Chapter 6 Technology Application Projects

Mathematica/Maple Module

Using Riemann Sums to Estimate Areas, Volumes, and Lengths of Curves

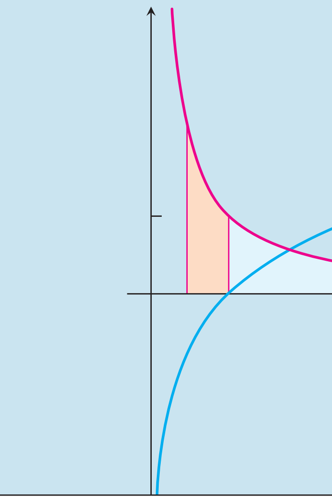
Visualize and approximate areas and volumes in **Part I** and **Part II**: Volumes of Revolution; and **Part III**: Lengths of Curves.

Mathematica/Maple Module

Modeling a Bungee Cord Jump

Collect data (or use data previously collected) to build and refine a model for the force exerted by a jumper's bungee cord. Use the work-energy theorem to compute the distance fallen for a given jumper and a given length of bungee cord.

Project Project Project



Chapter

7

TRANSCENDENTAL FUNCTIONS

OVERVIEW Functions can be classified into two broad groups (see Section 1.4). Polynomial functions are called *algebraic*, as are functions obtained from them by addition, multiplication, division, or taking powers and roots. Functions that are not algebraic are called *transcendental.* The trigonometric, exponential, logarithmic, and hyperbolic functions are transcendental, as are their inverses.

Transcendental functions occur frequently in many calculus settings and applications, including growths of populations, vibrations and waves, efficiencies of computer algorithms, and the stability of engineered structures. In this chapter we introduce several important transcendental functions and investigate their graphs, properties, derivatives, and integrals.

7.1 Inverse Functions and Their Derivatives

A function that undoes, or inverts, the effect of a function f is called the *inverse* of f. Many common functions, though not all, are paired with an inverse. Important inverse functions often show up in formulas for antiderivatives and solutions of differential equations. Inverse functions also play a key role in the development and properties of the logarithmic and exponential functions, as we will see in Section 7.3.

One-to-One Functions

A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and $+1$; the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called one-to-one. These functions take on any one value in their range exactly once.

DEFINITION One-to-One Function

A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

EXAMPLE 1 Domains of One-to-One Functions

(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. The sine *is* one-to-one on $[0, \pi/2]$, however, because it is a strictly increasing function on $[0, \pi/2]$. ■

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If it intersects the line more than once, it assumes the same y-value more than once, and is therefore not one-to-one (Figure 7.1).

The Horizontal Line Test for One-to-One Functions
A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

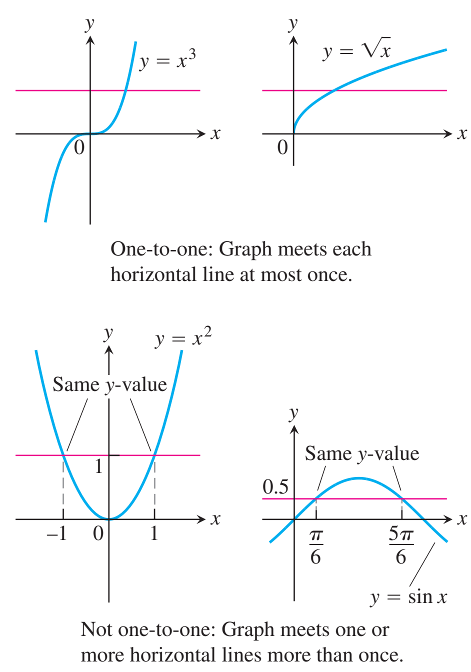


FIGURE 7.1 Using the horizontal line test, we see that $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$, but $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send an output back to the input from which it came.

DEFINITION Inverse Function

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The domains and ranges of f and f^{-1} are interchanged. The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is *not* an exponent: $f^{-1}(x)$ does not mean $1/f(x)$.

If we apply f to send an input x to the output $f(x)$ and follow by applying f^{-1} to $f(x)$ we get right back to x, just where we started. Similarly, if we take some number y in the range of f, apply f^{-1} to it, and then apply f to the resulting value $f^{-1}(y)$, we get back the value y with which we began. Composing a function and its inverse has the same effect as doing nothing.

