

b. $f(x) = x$, $g(x) = x + \frac{1}{x}$

c. $f(x) = x/100$, $g(x) = xe^{-x}$

d. $f(x) = x$, $g(x) = \tan^{-1} x$

e. $f(x) = \csc^{-1} x$, $g(x) = 1/x$

f. $f(x) = \sinh x$, $g(x) = e^x$

98. Does f grow faster, slower, or at the same rate as g as $x \to \infty$? Give reasons for your answers.

a. $f(x) = 3^{-x}$, $g(x) = 2^{-x}$

b. $f(x) = \ln 2x$, $g(x) = \ln x^2$

c. $f(x) = 10x^3 + 2x^2$, $g(x) = e^x$

d. $f(x) = \tan^{-1}(1/x)$, $g(x) = 1/x$

e. $f(x) = \sin^{-1}(1/x)$, $g(x) = 1/x^2$

f. $f(x) = \operatorname{sech} x$, $g(x) = e^{-x}$

99. True, or false? Give reasons for your answers.

a. $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^2}\right)$ **b.** $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^4}\right)$

c. $x = o(x + \ln x)$ **d.** $\ln(\ln x) = o(\ln x)$

e. $\tan^{-1} x = O(1)$ **f.** $\cosh x = O(e^x)$

100. True, or false? Give reasons for your answers.

a. $\frac{1}{x^4} = O\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$ **b.** $\frac{1}{x^4} = o\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$

c. $\ln x = o(x + 1)$ **d.** $\ln 2x = O(\ln x)$

e. $\sec^{-1} x = O(1)$ **f.** $\sinh x = O(e^x)$

Theory and Applications

101. The function $f(x) = e^x + x$, being differentiable and one-to-one, has a differentiable inverse $f^{-1}(x)$. Find the value of df^{-1}/dx at the point $f(\ln 2)$.

102. Find the inverse of the function $f(x) = 1 + (1/x), x \neq 0$. Then show that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$ and that

$$\left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{1}{f'(x)}.$$

In Exercises 103 and 104, find the absolute maximum and minimum values of each function on the given interval.

103. $y = x \ln 2x - x, \quad \left[\frac{1}{2e}, \frac{e}{2}\right]$

104. $y = 10x(2 - \ln x), \quad (0, e^2]$

105. Area Find the area between the curve $y = 2(\ln x)/x$ and the x-axis from $x = 1$ to $x = e$.

106. Area

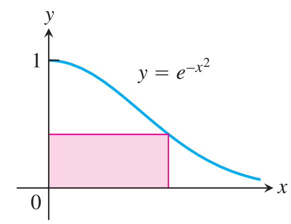
a. Show that the area between the curve $y = 1/x$ and the x-axis from $x = 10$ to $x = 20$ is the same as the area between the curve and the x-axis from $x = 1$ to $x = 2$.

b. Show that the area between the curve $y = 1/x$ and the x-axis from ka to kb is the same as the area between the curve and the x-axis from $x = a$ to $x = b$ $(0 < a < b, k > 0)$.

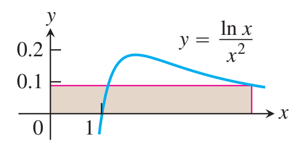
107. A particle is traveling upward and to the right along the curve $y = \ln x$. Its x-coordinate is increasing at the rate $(dx/dt) = \sqrt{x}$ m/sec. At what rate is the y-coordinate changing at the point $(e^2, 2)$?

108. A girl is sliding down a slide shaped like the curve $y = 9e^{-x/3}$. Her y-coordinate is changing at the rate $dy/dt = (-1/4)\sqrt{9 - y}$ ft/sec. At approximately what rate is her x-coordinate changing when she reaches the bottom of the slide at $x = 9$ ft? (Take e^3 to be 20 and round your answer to the nearest ft/sec.)

109. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What dimensions give the rectangle its largest area, and what is that area?



110. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = (\ln x)/x^2$. What dimensions give the rectangle its largest area, and what is that area?



111. The functions $f(x) = \ln 5x$ and $g(x) = \ln 3x$ differ by a constant. What constant? Give reasons for your answer.

112. a. If $(\ln x)/x = (\ln 2)/2$, must $x = 2$?

b. If $(\ln x)/x = -2 \ln 2$, must $x = 1/2$?

Give reasons for your answers.

