7.4 a^x and $\log_a x$

We have defined general exponential functions such as 2^x, 10^x, and π^x. In this section we compute their derivatives and integrals. We also define the general logarithmic functions such as $\log_2 x$, $\log_{10} x$, and $\log_\pi x$, and find their derivatives and integrals as well.

The Derivative of a^u

We start with the definition $a^x = e^{x \ln a}$:

$$\begin{aligned}\frac{d}{dx} a^x &= \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \frac{d}{dx} e^u = e^u \frac{du}{dx} \\ &= a^x \ln a.\end{aligned}$$

If $a > 0$, then

$$\frac{d}{dx} a^x = a^x \ln a.$$

With the Chain Rule, we get a more general form.

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}. \tag{1}$$

These equations show why e^x is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx} e^x = e^x \ln e = e^x.$$

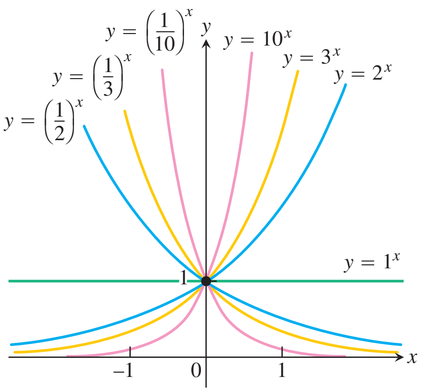


FIGURE 7.12 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

EXAMPLE 1 Differentiating General Exponential Functions

(a) $\frac{d}{dx} 3^x = 3^x \ln 3$

(b) $\frac{d}{dx} 3^{-x} = 3^{-x}(\ln 3) \frac{d}{dx}(-x) = -3^{-x} \ln 3$

(c) $\frac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3) \frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3) \cos x$ ■

From Equation (1), we see that the derivative of a^x is positive if $\ln a > 0$, or $a > 1$, and negative if $\ln a < 0$, or $0 < a < 1$. Thus, a^x is an increasing function of x if $a > 1$ and a decreasing function of x if $0 < a < 1$. In each case, a^x is one-to-one. The second derivative

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}(a^x \ln a) = (\ln a)^2 a^x$$

is positive for all x, so the graph of a^x is concave up on every interval of the real line (Figure 7.12).

Other Power Functions

The ability to raise positive numbers to arbitrary real powers makes it possible to define functions like x^x and $x^{\ln x}$ for $x > 0$. We find the derivatives of such functions by rewriting the functions as powers of e.

EXAMPLE 2 Differentiating a General Power Function

Find dy/dx if $y = x^x$, $x > 0$.

Solution Write x^x as a power of e:

$$y = x^x = e^{x \ln x}. \qquad a^x \text{ with } a = x.$$

Then differentiate as usual:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} e^{x \ln x} \\ &= e^{x \ln x} \frac{d}{dx}(x \ln x) \\ &= x^x \left(x \cdot \frac{1}{x} + \ln x \right) \\ &= x^x (1 + \ln x). \end{aligned}$$ ■

The Integral of a^u

If $a \neq 1$, so that $\ln a \neq 0$, we can divide both sides of Equation (1) by $\ln a$ to obtain

$$a^u \frac{du}{dx} = \frac{1}{\ln a} \frac{d}{dx}(a^u).$$

Integrating with respect to x then gives

$$\int a^u \frac{du}{dx}\,dx = \int \frac{1}{\ln a}\frac{d}{dx}(a^u)\,dx = \frac{1}{\ln a}\int \frac{d}{dx}(a^u)\,dx = \frac{1}{\ln a}a^u + C.$$

Writing the first integral in differential form gives

$$\int a^u\,du = \frac{a^u}{\ln a} + C. \tag{2}$$

EXAMPLE 3 Integrating General Exponential Functions

You Try It

(a) $\int 2^x\,dx = \frac{2^x}{\ln 2} + C$ — Eq. (2) with $a = 2$, $u = x$

(b) $\int 2^{\sin x} \cos x\,dx$

$= \int 2^u\,du = \frac{2^u}{\ln 2} + C$ — $u = \sin x$, $du = \cos x\,dx$, and Eq. (2)

$= \frac{2^{\sin x}}{\ln 2} + C$ — u replaced by $\sin x$ ■

Logarithms with Base *a*

As we saw earlier, if a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the **logarithm of** $\boldsymbol{x}$ **with base** $\boldsymbol{a}$ and denote it by $\boldsymbol{\log_a x}$.

DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$\log_a x$ is the inverse function of a^x.

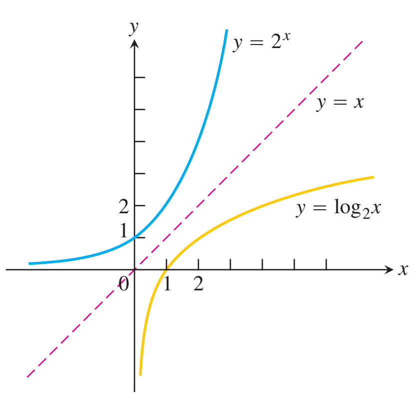


FIGURE 7.13 The graph of 2^x and its inverse, $\log_2 x$.

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the 45° line $y = x$ (Figure 7.13). When $a = e$, we have $\log_e x =$ inverse of $e^x = \ln x$. Since $\log_a x$ and a^x are inverses of one another, composing them in either order gives the identity function.

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \tag{3}$$

$$\log_a(a^x) = x \qquad (\text{all } x) \tag{4}$$

EXAMPLE 4 Applying the Inverse Equations

(a) $\log_2(2^5) = 5$ **(b)** $\log_{10}(10^{-7}) = -7$

(c) $2^{\log_2(3)} = 3$ **(d)** $10^{\log_{10}(4)} = 4$ ■

Evaluation of $\log_a x$

The evaluation of $\log_a x$ is simplified by the observation that $\log_a x$ is a numerical multiple of $\ln x$.

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \qquad (5)$$

We can derive this equation from Equation (3):

$$a^{\log_a(x)} = x \qquad \text{Eq. (3)}$$

$$\ln a^{\log_a(x)} = \ln x \qquad \text{Take the natural logarithm of both sides.}$$

$$\log_a(x) \cdot \ln a = \ln x \qquad \text{The Power Rule in Theorem 2}$$

$$\log_a x = \frac{\ln x}{\ln a} \qquad \text{Solve for } \log_a x.$$

For example,

$$\log_{10} 2 = \frac{\ln 2}{\ln 10} \approx \frac{0.69315}{2.30259} \approx 0.30103$$

The arithmetic rules satisfied by $\log_a x$ are the same as the ones for $\ln x$ (Theorem 2). These rules, given in Table 7.2, can be proved by dividing the corresponding rules for the natural logarithm function by $\ln a$. For example,

$$\ln xy = \ln x + \ln y \qquad \text{Rule 1 for natural logarithms} \ldots$$

$$\frac{\ln xy}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} \qquad \ldots \text{divided by } \ln a \ldots$$

$$\log_a xy = \log_a x + \log_a y. \qquad \ldots \text{gives Rule 1 for base } a \text{ logarithms.}$$

Derivatives and Integrals Involving $\log_a x$

To find derivatives or integrals involving base a logarithms, we convert them to natural logarithms.

If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$$

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*
   $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:*
   $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:*
   $\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:*
   $\log_a x^y = y \log_a x$

EXAMPLE 5

(a) $$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$$

(b) $$\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x, \quad du = \frac{1}{x}dx$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$ ■

Base 10 Logarithms

Base 10 logarithms, often called **common logarithms**, appear in many scientific formulas. For example, earthquake intensity is often reported on the logarithmic **Richter scale**. Here the formula is

$$\text{Magnitude } R = \log_{10}\left(\frac{a}{T}\right) + B,$$

where a is the amplitude of the ground motion in microns at the receiving station, T is the period of the seismic wave in seconds, and B is an empirical factor that accounts for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.