$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in the domain of } f$$

$$(f \circ f^{-1})(y) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f)$$

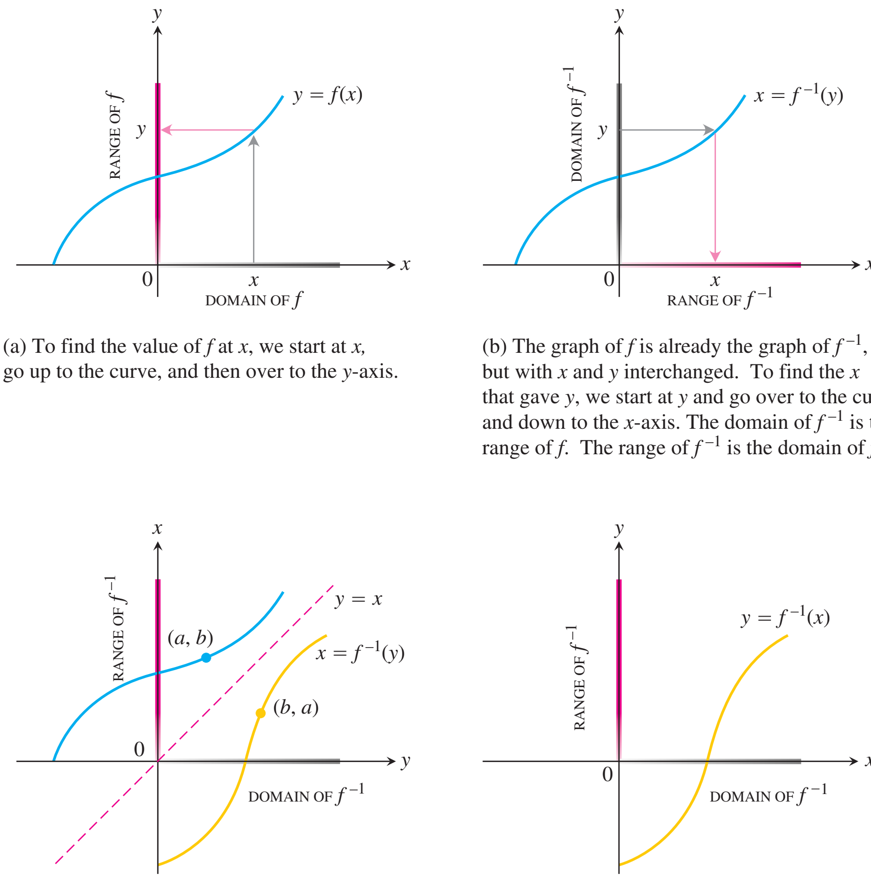
Only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2, then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$.

A function that is increasing on an interval, satisfying $f(x_2) > f(x_1)$ when $x_2 > x_1$, is one-to-one and has an inverse. Decreasing functions also have an inverse (Exercise 39). Functions that have positive derivatives at all x are increasing (Corollary 3 of the Mean Value Theorem, Section 4.2), and so they have inverses. Similarly, functions with negative derivatives at all x are decreasing and have inverses. Functions that are neither increasing nor decreasing may still be one-to-one and have an inverse, as with the function $\sec^{-1} x$ in Section 7.7.

Finding Inverses

The graphs of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x-axis, go vertically to the graph, and then move horizontally to the y-axis to read the value of y. The inverse function can be read from the graph by reversing this process. Start with a point y on the y-axis, go horizontally to the graph, and then move vertically to the x-axis to read the value of $x = f^{-1}(y)$ (Figure 7.2).

We want to set up the graph of f^{-1} so that its input values lie along the x-axis, as is usually done for functions, rather then on the y-axis. To achieve this we interchange the x and y axes by reflecting across the 45° line $y = x$. After this reflection we have a new graph that represents f^{-1}. The value of $f^{-1}(x)$ can now be read from the graph in the usual way, by starting with a point x on the x-axis, going vertically to the graph and then horizontally to the y-axis to get the value of $f^{-1}(x)$. Figure 7.2 indicates the relation between the graphs of f and f^{-1}. The graphs are interchanged by reflection through the line $y = x$.



(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.

(b) The graph of f is already the graph of f^{-1}, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.

(c) To draw the graph of f^{-1} in the more usual way, we reflect the system in the line $y = x$.

(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 7.2 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$.

The process of passing from f to f^{-1} can be summarized as a two-step process.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

EXAMPLE 2 Finding an Inverse Function

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution

1. *Solve for x in terms of y:*

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2.$$

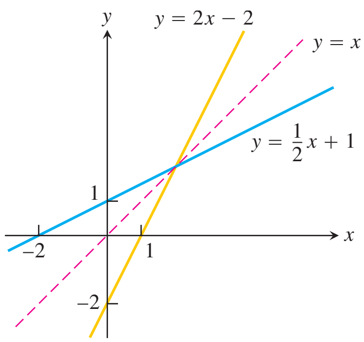


FIGURE 7.3 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$. The slopes are reciprocals of each other (Example 2).