113. The quotient $(\log_4 x)/(\log_2 x)$ has a constant value. What value? Give reasons for your answer.

T **114.** $\log_x(2)$ **vs.** $\log_2(x)$ How does $f(x) = \log_x(2)$ compare with $g(x) = \log_2(x)$? Here is one way to find out.

a. Use the equation $\log_a b = (\ln b)/(\ln a)$ to express $f(x)$ and $g(x)$ in terms of natural logarithms.

b. Graph f and g together. Comment on the behavior of f in relation to the signs and values of g.

T **115.** Graph the following functions and use what you see to locate and estimate the extreme values, identify the coordinates of the inflection points, and identify the intervals on which the graphs are concave up and concave down. Then confirm your estimates by working with the functions' derivatives.

a. $y = (\ln x)/\sqrt{x}$ **b.** $y = e^{-x^2}$ **c.** $y = (1 + x)e^{-x}$

T **116.** Graph $f(x) = x \ln x$. Does the function appear to have an absolute minimum value? Confirm your answer with calculus.

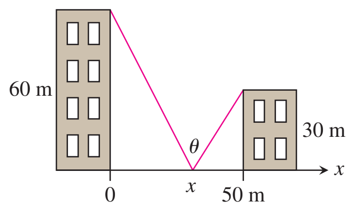
117. What is the age of a sample of charcoal in which 90% of the carbon-14 originally present has decayed?

118. Cooling a pie A deep-dish apple pie, whose internal temperature was 220°F when removed from the oven, was set out on a breezy 40°F porch to cool. Fifteen minutes later, the pie's internal temperature was 180°F. How long did it take the pie to cool from there to 70°F?

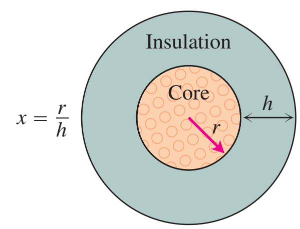
119. Locating a solar station You are under contract to build a solar station at ground level on the east–west line between the two buildings shown here. How far from the taller building should you place the station to maximize the number of hours it will be in the sun on a day when the sun passes directly overhead? Begin by observing that

$$\theta = \pi - \cot^{-1}\frac{x}{60} - \cot^{-1}\frac{50 - x}{30}.$$

Then find the value of x that maximizes θ.



120. A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If x denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $v = x^2 \ln(1/x)$. If the radius of the core is 1 cm, what insulation thickness h will allow the greatest transmission speed?



Chapter 7 Additional and Advanced Exercises

Limits

Find the limits in Exercises 1–6.

1. $\lim_{b\to 1^-}\int_0^b \frac{dx}{\sqrt{1-x^2}}$

2. $\lim_{x\to\infty}\frac{1}{x}\int_0^x \tan^{-1} t\,dt$

3. $\lim_{x\to 0^+}(\cos\sqrt{x})^{1/x}$

4. $\lim_{x\to\infty}(x+e^x)^{2/x}$

5. $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$

6. $\lim_{n\to\infty}\frac{1}{n}\left(e^{1/n}+e^{2/n}+\cdots+e^{(n-1)/n}+e^{n/n}\right)$

7. Let $A(t)$ be the area of the region in the first quadrant enclosed by the coordinate axes, the curve $y=e^{-x}$, and the vertical line $x=t, t>0$. Let $V(t)$ be the volume of the solid generated by revolving the region about the x-axis. Find the following limits.

a. $\lim_{t\to\infty}A(t)$ **b.** $\lim_{t\to\infty}V(t)/A(t)$ **c.** $\lim_{t\to 0^+}V(t)/A(t)$

8. Varying a logarithm's base

a. Find $\lim \log_a 2$ as $a\to 0^+, 1^-, 1^+$, and ∞.

T **b.** Graph $y=\log_a 2$ as a function of a over the interval $0<a\le 4$.

Theory and Examples

9. Find the areas between the curves $y=2(\log_2 x)/x$ and $y=2(\log_4 x)/x$ and the x-axis from $x=1$ to $x=e$. What is the ratio of the larger area to the smaller?

T **10.** Graph $f(x)=\tan^{-1}x+\tan^{-1}(1/x)$ for $-5\le x\le 5$. Then use calculus to explain what you see. How would you expect f to behave beyond the interval $[-5,5]$? Give reasons for your answer.

11. For what $x>0$ does $x^{(x^x)}=(x^x)^x$? Give reasons for your answer.

T **12.** Graph $f(x)=(\sin x)^{\sin x}$ over $[0,3\pi]$. Explain what you see.