EXAMPLE 6 Earthquake Intensity

For an earthquake 10,000 km from the receiving station, $B = 6.8$. If the recorded vertical ground motion is $a = 10$ microns and the period is $T = 1$ sec, the earthquake's magnitude is

$$R = \log_{10}\left(\frac{10}{1}\right) + 6.8 = 1 + 6.8 = 7.8.$$

An earthquake of this magnitude can do great damage near its epicenter. ■

The **pH scale** for measuring the acidity of a solution is a base 10 logarithmic scale. The pH value (hydrogen potential) of the solution is the common logarithm of the reciprocal of the solution's hydronium ion concentration, $[H_3O^+]$:

$$\text{pH} = \log_{10}\frac{1}{[H_3O^+]} = -\log_{10}[H_3O^+].$$

Most foods are acidic (pH < 7).

Food	pH Value
Bananas	4.5–4.7
Grapefruit	3.0–3.3
Oranges	3.0–4.0
Limes	1.8–2.0
Milk	6.3–6.6
Soft drinks	2.0–4.0
Spinach	5.1–5.7

The hydronium ion concentration is measured in moles per liter. Vinegar has a pH of three, distilled water a pH of 7, seawater a pH of 8.15, and household ammonia a pH of 12. The total scale ranges from about 0.1 for normal hydrochloric acid to 14 for a normal solution of sodium hydroxide.

Another example of the use of common logarithms is the **decibel** or dB ("dee bee") **scale** for measuring loudness. If I is the **intensity** of sound in watts per square meter, the decibel level of the sound is

$$\textbf{Sound level} = 10\log_{10}(I \times 10^{12})\text{ dB}. \qquad (6)$$

Typical sound levels

Threshold of hearing	0 dB
Rustle of leaves	10 dB
Average whisper	20 dB
Quiet automobile	50 dB
Ordinary conversation	65 dB
Pneumatic drill 10 feet away	90 dB
Threshold of pain	120 dB

If you ever wondered why doubling the power of your audio amplifier increases the sound level by only a few decibels, Equation (6) provides the answer. As the following example shows, doubling I adds only about 3 dB.

EXAMPLE 7 Sound Intensity

Doubling I in Equation (6) adds about 3 dB. Writing log for $\log_{10}$ (a common practice), we have

$$\begin{aligned}
\text{Sound level with } I \text{ doubled} &= 10 \log (2I \times 10^{12}) && \text{Eq. (6) with } 2I \text{ for } I \\
&= 10 \log (2 \cdot I \times 10^{12}) \\
&= 10 \log 2 + 10 \log (I \times 10^{12}) \\
&= \text{original sound level} + 10 \log 2 \\
&\approx \text{original sound level} + 3. && \log_{10} 2 \approx 0.30
\end{aligned}$$

■

EXERCISES 7.4

Algebraic Calculations With a^x and $\log_a x$

Simplify the expressions in Exercises 1–4.

1. **a.** $5^{\log_5 7}$ **b.** $8^{\log_8 \sqrt{2}}$ **c.** $1.3^{\log_{1.3} 75}$

d. $\log_4 16$ **e.** $\log_3 \sqrt{3}$ **f.** $\log_4 \left(\frac{1}{4}\right)$

2. **a.** $2^{\log_2 3}$ **b.** $10^{\log_{10}(1/2)}$ **c.** $\pi^{\log_\pi 7}$

d. $\log_{11} 121$ **e.** $\log_{121} 11$ **f.** $\log_3 \left(\frac{1}{9}\right)$

3. **a.** $2^{\log_4 x}$ **b.** $9^{\log_3 x}$ **c.** $\log_2\left(e^{(\ln 2)(\sin x)}\right)$

4. **a.** $25^{\log_5(3x^2)}$ **b.** $\log_e(e^x)$ **c.** $\log_4\left(2^{e^x \sin x}\right)$

Express the ratios in Exercises 5 and 6 as ratios of natural logarithms and simplify.

5. **a.** $\dfrac{\log_2 x}{\log_3 x}$ **b.** $\dfrac{\log_2 x}{\log_8 x}$ **c.** $\dfrac{\log_x a}{\log_{x^2} a}$

6. **a.** $\dfrac{\log_9 x}{\log_3 x}$ **b.** $\dfrac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$ **c.** $\dfrac{\log_a b}{\log_b a}$

Solve the equations in Exercises 7–10 for x.

7. $3^{\log_3(7)} + 2^{\log_2(5)} = 5^{\log_5(x)}$

8. $8^{\log_8(3)} - e^{\ln 5} = x^2 - 7^{\log_7(3x)}$

9. $3^{\log_3(x^2)} = 5e^{\ln x} - 3 \cdot 10^{\log_{10}(2)}$

10. $\ln e + 4^{-2\log_4(x)} = \frac{1}{x}\log_{10}(100)$

Derivatives

In Exercises 11–38, find the derivative of y with respect to the given independent variable.

11. $y = 2^x$

12. $y = 3^{-x}$

13. $y = 5^{\sqrt{s}}$

14. $y = 2^{(s^2)}$

15. $y = x^\pi$

16. $y = t^{1-e}$

17. $y = (\cos\theta)^{\sqrt{2}}$

18. $y = (\ln\theta)^\pi$

19. $y = 7^{\sec\theta}\ln 7$

20. $y = 3^{\tan\theta}\ln 3$

21. $y = 2^{\sin 3t}$

22. $y = 5^{-\cos 2t}$

23. $y = \log_2 5\theta$

24. $y = \log_3(1 + \theta\ln 3)$

25. $y = \log_4 x + \log_4 x^2$

26. $y = \log_{25} e^x - \log_5\sqrt{x}$

27. $y = \log_2 r \cdot \log_4 r$

28. $y = \log_3 r \cdot \log_9 r$

29. $y = \log_3\left(\left(\dfrac{x+1}{x-1}\right)^{\ln 3}\right)$

30. $y = \log_5\sqrt{\left(\dfrac{7x}{3x+2}\right)^{\ln 5}}$

31. $y = \theta\sin(\log_7\theta)$

32. $y = \log_7\left(\dfrac{\sin\theta\cos\theta}{e^\theta 2^\theta}\right)$

33. $y = \log_5 e^x$

34. $y = \log_2\left(\dfrac{x^2e^2}{2\sqrt{x+1}}\right)$

35. $y = 3^{\log_2 t}$

36. $y = 3\log_8(\log_2 t)$

37. $y = \log_2(8t^{\ln 2})$

38. $y = t\log_3\left(e^{(\sin t)(\ln 3)}\right)$

Logarithmic Differentiation

In Exercises 39–46, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

39. $y = (x+1)^x$

40. $y = x^{(x+1)}$

41. $y = (\sqrt{t})^t$

42. $y = t^{\sqrt{t}}$

43. $y = (\sin x)^x$

44. $y = x^{\sin x}$

45. $y = x^{\ln x}$

46. $y = (\ln x)^{\ln x}$

Integration

Evaluate the integrals in Exercises 47–56.

47. $\displaystyle\int 5^x\,dx$

48. $\displaystyle\int (1.3)^x\,dx$

49. $\int_0^1 2^{-\theta}\, d\theta$ **50.** $\int_{-2}^0 5^{-\theta}\, d\theta$

51. $\int_1^{\sqrt{2}} x2^{(x^2)}\, dx$ **52.** $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\, dx$

53. $\int_0^{\pi/2} 7^{\cos t} \sin t\, dt$ **54.** $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t\, dt$

55. $\int_2^4 x^{2x}(1 + \ln x)\, dx$ **56.** $\int_1^2 \frac{2^{\ln x}}{x}\, dx$

Evaluate the integrals in Exercises 57–60.