2. *Interchange x and y:* $y = 2x - 2$.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

See Figure 7.3. ■

EXAMPLE 3 Finding an Inverse Function

Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution We first solve for x in terms of y:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$ (Figure 7.4).

Notice that, unlike the restricted function $y = x^2, x \geq 0$, the unrestricted function $y = x^2$ is not one-to-one and therefore has no inverse. ■

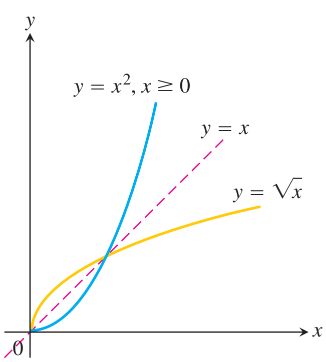


FIGURE 7.4 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 3).

Derivatives of Inverses of Differentiable Functions

If we calculate the derivatives of $f(x) = (1/2)x + 1$ and its inverse $f^{-1}(x) = 2x - 2$ from Example 2, we see that

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$

The derivatives are reciprocals of one another. The graph of f is the line $y = (1/2)x + 1$, and the graph of f^{-1} is the line $y = 2x - 2$ (Figure 7.3). Their slopes are reciprocals of one another.

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$ (Figure 7.5), the reflected line has slope $1/m$ (Exercise 36).

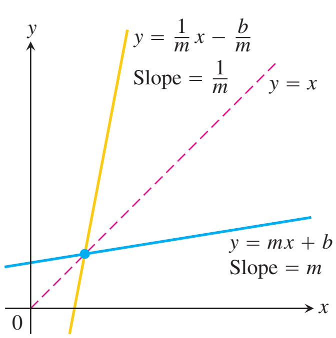
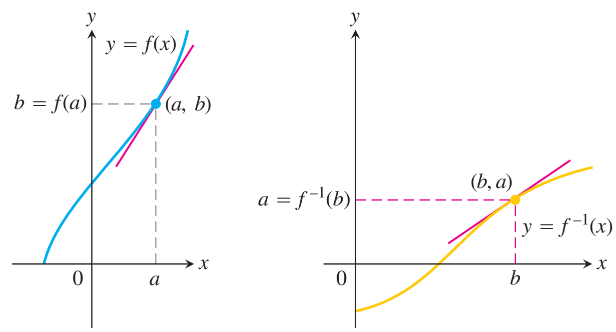


FIGURE 7.5 The slopes of nonvertical lines reflected through the line $y = x$ are reciprocals of each other.

The reciprocal relationship between the slopes of f and f^{-1} holds for other functions as well, but we must be careful to compare slopes at corresponding points. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$ and $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the point $(f(a), a)$ is the reciprocal $1/f'(a)$ (Figure 7.6). If we set $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

If $y = f(x)$ has a horizontal tangent line at $(a, f(a))$ then the inverse function f^{-1} has a vertical tangent line at $(f(a), a)$, and this infinite slope implies that f^{-1} is not differentiable



The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

FIGURE 7.6 The graphs of inverse functions have reciprocal slopes at corresponding points.

at $f(a)$. Theorem 1 gives the conditions under which f^{-1} is differentiable in its domain, which is the same as the range of f.

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}} \tag{1}$$

The proof of Theorem 1 is omitted, but here is another way to view it. When $y = f(x)$ is differentiable at $x = a$ and we change x by a small amount dx, the corresponding change in y is approximately

$$dy = f'(a)\,dx.$$

This means that y changes about $f'(a)$ times as fast as x when $x = a$ and that x changes about $1/f'(a)$ times as fast as y when $y = b$. It is reasonable that the derivative of f^{-1} at b is the reciprocal of the derivative of f at a.

EXAMPLE 4 Applying Theorem 1

The function $f(x) = x^2, x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

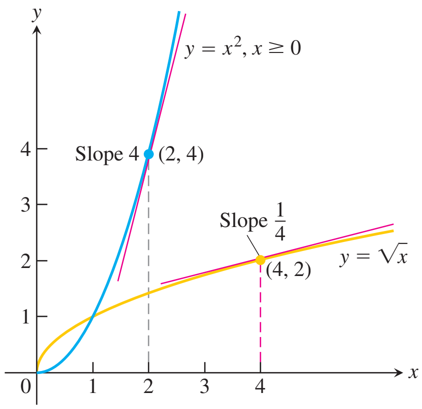


FIGURE 7.7 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 4).