13. Find $f'(2)$ if $f(x)=e^{g(x)}$ and $g(x)=\int_2^x \frac{t}{1+t^4}\,dt$.

14. a. Find df/dx if

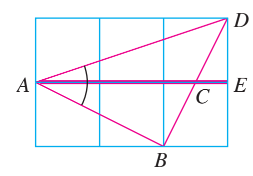
$$f(x)=\int_1^{e^x}\frac{2\ln t}{t}\,dt.$$

b. Find $f(0)$.

c. What can you conclude about the graph of f? Give reasons for your answer.

15. The figure here shows an informal proof that

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}.$$

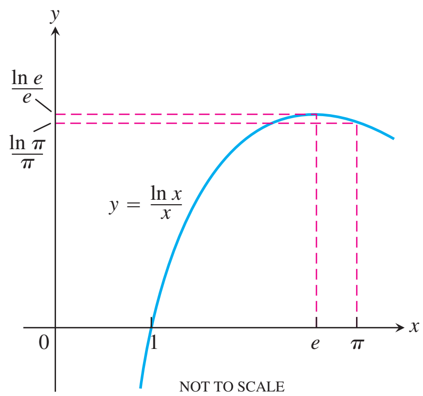


How does the argument go? (*Source*: "Behold! Sums of Arctan," by Edward M. Harris, *College Mathematics Journal*, Vol. 18, No. 2, Mar. 1987, p. 141.)

16. $\pi^e < e^\pi$

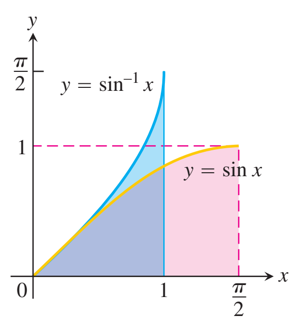
a. Why does the accompanying figure "prove" that $\pi^e<e^\pi$? (*Source:* "Proof Without Words," by Fouad Nakhil, *Mathematics Magazine*, Vol. 60, No. 3, June 1987, p. 165.)

b. The accompanying figure assumes that $f(x)=(\ln x)/x$ has an absolute maximum value at $x=e$. How do you know it does?



17. Use the accompanying figure to show that

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x \, dx.$$

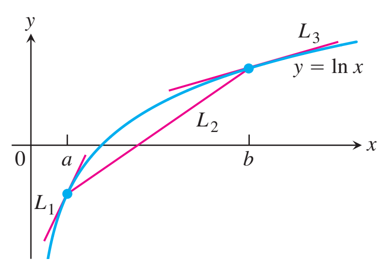


18. Napier's inequality Here are two pictorial proofs that

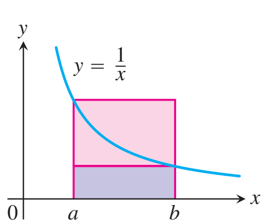
$$b > a > 0 \quad \Rightarrow \quad \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}.$$

Explain what is going on in each case.

a.



b.



(*Source:* Roger B. Nelson, *College Mathematics Journal*, Vol. 24, No. 2, March 1993, p. 165.)

19. Even-odd decompositions

a. Suppose that g is an even function of x and h is an odd function of x. Show that if $g(x) + h(x) = 0$ for all x then $g(x) = 0$ for all x and $h(x) = 0$ for all x.

b. Use the result in part (a) to show that if $f(x) = f_E(x) + f_O(x)$ is the sum of an even function $f_E(x)$ and an odd function $f_O(x)$, then

$$f_E(x) = (f(x) + f(-x))/2 \quad \text{and} \quad f_O(x) = (f(x) - f(-x))/2.$$

c. What is the significance of the result in part (b)?

20. Let g be a function that is differentiable throughout an open interval containing the origin. Suppose g has the following properties:

i. $g(x + y) = \dfrac{g(x) + g(y)}{1 - g(x)g(y)}$ for all real numbers x, y, and $x + y$ in the domain of g.

ii. $\lim_{h \to 0} g(h) = 0$

iii. $\lim_{h \to 0} \dfrac{g(h)}{h} = 1$

a. Show that $g(0) = 0$.

b. Show that $g'(x) = 1 + [g(x)]^2$.

c. Find $g(x)$ by solving the differential equation in part (b).