57. $\int 3x^{\sqrt{3}}\, dx$ **58.** $\int x^{\sqrt{2}-1}\, dx$

59. $\int_0^3 (\sqrt{2} + 1)x^{\sqrt{2}}\, dx$ **60.** $\int_1^e x^{(\ln 2)-1}\, dx$

Evaluate the integrals in Exercises 61–70.

61. $\int \frac{\log_{10} x}{x}\, dx$ **62.** $\int_1^4 \frac{\log_2 x}{x}\, dx$

63. $\int_1^4 \frac{\ln 2 \log_2 x}{x}\, dx$ **64.** $\int_1^e \frac{2 \ln 10 \log_{10} x}{x}\, dx$

65. $\int_0^2 \frac{\log_2 (x + 2)}{x + 2}\, dx$ **66.** $\int_{1/10}^{10} \frac{\log_{10} (10x)}{x}\, dx$

67. $\int_0^9 \frac{2 \log_{10} (x + 1)}{x + 1}\, dx$ **68.** $\int_2^3 \frac{2 \log_2 (x - 1)}{x - 1}\, dx$

69. $\int \frac{dx}{x \log_{10} x}$ **70.** $\int \frac{dx}{x(\log_8 x)^2}$

Evaluate the integrals in Exercises 71–74.

71. $\int_1^{\ln x} \frac{1}{t}\, dt, \quad x > 1$ **72.** $\int_1^{e^x} \frac{1}{t}\, dt$

73. $\int_1^{1/x} \frac{1}{t}\, dt, \quad x > 0$ **74.** $\frac{1}{\ln a} \int_1^x \frac{1}{t}\, dt, \quad x > 0$

Theory and Applications

75. Find the area of the region between the curve $y = 2x/(1 + x^2)$ and the interval $-2 \le x \le 2$ of the x-axis.

76. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \le x \le 1$ of the x-axis.

77. Blood pH The pH of human blood normally falls between 7.37 and 7.44. Find the corresponding bounds for $[H_3O^+]$.

78. Brain fluid pH The cerebrospinal fluid in the brain has a hydronium ion concentration of about $[H_3O^+] = 4.8 \times 10^{-8}$ moles per liter. What is the pH?

79. Audio amplifiers By what factor k do you have to multiply the intensity of I of the sound from your audio amplifier to add 10 dB to the sound level?

80. Audio amplifiers You multiplied the intensity of the sound of your audio system by a factor of 10. By how many decibels did this increase the sound level?

81. In any solution, the product of the hydronium ion concentration $[H_3O^+]$ (moles/L) and the hydroxyl ion concentration $[OH^-]$ (moles/L) is about 10^{-14}.

a. What value of $[H_3O^+]$ minimizes the sum of the concentrations, $S = [H_3O^+] + [OH^-]$? (*Hint:* Change notation. Let $x = [H_3O^+]$.)

b. What is the pH of a solution in which S has this minimum value?

c. What ratio of $[H_3O^+]$ to $[OH^-]$ minimizes S?

82. Could $\log_a b$ possibly equal $1/\log_b a$? Give reasons for your answer.

T **83.** The equation $x^2 = 2^x$ has three solutions: $x = 2, x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

T **84.** Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for $x > 0$? Graph the two functions and explain what you see.

85. The linearization of 2^x

a. Find the linearization of $f(x) = 2^x$ at $x = 0$. Then round its coefficients to two decimal places.

T **b.** Graph the linearization and function together for $-3 \le x \le 3$ and $-1 \le x \le 1$.

86. The linearization of $\log_3 x$

a. Find the linearization of $f(x) = \log_3 x$ at $x = 3$. Then round its coefficients to two decimal places.

T **b.** Graph the linearization and function together in the window $0 \le x \le 8$ and $2 \le x \le 4$.

Calculations with Other Bases

T **87.** Most scientific calculators have keys for $\log_{10} x$ and $\ln x$. To find logarithms to other bases, we use the Equation (5), $\log_a x = (\ln x)/(\ln a)$.

Find the following logarithms to five decimal places.

a. $\log_3 8$ **b.** $\log_7 0.5$

c. $\log_{20} 17$ **d.** $\log_{0.5} 7$

e. $\ln x$, given that $\log_{10} x = 2.3$

f. $\ln x$, given that $\log_2 x = 1.4$

g. $\ln x$, given that $\log_2 x = -1.5$

h. $\ln x$, given that $\log_{10} x = -0.7$

88. Conversion factors

a. Show that the equation for converting base 10 logarithms to base 2 logarithms is

$$\log_2 x = \frac{\ln 10}{\ln 2} \log_{10} x.$$

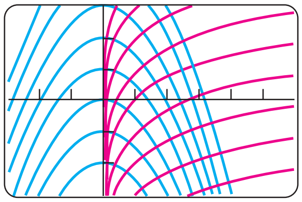
b. Show that the equation for converting base a logarithms to base b logarithms is

$$\log_b x = \frac{\ln a}{\ln b} \log_a x.$$

89. Orthogonal families of curves Prove that all curves in the family

$$y = -\frac{1}{2}x^2 + k$$

(k any constant) are perpendicular to all curves in the family $y = \ln x + c$ (c any constant) at their points of intersection. (See the accompanying figure.)



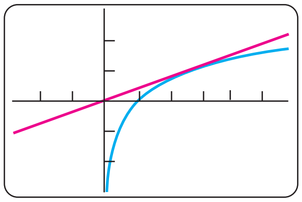
T 90. The inverse relation between e^x and ln x Find out how good your calculator is at evaluating the composites

$$e^{\ln x} \quad \text{and} \quad \ln(e^x).$$

T 91. A decimal representation of e Find e to as many decimal places as your calculator allows by solving the equation $\ln x = 1$.

T 92. Which is bigger, π^e or e^π? Calculators have taken some of the mystery out of this once-challenging question. (Go ahead and check; you will see that it is a surprisingly close call.) You can answer the question without a calculator, though.

a. Find an equation for the line through the origin tangent to the graph of $y = \ln x$.



[–3, 6] by [–3, 3]

b. Give an argument based on the graphs of $y = \ln x$ and the tangent line to explain why $\ln x < x/e$ for all positive $x \neq e$.

c. Show that $\ln(x^e) < x$ for all positive $x \neq e$.

d. Conclude that $x^e < e^x$ for all positive $x \neq e$.

e. So which is bigger, π^e or e^π?

7.5 Exponential Growth and Decay

Exponential functions increase or decrease very rapidly with changes in the independent variable. They describe growth or decay in a wide variety of natural and industrial situations. The variety of models based on these functions partly accounts for their importance.

The Law of Exponential Change

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t. Examples of such quantities include the amount of a decaying radioactive material, funds earning interest in a bank account, the size of a population, and the temperature difference between a hot cup of coffee and the room in which it sits. Such quantities change according to the *law of exponential change*, which we derive in this section.

If the amount present at time $t = 0$ is called y_0, then we can find y as a function of t by solving the following initial value problem:

$$\begin{aligned} &\text{Differential equation:} \quad && \frac{dy}{dt} = ky \\ &\text{Initial condition:} \quad && y = y_0 \quad \text{when} \quad t = 0. \end{aligned} \tag{1}$$

If y is positive and increasing, then k is positive, and we use Equation (1) to say that the rate of growth is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative, and we use Equation (1) to say that the rate of decay is proportional to the amount still left.