Theorem 1 predicts that the derivative of $f^{-1}(x)$ is

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))}\\ &= \frac{1}{2(f^{-1}(x))}\\ &= \frac{1}{2(\sqrt{x})}.\end{aligned}$$

Theorem 1 gives a derivative that agrees with our calculation using the Power Rule for the derivative of the square root function.

Let's examine Theorem 1 at a specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem 1 says that the derivative of f at 2, $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, $(f^{-1})'(4)$, are reciprocals. It states that

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \left.\frac{1}{2x}\right|_{x=2} = \frac{1}{4}.$$

See Figure 7.7. ■

Equation (1) sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1}.

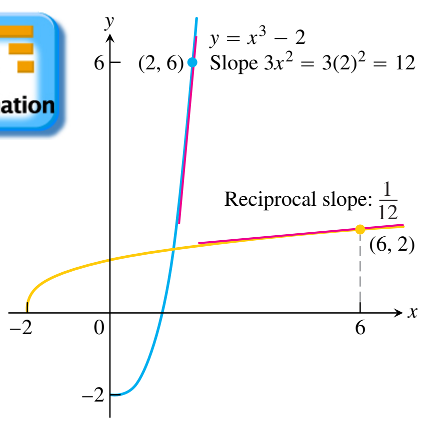


FIGURE 7.8 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 5).

EXAMPLE 5 Finding a Value of the Inverse Derivative

Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution

$$\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12} \qquad \text{Eq. (1)}$$

See Figure 7.8. ■

Parametrizing Inverse Functions

We can graph or represent any function $y = f(x)$ parametrically as

$$x = t \quad \text{and} \quad y = f(t).$$

Interchanging t and $f(t)$ produces parametric equations for the inverse:

$$x = f(t) \quad \text{and} \quad y = t$$

(see Section 3.5).

For example, to graph the one-to-one function $f(x) = x^2, x \geq 0$, on a grapher together with its inverse and the line $y = x, x \geq 0$, use the parametric graphing option with

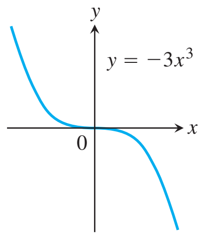
$$\begin{aligned}&\text{Graph of } f: && x_1 = t, && y_1 = t^2, && t \geq 0\\ &\text{Graph of } f^{-1}: && x_2 = t^2, && y_2 = t\\ &\text{Graph of } y = x: && x_3 = t, && y_3 = t\end{aligned}$$

EXERCISES 7.1

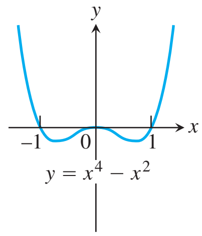
Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–6 are one-to-one, and which are not?

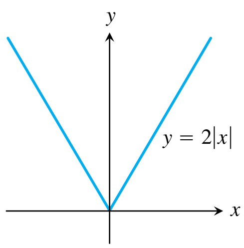
1.



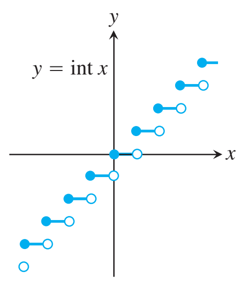
2.



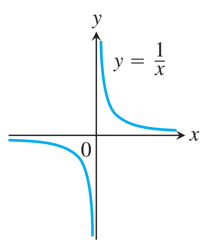
3.



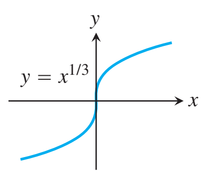
4.



5.



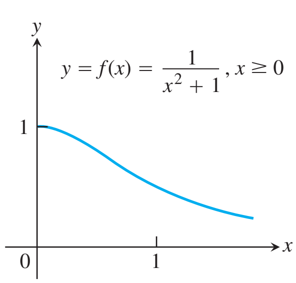
6.



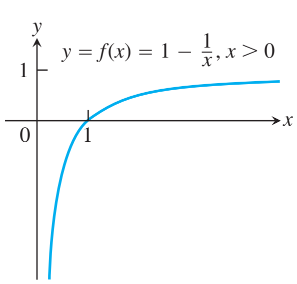
Graphing Inverse Functions

Each of Exercises 7–10 shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use symmetry with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1}.) Identify the domain and range of f^{-1}.