Applications

21. Center of mass Find the center of mass of a thin plate of constant density covering the region in the first and fourth quadrants enclosed by the curves $y = 1/(1 + x^2)$ and $y = -1/(1 + x^2)$ and by the lines $x = 0$ and $x = 1$.

22. Solid of revolution The region between the curve $y = 1/(2\sqrt{x})$ and the x-axis from $x = 1/4$ to $x = 4$ is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

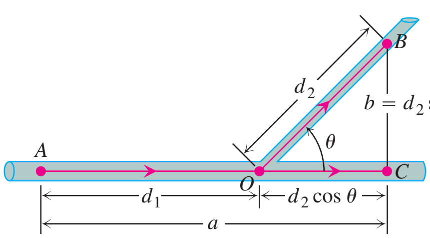
b. Find the centroid of the region.

23. The Rule of 70 If you use the approximation $\ln 2 \approx 0.70$ (in place of $0.69314\ldots$), you can derive a rule of thumb that says, "To estimate how many years it will take an amount of money to double when invested at r percent compounded continuously, divide r into 70." For instance, an amount of money invested at 5% will double in about $70/5 = 14$ years. If you want it to double in 10 years instead, you have to invest it at $70/10 = 7\%$. Show how the Rule of 70 is derived. (A similar "Rule of 72" uses 72 instead of 70, because 72 has more integer factors.)

24. Free fall in the fourteenth century In the middle of the fourteenth century, Albert of Saxony (1316–1390) proposed a model of free fall that assumed that the velocity of a falling body was proportional to the distance fallen. It seemed reasonable to think that a body that had fallen 20 ft might be moving twice as fast as a body that had fallen 10 ft. And besides, none of the instruments in use at the time were accurate enough to prove otherwise. Today we can see just how far off Albert of Saxony's model was by solving the initial value problem implicit in his model. Solve the problem and compare your solution graphically with the equation $s = 16t^2$. You will see that it describes a motion that starts too slowly at first and then becomes too fast too soon to be realistic.

25. The best branching angles for blood vessels and pipes When a smaller pipe branches off from a larger one in a flow system, we may want it to run off at an angle that is best from some energy-saving point of view. We might require, for instance, that energy loss due to friction be minimized along the section AOB shown in the accompanying figure. In this diagram, B is a given point to be reached by the smaller pipe, A is a point in the larger pipe upstream from B, and O is the point where the branching occurs. A law due to Poiseuille states that the loss of energy due to friction in nonturbulent flow is proportional to the length of the path and inversely proportional to the fourth power of the radius. Thus, the loss along AO is $(kd_1)/R^4$ and along OB is $(kd_2)/r^4$, where k is a constant, d_1 is the length of AO, d_2 is the length of OB, R is the radius of the larger pipe, and r is the radius of the smaller pipe. The angle θ is to be chosen to minimize the sum of these two losses:

$$L = k\frac{d_1}{R^4} + k\frac{d_2}{r^4}.$$



In our model, we assume that $AC = a$ and $BC = b$ are fixed. Thus we have the relations

$$d_1 + d_2 \cos\theta = a \qquad d_2 \sin\theta = b,$$

so that

$$d_2 = b \csc\theta,$$
$$d_1 = a - d_2 \cos\theta = a - b \cot\theta.$$

We can express the total loss L as a function of θ:

$$L = k\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right).$$

a. Show that the critical value of θ for which $dL/d\theta$ equals zero is

$$\theta_c = \cos^{-1}\frac{r^4}{R^4}.$$

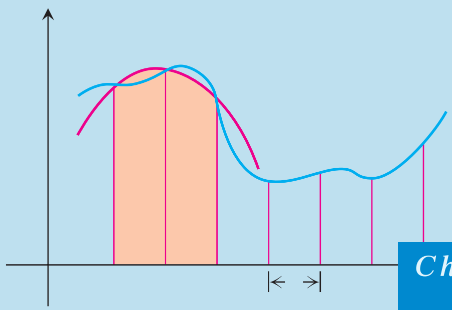
b. If the ratio of the pipe radii is $r/R = 5/6$, estimate to the nearest degree the optimal branching angle given in part (a).

The mathematical analysis described here is also used to explain the angles at which arteries branch in an animal's body. (See *Introduction to Mathematics for Life Scientists*, Second Edition, by E. Batschelet [New York: Springer-Verlag, 1976].)