We see right away that the constant function $y = 0$ is a solution of Equation (1) if $y_0 = 0$. To find the nonzero solutions, we divide Equation (1) by y:

$$\begin{aligned}
\frac{1}{y} \cdot \frac{dy}{dt} &= k \\
\int \frac{1}{y}\frac{dy}{dt}\,dt &= \int k\,dt && \text{Integrate with respect to } t; \\
\ln |y| &= kt + C && \textstyle\int (1/u)\,du = \ln |u| + C. \\
|y| &= e^{kt+C} && \text{Exponentiate.} \\
|y| &= e^{C} \cdot e^{kt} && e^{a+b} = e^{a} \cdot e^{b} \\
y &= \pm e^{C} e^{kt} && \text{If } |y| = r, \text{ then } y = \pm r. \\
y &= A e^{kt}. && A \text{ is a shorter name for } \pm e^{C}.
\end{aligned}$$

By allowing A to take on the value 0 in addition to all possible values $\pm e^{C}$, we can include the solution $y = 0$ in the formula.

We find the value of A for the initial value problem by solving for A when $y = y_0$ and $t = 0$:

$$y_0 = A e^{k \cdot 0} = A.$$

The solution of the initial value problem is therefore $y = y_0 e^{kt}$.

Quantities changing in this way are said to undergo **exponential growth** if $k > 0$, and **exponential decay** if $k < 0$.

Animation

The Law of Exponential Change

$$y = y_0 e^{kt} \tag{2}$$

$$\text{Growth:} \quad k > 0 \qquad \text{Decay:} \quad k < 0$$

The number k is the **rate constant** of the equation.

The derivation of Equation (2) shows that the only functions that are their own derivatives are constant multiples of the exponential function.

Unlimited Population Growth

Strictly speaking, the number of individuals in a population (of people, plants, foxes, or bacteria, for example) is a discontinuous function of time because it takes on discrete values. However, when the number of individuals becomes large enough, the population can be approximated by a continuous function. Differentiability of the approximating function is another reasonable hypothesis in many settings, allowing for the use of calculus to model and predict population sizes.

If we assume that the proportion of reproducing individuals remains constant and assume a constant fertility, then at any instant t the birth rate is proportional to the number $y(t)$ of individuals present. Let's assume, too, that the death rate of the population is stable and proportional to $y(t)$. If, further, we neglect departures and arrivals, the growth rate

dy/dt is the birth rate minus the death rate, which is the difference of the two proportionalities under our assumptions. In other words, $dy/dt = ky$, so that $y = y_0 e^{kt}$, where y_0 is the size of the population at time $t = 0$. As with all kinds of growth, there may be limitations imposed by the surrounding environment, but we will not go into these here. (This situation is analyzed in Section 9.5.)

In the following example we assume this population model to look at how the number of individuals infected by a disease within a given population decreases as the disease is appropriately treated.

EXAMPLE 1 Reducing the Cases of an Infectious Disease

You Try It

One model for the way diseases die out when properly treated assumes that the rate dy/dt at which the number of infected people changes is proportional to the number y. The number of people cured is proportional to the number that have the disease. Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?

Solution We use the equation $y = y_0 e^{kt}$. There are three things to find: the value of y_0, the value of k, and the time t when $y = 1000$.

The value of y_0. We are free to count time beginning anywhere we want. If we count from today, then $y = 10{,}000$ when $t = 0$, so $y_0 = 10{,}000$. Our equation is now

$$y = 10{,}000 e^{kt}. \tag{3}$$

The value of k. When $t = 1$ year, the number of cases will be 80% of its present value, or 8000. Hence,

$$\begin{aligned} 8000 &= 10{,}000 e^{k(1)} && \text{Eq. (3) with } t = 1 \text{ and } y = 8000 \\ e^k &= 0.8 \\ \ln(e^k) &= \ln 0.8 && \text{Logs of both sides} \\ k &= \ln 0.8 < 0. \end{aligned}$$

At any given time t,

$$y = 10{,}000 e^{(\ln 0.8)t}. \tag{4}$$

The value of t *that makes* $y = 1000$. We set y equal to 1000 in Equation (4) and solve for t:

$$\begin{aligned} 1000 &= 10{,}000 e^{(\ln 0.8)t} \\ e^{(\ln 0.8)t} &= 0.1 \\ (\ln 0.8)t &= \ln 0.1 && \text{Logs of both sides} \\ t &= \frac{\ln 0.1}{\ln 0.8} \approx 10.32 \text{ years}. \end{aligned}$$

It will take a little more than 10 years to reduce the number of cases to 1000. ■

Continuously Compounded Interest

If you invest an amount A_0 of money at a fixed annual interest rate r (expressed as a decimal) and if interest is added to your account k times a year, the formula for the amount of money you will have at the end of t years is

$$A_t = A_0\left(1 + \frac{r}{k}\right)^{kt}. \tag{5}$$

The interest might be added ("compounded," bankers say) monthly ($k = 12$), weekly ($k = 52$), daily ($k = 365$), or even more frequently, say by the hour or by the minute. By taking the limit as interest is compounded more and more often, we arrive at the following formula for the amount after t years,

$$\begin{aligned}
\lim_{k\to\infty} A_t &= \lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} \\
&= A_0 \lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{\frac{k}{r}\cdot rt} \\
&= A_0\left[\lim_{\frac{r}{k}\to 0}\left(1+\frac{r}{k}\right)^{\frac{k}{r}}\right]^{rt} && \text{As } k\to\infty,\ \frac{r}{k}\to 0 \\
&= A_0\left[\lim_{x\to 0}(1+x)^{1/x}\right]^{rt} && \text{Substitute } x=\frac{r}{k} \\
&= A_0 e^{rt} && \text{Theorem 4}
\end{aligned}$$

The resulting formula for the amount of money in your account after t years is

$$A(t) = A_0 e^{rt}. \tag{6}$$

Interest paid according to this formula is said to be **compounded continuously**. The number r is called the **continuous interest rate.** The amount of money after t years is calculated with the law of exponential change given in Equation (6).

EXAMPLE 2 A Savings Account

Suppose you deposit \$621 in a bank account that pays 6% compounded continuously. How much money will you have 8 years later?

Solution We use Equation (6) with $A_0 = 621$, $r = 0.06$, and $t = 8$:

$$A(8) = 621e^{(0.06)(8)} = 621e^{0.48} = 1003.58 \qquad \text{Nearest cent}$$

Had the bank paid interest quarterly ($k = 4$ in Equation 5), the amount in your account would have been \$1000.01. Thus the effect of continuous compounding, as compared with quarterly compounding, has been an addition of \$3.57. A bank might decide it would be worth this additional amount to be able to advertise, "We compound interest every second, night and day—better yet, we compound the interest continuously." ■

Radioactivity

Some atoms are unstable and can spontaneously emit mass or radiation. This process is called **radioactive decay**, and an element whose atoms go spontaneously through this process is called **radioactive**. Sometimes when an atom emits some of its mass through this process of radioactivity, the remainder of the atom re-forms to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium, through a number of intermediate radioactive steps, decays into lead.

For radon-222 gas, t is measured in days and $k = 0.18$. For radium-226, which used to be painted on watch dials to make them glow at night (a dangerous practice), t is measured in years and $k = 4.3 \times 10^{-4}$.

Experiments have shown that at any given time the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. It is conventional to use

$-k(k > 0)$ here instead of $k(k < 0)$ to emphasize that y is decreasing. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0 e^{-kt}, \qquad k > 0.$$

EXAMPLE 3 Half-Life of a Radioactive Element

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. It is an interesting fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

To see why, let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0 e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$\begin{aligned} y_0 e^{-kt} &= \frac{1}{2} y_0 \\ e^{-kt} &= \frac{1}{2} \\ -kt &= \ln \frac{1}{2} = -\ln 2 && \text{Reciprocal Rule for logarithms} \\ t &= \frac{\ln 2}{k} \end{aligned}$$

This value of t is the half-life of the element. It depends only on the value of k; the number y_0 does not enter in.

$$\text{Half-life} = \frac{\ln 2}{k} \tag{7}$$

■

EXAMPLE 4 Half-Life of Polonium-210

You Try It

The effective radioactive lifetime of polonium-210 is so short we measure it in days rather than years. The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0 e^{-5 \times 10^{-3} t}.$$

Find the element's half-life.