T **26. Group blood testing** During World War II it was necessary to administer blood tests to large numbers of recruits. There are two standard ways to administer a blood test to N people. In method 1, each person is tested separately. In method 2, the blood samples of x people are pooled and tested as one large sample. If the test is negative, this one test is enough for all x people. If the test is positive, then each of the x people is tested separately, requiring a total of $x + 1$ tests. Using the second method and some probability theory it can be shown that, on the average, the total number of tests y will be

$$y = N\left(1 - q^x + \frac{1}{x}\right).$$

With $q = 0.99$ and $N = 1000$, find the integer value of x that minimizes y. Also find the integer value of x that maximizes y. (This second result is not important to the real-life situation.) The group testing method was used in World War II with a savings of 80% over the individual testing method, but not with the given value of q.



Chapter 8

TECHNIQUES OF INTEGRATION

OVERVIEW The Fundamental Theorem connects antiderivatives and the definite integral. Evaluating the indefinite integral

$$\int f(x)\,dx$$

is equivalent to finding a function F such that $F'(x) = f(x)$, and then adding an arbitrary constant C:

$$\int f(x)\,dx = F(x) + C.$$

In this chapter we study a number of important techniques for finding indefinite integrals of more complicated functions than those seen before. The goal of this chapter is to show how to change unfamiliar integrals into integrals we can recognize, find in a table, or evaluate with a computer. We also extend the idea of the definite integral to *improper integrals* for which the integrand may be unbounded over the interval of integration, or the interval itself may no longer be finite.

8.1 Basic Integration Formulas

To help us in the search for finding indefinite integrals, it is useful to build up a table of integral formulas by inverting formulas for derivatives, as we have done in previous chapters. Then we try to match any integral that confronts us against one of the standard types. This usually involves a certain amount of algebraic manipulation as well as use of the Substitution Rule.

Recall the Substitution Rule from Section 5.5:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I. Success in integration often hinges on the ability to spot what part of the integrand should be called u in order that one will also have du, so that a known formula can be applied. This means that the first requirement for skill in integration is a thorough mastery of the formulas for differentiation.

Table 8.1 shows the basic forms of integrals we have evaluated so far. In this section we present several algebraic or substitution methods to help us use this table. There is a more extensive table at the back of the book; we discuss its use in Section 8.6.

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\, du = ku + C \qquad \text{(any number } k\text{)}$
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\, du = \dfrac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \dfrac{du}{u} = \ln |u| + C$
6. $\int \sin u\, du = -\cos u + C$
7. $\int \cos u\, du = \sin u + C$
8. $\int \sec^2 u\, du = \tan u + C$
9. $\int \csc^2 u\, du = -\cot u + C$
10. $\int \sec u \tan u\, du = \sec u + C$
11. $\int \csc u \cot u\, du = -\csc u + C$
12. $\int \tan u\, du = -\ln |\cos u| + C$ $= \ln |\sec u| + C$
13. $\int \cot u\, du = \ln |\sin u| + C$ $= -\ln |\csc u| + C$
14. $\int e^u\, du = e^u + C$
15. $\int a^u\, du = \dfrac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\, du = \cosh u + C$
17. $\int \cosh u\, du = \sinh u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$
21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (u > a > 0)$

We often have to rewrite an integral to match it to a standard formula.

EXAMPLE 1 Making a Simplifying Substitution

Evaluate

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\, dx.$$

Solution

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx = \int \frac{du}{\sqrt{u}} \qquad u = x^2 - 9x + 1,\ du = (2x - 9)\,dx.$$

$$= \int u^{-1/2}\,du$$

$$= \frac{u^{(-1/2)+1}}{(-1/2) + 1} + C \qquad \text{Table 8.1 Formula 4, with } n = -1/2$$

$$= 2u^{1/2} + C$$

$$= 2\sqrt{x^2 - 9x + 1} + C$$

■

EXAMPLE 2 Completing the Square

Evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$8x - x^2 = -(x^2 - 8x) = -(x^2 - 8x + 16 - 16)$$
$$= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2.$$

Then

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}} \qquad a = 4,\ u = (x - 4),\ du = dx$$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 8.1, Formula 18}$$

$$= \sin^{-1}\left(\frac{x - 4}{4}\right) + C.$$

■

EXAMPLE 3 Expanding a Power and Using a Trigonometric Identity

Evaluate

$$\int (\sec x + \tan x)^2\,dx.$$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are familiar; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned}\int (\sec x + \tan x)^2\,dx &= \int (\sec^2 x + 2\sec x \tan x + \sec^2 x - 1)\,dx \\ &= 2\int \sec^2 x\,dx + 2\int \sec x \tan x\,dx - \int 1\,dx \\ &= 2\tan x + 2\sec x - x + C.\end{aligned}$$