Solution

$$\begin{aligned} \text{Half-life} &= \frac{\ln 2}{k} && \text{Eq. (7)} \\ &= \frac{\ln 2}{5 \times 10^{-3}} && \text{The } k \text{ from polonium's decay equation} \\ &\approx 139 \text{ days} \end{aligned}$$

■

EXAMPLE 5 Carbon-14 Dating

You Try It

The decay of radioactive elements can sometimes be used to date events from the Earth's past. In a living organism, the ratio of radioactive carbon, carbon-14, to ordinary carbon stays fairly constant during the lifetime of the organism, being approximately equal to the ratio in the organism's surroundings at the time. After the organism's death, however, no new carbon is ingested, and the proportion of carbon-14 in the organism's remains decreases as the carbon-14 decays.

Scientists who do carbon-14 dating use a figure of 5700 years for its half-life (more about carbon-14 dating in the exercises). Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Solution We use the decay equation $y = y_0 e^{-kt}$. There are two things to find: the value of k and the value of t when y is $0.9y_0$ (90% of the radioactive nuclei are still present). That is, find t when $y_0 e^{-kt} = 0.9y_0$, or $e^{-kt} = 0.9$.

The value of k. We use the half-life Equation (7):

$$k = \frac{\ln 2}{\text{half-life}} = \frac{\ln 2}{5700} \qquad (\text{about } 1.2 \times 10^{-4})$$

The value of t that makes $e^{-kt} = 0.9$:

$$\begin{aligned} e^{-kt} &= 0.9 \\ e^{-(\ln 2/5700)t} &= 0.9 \\ -\frac{\ln 2}{5700}t &= \ln 0.9 && \text{Logs of both sides} \\ t &= -\frac{5700 \ln 0.9}{\ln 2} \approx 866 \text{ years}. \end{aligned}$$

The sample is about 866 years old. ■

Heat Transfer: Newton's Law of Cooling

Hot soup left in a tin cup cools to the temperature of the surrounding air. A hot silver ingot immersed in a large tub of water cools to the temperature of the surrounding water. In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. This observation is called *Newton's law of cooling*, although it applies to warming as well, and there is an equation for it.

If H is the temperature of the object at time t and H_S is the constant surrounding temperature, then the differential equation is

$$\frac{dH}{dt} = -k(H - H_S). \tag{8}$$

If we substitute y for $(H - H_S)$, then

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}(H - H_S) = \frac{dH}{dt} - \frac{d}{dt}(H_S) \\ &= \frac{dH}{dt} - 0 && H_S \text{ is a constant.} \\ &= \frac{dH}{dt} \\ &= -k(H - H_S) && \text{Eq. (8)} \\ &= -ky. && H - H_S = y. \end{aligned}$$

Now we know that the solution of $dy/dt = -ky$ is $y = y_0 e^{-kt}$, where $y(0) = y_0$. Substituting $(H - H_S)$ for y, this says that

$$H - H_S = (H_0 - H_S)e^{-kt}, \tag{9}$$

where H_0 is the temperature at $t = 0$. This is the equation for Newton's Law of Cooling.

EXAMPLE 6 Cooling a Hard-Boiled Egg

A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 min, the egg's temperature is 38°C. Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach 20°C?

Solution We find how long it would take the egg to cool from 98°C to 20°C and subtract the 5 min that have already elapsed. Using Equation (9) with $H_S = 18$ and $H_0 = 98$, the egg's temperature t min after it is put in the sink is

$$H = 18 + (98 - 18)e^{-kt} = 18 + 80e^{-kt}.$$

To find k, we use the information that $H = 38$ when $t = 5$:

$$\begin{aligned}
38 &= 18 + 80e^{-5k} \\
e^{-5k} &= \frac{1}{4} \\
-5k &= \ln\frac{1}{4} = -\ln 4 \\
k &= \frac{1}{5}\ln 4 = 0.2\ln 4 \qquad (\text{about } 0.28).
\end{aligned}$$

The egg's temperature at time t is $H = 18 + 80e^{-(0.2\ln 4)t}$. Now find the time t when $H = 20$:

$$\begin{aligned}
20 &= 18 + 80e^{-(0.2\ln 4)t} \\
80e^{-(0.2\ln 4)t} &= 2 \\
e^{-(0.2\ln 4)t} &= \frac{1}{40} \\
-(0.2\ln 4)t &= \ln\frac{1}{40} = -\ln 40 \\
t &= \frac{\ln 40}{0.2\ln 4} \approx 13 \text{ min}.
\end{aligned}$$

The egg's temperature will reach 20°C about 13 min after it is put in the water to cool. Since it took 5 min to reach 38°C, it will take about 8 min more to reach 20°C. ■

EXERCISES 7.5

The answers to most of the following exercises are in terms of logarithms and exponentials. A calculator can be helpful, enabling you to express the answers in decimal form.

1. **Human evolution continues** The analysis of tooth shrinkage by C. Loring Brace and colleagues at the University of Michigan's Museum of Anthropology indicates that human tooth size is continuing to decrease and that the evolutionary process did not come to a halt some 30,000 years ago as many scientists contend. In northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

a. If t represents time in years and y represents tooth size, use the condition that $y = 0.99y_0$ when $t = 1000$ to find the value of k in the equation $y = y_0 e^{kt}$. Then use this value of k to answer the following questions.

b. In about how many years will human teeth be 90% of their present size?

c. What will be our descendants' tooth size 20,000 years from now (as a percentage of our present tooth size)?

(*Source: LSA Magazine*, Spring 1989, Vol. 12, No. 2, p. 19, Ann Arbor, MI.)

2. Atmospheric pressure The earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1013 millibars (about 14.7 pounds per square inch) and that the pressure at an altitude of 20 km is 90 millibars.

a. Solve the initial value problem

Differential equation: $dp/dh = kp$ (k a constant)
Initial condition: $p = p_0$ when $h = 0$

to express p in terms of h. Determine the values of p_0 and k from the given altitude-pressure data.

b. What is the atmospheric pressure at $h = 50$ km?

c. At what altitude does the pressure equal 900 millibars?

Exercises

3. First-order chemical reactions In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

when t is measured in hours. If there are 100 grams of δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

4. The inversion of sugar The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 hours, how much raw sugar will remain after another 14 hours?

5. Working underwater The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL.$$

As a diver, you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can you expect to work without artificial light?

6. Voltage in a discharging capacitor Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage V across its terminals and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

Solve this equation for V, using V_0 to denote the value of V when $t = 0$. How long will it take the voltage to drop to 10% of its original value?

7. Cholera bacteria Suppose that the bacteria in a colony can grow unchecked, by the law of exponential change. The colony starts with 1 bacterium and doubles every half-hour. How many bacteria will the colony contain at the end of 24 hours? (Under favorable laboratory conditions, the number of cholera bacteria can double every 30 min. In an infected person, many bacteria are destroyed, but this example helps explain why a person who feels well in the morning may be dangerously ill by evening.)

8. Growth of bacteria A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000. How many bacteria were present initially?

9. The incidence of a disease (*Continuation of Example 1.*) Suppose that in any given year the number of cases can be reduced by 25% instead of 20%.

a. How long will it take to reduce the number of cases to 1000?

b. How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1?