■

EXAMPLE 4 Eliminating a Square Root

You Try It

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx.$$

Solution We use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Hence,

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx &= \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\,dx \\ &= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\,dx && \sqrt{u^2} = |u| \\ &= \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx && \text{On } [0, \pi/4],\ \cos 2x \ge 0, \text{ so } |\cos 2x| = \cos 2x. \\ &= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} && \text{Table 8.1, Formula 7, with } u = 2x \text{ and } du = 2\,dx \\ &= \sqrt{2}\left[\frac{1}{2} - 0\right] = \frac{\sqrt{2}}{2}.\end{aligned}$$

■

EXAMPLE 5 Reducing an Improper Fraction

Video

You Try It

Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx.$$

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\begin{array}{r} x - 3 \\ 3x + 2\,\overline{)\,3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ +\,6 \end{array}$$

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx = \int \left(x - 3 + \frac{6}{3x + 2}\right)dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C.$$

■

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

EXAMPLE 6 Separating a Fraction

Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = 3\int \frac{x\, dx}{\sqrt{1 - x^2}} + 2\int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\, dx, \qquad \text{and} \qquad x\, dx = -\frac{1}{2}\, du.$$

$$\begin{aligned} 3\int \frac{x\, dx}{\sqrt{1 - x^2}} &= 3\int \frac{(-1/2)\, du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\, du \\ &= -\frac{3}{2}\cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1 \end{aligned}$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1 - x^2}} = 2 \sin^{-1} x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = -3\sqrt{1 - x^2} + 2 \sin^{-1} x + C.$$

■

The final example of this section calculates an important integral by the algebraic technique of multiplying the integrand by a form of 1 to change the integrand into one we can integrate.

HISTORICAL BIOGRAPHY

George David Birkhoff (1884–1944)

EXAMPLE 7 Integral of $y = \sec x$—Multiplying by a Form of 1

Evaluate

$$\int \sec x\, dx.$$

Solution

$$\begin{aligned} \int \sec x\, dx &= \int (\sec x)(1)\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx \\ &= \int \frac{du}{u} \qquad\qquad u = \tan x + \sec x,\ du = (\sec^2 x + \sec x \tan x)\, dx \\ &= \ln |u| + C = \ln |\sec x + \tan x| + C. \end{aligned}$$

■

With cosecants and cotangents in place of secants and tangents, the method of Example 7 leads to a companion formula for the integral of the cosecant (see Exercise 95).

TABLE 8.2 The secant and cosecant integrals

1. $\displaystyle\int \sec u\, du = \ln|\sec u + \tan u| + C$

2. $\displaystyle\int \csc u\, du = -\ln|\csc u + \cot u| + C$

Procedures for Matching Integrals to Basic Formulas

PROCEDURE	EXAMPLE
Making a simplifying substitution	$\dfrac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x\tan x + \tan^2 x$ $= \sec^2 x + 2\sec x\tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x\tan x - 1$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,|\cos 2x|$
Reducing an improper fraction	$\dfrac{3x^2-7x}{3x+2} = x - 3 + \dfrac{6}{3x+2}$
Separating a fraction	$\dfrac{3x+2}{\sqrt{1-x^2}} = \dfrac{3x}{\sqrt{1-x^2}} + \dfrac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\sec x = \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}$ $= \dfrac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}$

EXERCISES 8.1

Basic Substitutions

Evaluate each integral in Exercises 1–36 by using a substitution to reduce it to standard form.

1. $\displaystyle\int \frac{16x\,dx}{\sqrt{8x^2+1}}$

2. $\displaystyle\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$

3. $\displaystyle\int 3\sqrt{\sin v}\cos v\,dv$

4. $\displaystyle\int \cot^3 y\csc^2 y\,dy$

5. $\displaystyle\int_0^1 \frac{16x\,dx}{8x^2+2}$

6. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z}\,dz$

7. $\displaystyle\int \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)}$

8. $\displaystyle\int \frac{dx}{x-\sqrt{x}}$

9. $\displaystyle\int \cot(3-7x)\,dx$

10. $\displaystyle\int \csc(\pi x-1)\,dx$

11. $\displaystyle\int e^{\theta}\csc(e^{\theta}+1)\,d\theta$

12. $\displaystyle\int \frac{\cot(3+\ln x)}{x}\,dx$

13. $\displaystyle\int \sec\frac{t}{3}\,dt$

14. $\displaystyle\int x\sec(x^2-5)\,dx$

15. $\displaystyle\int \csc(s-\pi)\,ds$

16. $\displaystyle\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta$

17. $\displaystyle\int_0^{\sqrt{\ln 2}} 2x\,e^{x^2}\,dx$

18. $\displaystyle\int_{\pi/2}^{\pi} (\sin y)e^{\cos y}\,dy$

19. $\displaystyle\int e^{\tan v}\sec^2 v\,dv$

20. $\displaystyle\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}}$

21. $\displaystyle\int 3^{x+1}\,dx$

22. $\displaystyle\int \frac{2^{\ln x}}{x}\,dx$

23. $\displaystyle\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}$

24. $\displaystyle\int 10^{2\theta}\,d\theta$

25. $\displaystyle\int \frac{9\,du}{1+9u^2}$

26. $\displaystyle\int \frac{4\,dx}{1+(2x+1)^2}$

27. $\displaystyle\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}}$

28. $\displaystyle\int_0^{1} \frac{dt}{\sqrt{4-t^2}}$

29. $\displaystyle\int \frac{2s\,ds}{\sqrt{1-s^4}}$

30. $\displaystyle\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$

31. $\displaystyle\int \frac{6\,dx}{x\sqrt{25x^2-1}}$

32. $\displaystyle\int \frac{dr}{r\sqrt{r^2-9}}$

33. $\displaystyle\int \frac{dx}{e^x+e^{-x}}$

34. $\displaystyle\int \frac{dy}{\sqrt{e^{2y}-1}}$

35. $\displaystyle\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}$

36. $\displaystyle\int \frac{\ln x\,dx}{x+4x\ln^2 x}$

Completing the Square

Evaluate each integral in Exercises 37–42 by completing the square and using a substitution to reduce it to standard form.

37. $\displaystyle\int_1^{2} \frac{8\,dx}{x^2-2x+2}$

38. $\displaystyle\int_2^{4} \frac{2\,dx}{x^2-6x+10}$

39. $\displaystyle\int \frac{dt}{\sqrt{-t^2+4t-3}}$

40. $\displaystyle\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$

41. $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

42. $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Trigonometric Identities

Evaluate each integral in Exercises 43–46 by using trigonometric identities and substitutions to reduce it to standard form.

43. $\displaystyle\int (\sec x+\cot x)^2\,dx$

44. $\displaystyle\int (\csc x-\tan x)^2\,dx$

45. $\displaystyle\int \csc x\sin 3x\,dx$

46. $\displaystyle\int (\sin 3x\cos 2x-\cos 3x\sin 2x)\,dx$

Improper Fractions

Evaluate each integral in Exercises 47–52 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

47. $\displaystyle\int \frac{x}{x+1}\,dx$

48. $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

49. $\displaystyle\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\,dx$

50. $\displaystyle\int_{-1}^{3} \frac{4x^2-7}{2x+3}\,dx$

51. $\displaystyle\int \frac{4t^3-t^2+16t}{t^2+4}\,dt$

52. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

Separating Fractions

Evaluate each integral in Exercises 53–56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

53. $\displaystyle\int \frac{1-x}{\sqrt{1-x^2}}\,dx$

54. $\displaystyle\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

55. $\displaystyle\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx$

56. $\displaystyle\int_0^{1/2} \frac{2-8x}{1+4x^2}\,dx$

Multiplying by a Form of 1

Evaluate each integral in Exercises 57–62 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

57. $\displaystyle\int \frac{1}{1+\sin x}\,dx$

58. $\displaystyle\int \frac{1}{1+\cos x}\,dx$

59. $\displaystyle\int \frac{1}{\sec\theta+\tan\theta}\,d\theta$

60. $\displaystyle\int \frac{1}{\csc\theta+\cot\theta}\,d\theta$

61. $\displaystyle\int \frac{1}{1-\sec x}\,dx$

62. $\displaystyle\int \frac{1}{1-\csc x}\,dx$

Eliminating Square Roots

Evaluate each integral in Exercises 63–70 by eliminating the square root.

63. $\displaystyle\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx$

64. $\displaystyle\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$