10. The U.S. population The Museum of Science in Boston displays a running total of the U.S. population. On May 11, 1993, the total was increasing at the rate of 1 person every 14 sec. The displayed population figure for 3:45 P.M. that day was 257,313,431.

a. Assuming exponential growth at a constant rate, find the rate constant for the population's growth (people per 365-day year).

b. At this rate, what will the U.S. population be at 3:45 P.M. Boston time on May 11, 2008?

11. Oil depletion Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

Exercises

12. Continuous price discounting To encourage buyers to place 100-unit orders, your firm's sales department applies a continuous discount that makes the unit price a function $p(x)$ of the number of units x ordered. The discount decreases the price at the rate of \$0.01 per unit ordered. The price per unit for a 100-unit order is $p(100) = \$20.09$.

a. Find $p(x)$ by solving the following initial value problem:

Differential equation: $\dfrac{dp}{dx} = -\dfrac{1}{100}p$

Initial condition: $p(100) = 20.09.$

b. Find the unit price $p(10)$ for a 10-unit order and the unit price $p(90)$ for a 90-unit order.

c. The sales department has asked you to find out if it is discounting so much that the firm's revenue, $r(x) = x \cdot p(x)$, will actually be less for a 100-unit order than, say, for a 90-unit order. Reassure them by showing that r has its maximum value at $x = 100$.

T **d.** Graph the revenue function $r(x) = xp(x)$ for $0 \le x \le 200$.

13. Continuously compounded interest You have just placed A_0 dollars in a bank account that pays 4% interest, compounded continuously.

a. How much money will you have in the account in 5 years?

b. How long will it take your money to double? To triple?

14. John Napier's question John Napier (1550–1617), the Scottish laird who invented logarithms, was the first person to answer the question, What happens if you invest an amount of money at 100% interest, compounded continuously?

a. What does happen?

b. How long does it take to triple your money?

c. How much can you earn in a year?

Give reasons for your answers.

15. Benjamin Franklin's will The Franklin Technical Institute of Boston owes its existence to a provision in a codicil to Benjamin Franklin's will. In part the codicil reads:

> I wish to be useful even after my Death, if possible, in forming and advancing other young men that may be serviceable to their Country in both Boston and Philadelphia. To this end I devote Two thousand Pounds Sterling, which I give, one thousand thereof to the Inhabitants of the Town of Boston in Massachusetts, and the other thousand to the inhabitants of the City of Philadelphia, in Trust and for the Uses, Interests and Purposes hereinafter mentioned and declared.

Franklin's plan was to lend money to young apprentices at 5% interest with the provision that each borrower should pay each year along

> . . . with the yearly Interest, one tenth part of the Principal, which sums of Principal and Interest shall be again let to fresh Borrowers. . . . If this plan is executed and succeeds as projected without interruption for one hundred Years, the Sum will then be one hundred and thirty-one thousand Pounds of which I would have the Managers of the Donation to the Inhabitants of the Town of Boston, then lay out at their discretion one hundred thousand Pounds in Public Works. . . . The remaining thirty-one thousand Pounds, I would have continued to be let out on Interest in the manner above directed for another hundred Years. . . . At the end of this second term if no unfortunate accident has prevented the operation the sum will be Four Millions and Sixty-one Thousand Pounds.

It was not always possible to find as many borrowers as Franklin had planned, but the managers of the trust did the best they could. At the end of 100 years from the reception of the Franklin gift, in January 1894, the fund had grown from 1000 pounds to almost exactly 90,000 pounds. In 100 years the original capital had multiplied about 90 times instead of the 131 times Franklin had imagined.

What rate of interest, compounded continuously for 100 years, would have multiplied Benjamin Franklin's original capital by 90?

16. (*Continuation of Exercise 15.*) In Benjamin Franklin's estimate that the original 1000 pounds would grow to 131,000 in 100 years, he was using an annual rate of 5% and compounding once each year. What rate of interest per year when compounded continuously for 100 years would multiply the original amount by 131?

17. Radon-222 The decay equation for radon-222 gas is known to be $y = y_0 e^{-0.18t}$, with t in days. About how long will it take the radon in a sealed sample of air to fall to 90% of its original value?

18. Polonium-210 The half-life of polonium is 139 days, but your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives has disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

19. The mean life of a radioactive nucleus Physicists using the radioactivity equation $y = y_0 e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon nucleus is about $1/0.18 = 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in a sample will disintegrate within three mean lifetimes, i.e., by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.

20. Californium-252 What costs \$27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope so rare that only 8 g of it have been made in the western world since its discovery by Glenn Seaborg in 1950. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per second.

a. What is the value of k in the decay equation for this isotope?

b. What is the isotope's mean life? (See Exercise 19.)

c. How long will it take 95% of a sample's radioactive nuclei to disintegrate?

21. Cooling soup Suppose that a cup of soup cooled from 90°C to 60°C after 10 min in a room whose temperature was 20°C. Use Newton's law of cooling to answer the following questions.

a. How much longer would it take the soup to cool to 35°C?

b. Instead of being left to stand in the room, the cup of 90°C soup is put in a freezer whose temperature is -15°C. How long will it take the soup to cool from 90°C to 35°C?

22. **A beam of unknown temperature** An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°F. After 10 min, the beam warmed to 35°F and after another 10 min it was 50°F. Use Newton's law of cooling to estimate the beam's initial temperature.

23. **Surrounding medium of unknown temperature** A pan of warm water (46°C) was put in a refrigerator. Ten minutes later, the water's temperature was 39°C; 10 min after that, it was 33°C. Use Newton's law of cooling to estimate how cold the refrigerator was.

24. **Silver cooling in air** The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

    a. 15 min from now?

    b. 2 hours from now?

    c. When will the silver be 10°C above room temperature?

25. **The age of Crater Lake** The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

26. **The sensitivity of carbon-14 dating to measurement** To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, consider this hypothetical situation:

    a. A fossilized bone found in central Illinois in the year A.D. 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.

    b. Repeat part (a) assuming 18% instead of 17%.

    c. Repeat part (a) assuming 16% instead of 17%.

27. **Art forgery** A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

7.6 Relative Rates of Growth

It is often important in mathematics, computer science, and engineering to compare the rates at which functions of x grow as x becomes large. Exponential functions are important in these comparisons because of their very fast growth, and logarithmic functions because of their very slow growth. In this section we introduce the *little-oh* and *big-oh* notation used to describe the results of these comparisons. We restrict our attention to functions whose values eventually become and remain positive as $x \to \infty$.

Growth Rates of Functions

You may have noticed that exponential functions like 2^x and e^x seem to grow more rapidly as x gets large than do polynomials and rational functions. These exponentials certainly grow more rapidly than x itself, and you can see 2^x outgrowing x^2 as x increases in Figure 7.14. In fact, as $x \to \infty$, the functions 2^x and e^x grow faster than any power of x, even $x^{1{,}000{,}000}$ (Exercise 19).

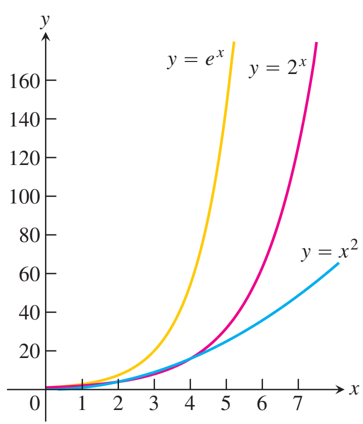


FIGURE 7.14 The graphs of e^x, 2^x, and x^2.

To get a feeling for how rapidly the values of $y = e^x$ grow with increasing x, think of graphing the function on a large blackboard, with the axes scaled in centimeters. At $x = 1$ cm, the graph is $e^1 \approx 3$ cm above the x-axis. At $x = 6$ cm, the graph is $e^6 \approx 403$ cm ≈ 4 m high (it is about to go through the ceiling if it hasn't done so already). At $x = 10$ cm, the graph is $e^{10} \approx 22{,}026$ cm ≈ 220 m high, higher than most buildings. At $x = 24$ cm, the graph is more than halfway to the moon, and at $x = 43$ cm from the origin, the graph is high enough to reach past the sun's closest stellar neighbor, the red dwarf star Proxima Centauri:

$$\begin{aligned} e^{43} &\approx 4.73 \times 10^{18} \text{ cm} \\ &= 4.73 \times 10^{13} \text{ km} \\ &\approx 1.58 \times 10^{8} \text{ light-seconds} \\ &\approx 5.0 \text{ light-years} \end{aligned}$$

In a vacuum, light travels at 300,000 km/sec.

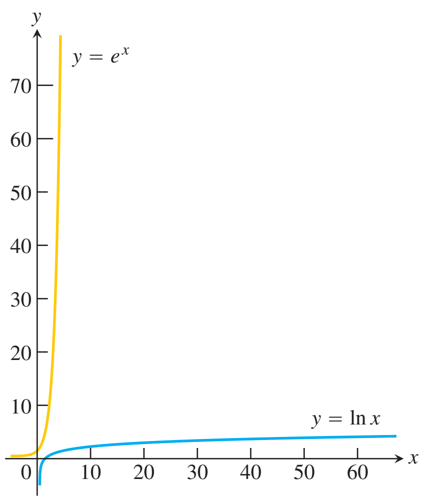


FIGURE 7.15 Scale drawings of the graphs of e^x and $\ln x$.

The distance to Proxima Centauri is about 4.22 light-years. Yet with $x = 43$ cm from the origin, the graph is still less than 2 feet to the right of the y-axis.

In contrast, logarithmic functions like $y = \log_2 x$ and $y = \ln x$ grow more slowly as $x \to \infty$ than any positive power of x (Exercise 21). With axes scaled in centimeters, you have to go nearly 5 light-years out on the x-axis to find a point where the graph of $y = \ln x$ is even $y = 43$ cm high. See Figure 7.15.

These important comparisons of exponential, polynomial, and logarithmic functions can be made precise by defining what it means for a function $f(x)$ to grow faster than another function $g(x)$ as $x \to \infty$.

DEFINITION Rates of Growth as $x \to \infty$

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

2. f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

According to these definitions, $y = 2x$ does not grow faster than $y = x$. The two functions grow at the same rate because

$$\lim_{x\to\infty} \frac{2x}{x} = \lim_{x\to\infty} 2 = 2,$$

which is a finite, nonzero limit. The reason for this apparent disregard of common sense is that we want "f grows faster than g" to mean that for large x-values g is negligible when compared with f.

EXAMPLE 1 Several Useful Comparisons of Growth Rates

You Try It

(a) e^x grows faster than x^2 as $x \to \infty$ because

$$\underbrace{\lim_{x\to\infty} \frac{e^x}{x^2}}_{\infty/\infty} = \underbrace{\lim_{x\to\infty} \frac{e^x}{2x}}_{\infty/\infty} = \lim_{x\to\infty} \frac{e^x}{2} = \infty. \quad \text{Using l'Hôpital's Rule twice}$$

(b) 3^x grows faster than 2^x as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{3^x}{2^x} = \lim_{x\to\infty} \left(\frac{3}{2}\right)^x = \infty.$$

(c) x^2 grows faster than $\ln x$ as $x \to \infty$, because

$$\lim_{x\to\infty} \frac{x^2}{\ln x} = \lim_{x\to\infty} \frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty. \qquad \text{l'Hôpital's Rule}$$

(d) $\ln x$ grows slower than x as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} \qquad \text{l'Hôpital's Rule}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0.$$

■

EXAMPLE 2 Exponential and Logarithmic Functions with Different Bases

(a) As Example 1b suggests, exponential functions with different bases never grow at the same rate as $x \to \infty$. If $a > b > 0$, then a^x grows faster than b^x. Since $(a/b) > 1$,

$$\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = \infty.$$

(b) In contrast to exponential functions, logarithmic functions with different bases a and b always grow at the same rate as $x \to \infty$:

$$\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x/\ln a}{\ln x/\ln b} = \frac{\ln b}{\ln a}.$$

The limiting ratio is always finite and never zero. ■

If f grows at the same rate as g as $x \to \infty$, and g grows at the same rate as h as $x \to \infty$, then f grows at the same rate as h as $x \to \infty$. The reason is that

$$\lim_{x\to\infty} \frac{f}{g} = L_1 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{g}{h} = L_2$$

together imply

$$\lim_{x\to\infty} \frac{f}{h} = \lim_{x\to\infty} \frac{f}{g} \cdot \frac{g}{h} = L_1 L_2.$$

If L_1 and L_2 are finite and nonzero, then so is $L_1 L_2$.

EXAMPLE 3 Functions Growing at the Same Rate

Show that $\sqrt{x^2+5}$ and $(2\sqrt{x}-1)^2$ grow at the same rate as $x \to \infty$.

Solution We show that the functions grow at the same rate by showing that they both grow at the same rate as the function $g(x) = x$:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{x} = \lim_{x\to\infty} \sqrt{1 + \frac{5}{x^2}} = 1,$$

$$\lim_{x\to\infty} \frac{(2\sqrt{x}-1)^2}{x} = \lim_{x\to\infty} \left(\frac{2\sqrt{x}-1}{\sqrt{x}}\right)^2 = \lim_{x\to\infty} \left(2 - \frac{1}{\sqrt{x}}\right)^2 = 4.$$

■

Order and Oh-Notation

Here we introduce the "little-oh" and "big-oh" notation invented by number theorists a hundred years ago and now commonplace in mathematical analysis and computer science.

DEFINITION Little-oh

A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\boldsymbol{f = o(g)}$ ("f is little-oh of g").

Notice that saying $f = o(g)$ as $x \to \infty$ is another way to say that f grows slower than g as $x \to \infty$.

EXAMPLE 4 Using Little-oh Notation

You Try It

(a) $\ln x = o(x)$ as $x \to \infty$ because $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

(b) $x^2 = o(x^3 + 1)$ as $x \to \infty$ because $\lim_{x\to\infty} \frac{x^2}{x^3 + 1} = 0$ ■

DEFINITION Big-oh

Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \le M,$$

for x sufficiently large. We indicate this by writing $\boldsymbol{f = O(g)}$ ("f is big-oh of g").

EXAMPLE 5 Using Big-oh Notation

(a) $x + \sin x = O(x)$ as $x \to \infty$ because $\frac{x + \sin x}{x} \le 2$ for x sufficiently large.

(b) $e^x + x^2 = O(e^x)$ as $x \to \infty$ because $\frac{e^x + x^2}{e^x} \to 1$ as $x \to \infty$.

(c) $x = O(e^x)$ as $x \to \infty$ because $\frac{x}{e^x} \to 0$ as $x \to \infty$. ■

If you look at the definitions again, you will see that $f = o(g)$ implies $f = O(g)$ for functions that are positive for x sufficiently large. Also, if f and g grow at the same rate, then $f = O(g)$ and $g = O(f)$ (Exercise 11).

Sequential vs. Binary Search

Computer scientists often measure the efficiency of an algorithm by counting the number of steps a computer must take to execute the algorithm. There can be significant differences in how efficiently algorithms perform, even if they are designed to accomplish the same task. These differences are often described in big-oh notation. Here is an example.

Webster's Third New International Dictionary lists about 26,000 words that begin with the letter a. One way to look up a word, or to learn if it is not there, is to read through the list one word at a time until you either find the word or determine that it is not there. This method, called sequential search, makes no particular use of the words' alphabetical arrangement. You are sure to get an answer, but it might take 26,000 steps.

Another way to find the word or to learn it is not there is to go straight to the middle of the list (give or take a few words). If you do not find the word, then go to the middle of the half that contains it and forget about the half that does not. (You know which half contains it because you know the list is ordered alphabetically.) This method eliminates roughly 13,000 words in a single step. If you do not find the word on the second try, then jump to the middle of the half that contains it. Continue this way until you have either found the word or divided the list in half so many times there are no words left. How many times do you have to divide the list to find the word or learn that it is not there? At most 15, because

$$(26{,}000/2^{15}) < 1.$$

That certainly beats a possible 26,000 steps.

For a list of length n, a sequential search algorithm takes on the order of n steps to find a word or determine that it is not in the list. A binary search, as the second algorithm is called, takes on the order of $\log_2 n$ steps. The reason is that if $2^{m-1} < n \leq 2^m$, then $m - 1 < \log_2 n \leq m$, and the number of bisections required to narrow the list to one word will be at most $m = \lceil \log_2 n \rceil$, the integer ceiling for $\log_2 n$.

Big-oh notation provides a compact way to say all this. The number of steps in a sequential search of an ordered list is $O(n)$; the number of steps in a binary search is $O(\log_2 n)$. In our example, there is a big difference between the two (26,000 vs. 15), and the difference can only increase with n because n grows faster than $\log_2 n$ as $n \to \infty$ (as in Example 1d).

EXERCISES 7.6

Comparisons with the Exponential e^x

1. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate as e^x? Which grow slower?

a. $x + 3$

b. $x^3 + \sin^2 x$

c. $\sqrt{x}$

d. 4^x

e. $(3/2)^x$

f. $e^{x/2}$

g. $e^x/2$

h. $\log_{10} x$

2. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate as e^x? Which grow slower?

a. $10x^4 + 30x + 1$

b. $x \ln x - x$

c. $\sqrt{1 + x^4}$

d. $(5/2)^x$

e. e^{-x}

f. xe^x

g. $e^{\cos x}$

h. e^{x-1}

Comparisons with the Power x^2

3. Which of the following functions grow faster than x^2 as $x \to \infty$? Which grow at the same rate as x^2? Which grow slower?

a. $x^2 + 4x$

b. $x^5 - x^2$

c. $\sqrt{x^4 + x^3}$

d. $(x + 3)^2$

e. $x \ln x$

f. 2^x

g. $x^3 e^{-x}$

h. $8x^2$

4. Which of the following functions grow faster than x^2 as $x \to \infty$? Which grow at the same rate as x^2? Which grow slower?

a. $x^2 + \sqrt{x}$

b. $10x^2$

c. $x^2 e^{-x}$

d. $\log_{10}(x^2)$

e. $x^3 - x^2$

f. $(1/10)^x$

g. $(1.1)^x$

h. $x^2 + 100x$

Comparisons with the Logarithm ln x

5. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

a. $\log_3 x$ **b.** $\ln 2x$

c. $\ln \sqrt{x}$ **d.** $\sqrt{x}$

e. x **f.** $5 \ln x$

g. $1/x$ **h.** e^x

6. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

a. $\log_2 (x^2)$ **b.** $\log_{10} 10x$

c. $1/\sqrt{x}$ **d.** $1/x^2$

e. $x - 2 \ln x$ **f.** e^{-x}

g. $\ln (\ln x)$ **h.** $\ln (2x + 5)$

Ordering Functions by Growth Rates

7. Order the following functions from slowest growing to fastest growing as $x \to \infty$.

a. e^x **b.** x^x

c. $(\ln x)^x$ **d.** $e^{x/2}$

8. Order the following functions from slowest growing to fastest growing as $x \to \infty$.

a. 2^x **b.** x^2

c. $(\ln 2)^x$ **d.** e^x

Big-oh and Little-oh; Order

9. True, or false? As $x \to \infty$,

a. $x = o(x)$ **b.** $x = o(x + 5)$

c. $x = O(x + 5)$ **d.** $x = O(2x)$

e. $e^x = o(e^{2x})$ **f.** $x + \ln x = O(x)$

g. $\ln x = o(\ln 2x)$ **h.** $\sqrt{x^2 + 5} = O(x)$

10. True, or false? As $x \to \infty$,

a. $\dfrac{1}{x + 3} = O\left(\dfrac{1}{x}\right)$ **b.** $\dfrac{1}{x} + \dfrac{1}{x^2} = O\left(\dfrac{1}{x}\right)$

c. $\dfrac{1}{x} - \dfrac{1}{x^2} = o\left(\dfrac{1}{x}\right)$ **d.** $2 + \cos x = O(2)$

e. $e^x + x = O(e^x)$ **f.** $x \ln x = o(x^2)$

g. $\ln (\ln x) = O(\ln x)$ **h.** $\ln (x) = o(\ln (x^2 + 1))$

11. Show that if positive functions $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$, then $f = O(g)$ and $g = O(f)$.

12. When is a polynomial $f(x)$ of smaller order than a polynomial $g(x)$ as $x \to \infty$? Give reasons for your answer.

13. When is a polynomial $f(x)$ of at most the order of a polynomial $g(x)$ as $x \to \infty$? Give reasons for your answer.

14. What do the conclusions we drew in Section 2.4 about the limits of rational functions tell us about the relative growth of polynomials as $x \to \infty$?

Other Comparisons

15. Investigate

$$\lim_{x\to\infty} \frac{\ln (x + 1)}{\ln x} \quad \text{and} \quad \lim_{x\to\infty} \frac{\ln (x + 999)}{\ln x}.$$

Then use l'Hôpital's Rule to explain what you find.

16. (*Continuation of Exercise 15.*) Show that the value of

$$\lim_{x\to\infty} \frac{\ln (x + a)}{\ln x}$$

is the same no matter what value you assign to the constant a. What does this say about the relative rates at which the functions $f(x) = \ln (x + a)$ and $g(x) = \ln x$ grow?

17. Show that $\sqrt{10x + 1}$ and $\sqrt{x + 1}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as $\sqrt{x}$ as $x \to \infty$.

18. Show that $\sqrt{x^4 + x}$ and $\sqrt{x^4 - x^3}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as x^2 as $x \to \infty$.

19. Show that e^x grows faster as $x \to \infty$ than x^n for any positive integer n, even $x^{1{,}000{,}000}$. (*Hint:* What is the nth derivative of x^n?)

20. The function e^x outgrows any polynomial Show that e^x grows faster as $x \to \infty$ than any polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

21. a. Show that $\ln x$ grows slower as $x \to \infty$ than $x^{1/n}$ for any positive integer n, even $x^{1/1{,}000{,}000}$.

b. Although the values of $x^{1/1{,}000{,}000}$ eventually overtake the values of $\ln x$, you have to go way out on the x-axis before this happens. Find a value of x greater than 1 for which $x^{1/1{,}000{,}000} > \ln x$. You might start by observing that when $x > 1$ the equation $\ln x = x^{1/1{,}000{,}000}$ is equivalent to the equation $\ln (\ln x) = (\ln x)/1{,}000{,}000$.

c. Even $x^{1/10}$ takes a long time to overtake $\ln x$. Experiment with a calculator to find the value of x at which the graphs of $x^{1/10}$ and $\ln x$ cross, or, equivalently, at which $\ln x = 10 \ln (\ln x)$. Bracket the crossing point between powers of 10 and then close in by successive halving.

d. (*Continuation of part (c).*) The value of x at which $\ln x = 10 \ln (\ln x)$ is too far out for some graphers and root finders to identify. Try it on the equipment available to you and see what happens.

22. The function ln x grows slower than any polynomial Show that $\ln x$ grows slower as $x \to \infty$ than any nonconstant polynomial.