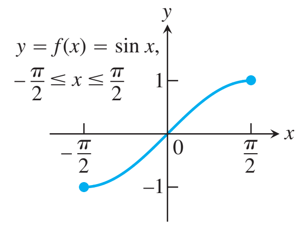
7.



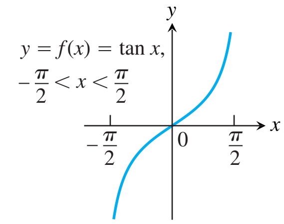
8.



9.



10.



11. a. Graph the function $f(x) = \sqrt{1 - x^2}, 0 \leq x \leq 1$. What symmetry does the graph have?

b. Show that f is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \geq 0$.)

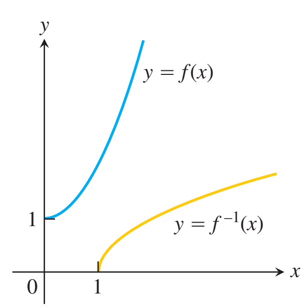
12. a. Graph the function $f(x) = 1/x$. What symmetry does the graph have?

b. Show that f is its own inverse.

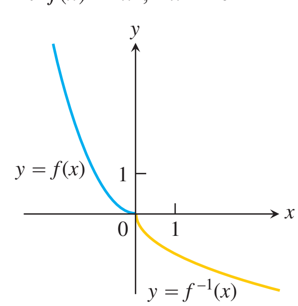
Formulas for Inverse Functions

Each of Exercises 13–18 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

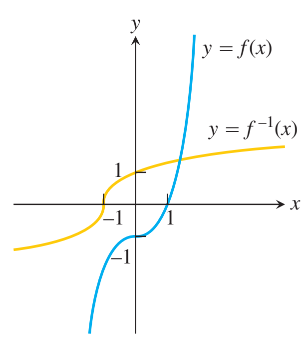
13. $f(x) = x^2 + 1, \quad x \geq 0$



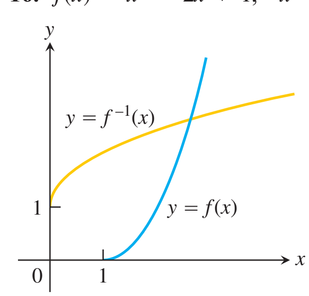
14. $f(x) = x^2, \quad x \leq 0$



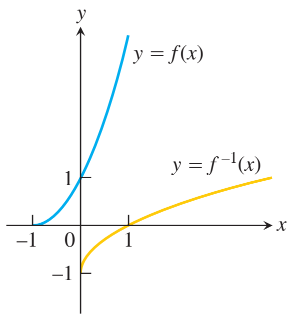
15. $f(x) = x^3 - 1$



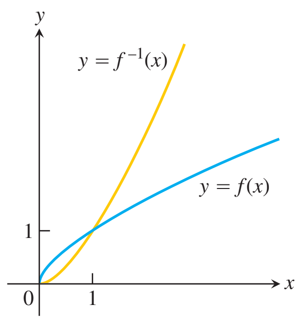
16. $f(x) = x^2 - 2x + 1, \quad x \geq 1$



17. $f(x) = (x + 1)^2, \quad x \geq -1$



18. $f(x) = x^{2/3}, \quad x \geq 0$



Each of Exercises 19–24 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

19. $f(x) = x^5$ **20.** $f(x) = x^4, \quad x \geq 0$

21. $f(x) = x^3 + 1$ **22.** $f(x) = (1/2)x - 7/2$

23. $f(x) = 1/x^2, \quad x > 0$ **24.** $f(x) = 1/x^3, \quad x \neq 0$

Derivatives of Inverse Functions

In Exercises 25–28:

a. Find $f^{-1}(x)$.

b. Graph f and f^{-1} together.

c. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

25. $f(x) = 2x + 3, \quad a = -1$ **26.** $f(x) = (1/5)x + 7, \quad a = -1$

27. $f(x) = 5 - 4x, \quad a = 1/2$ **28.** $f(x) = 2x^2, \quad x \geq 0, \quad a = 5$

29. a. Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

b. Graph f and g over an x-interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry about the line $y = x$.

c. Find the slopes of the tangents to the graphs of f and g at $(1, 1)$ and $(-1, -1)$ (four tangents in all).

d. What lines are tangent to the curves at the origin?

30. a. Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.

b. Graph h and k over an x-interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.

c. Find the slopes of the tangents to the graphs at h and k at $(2, 2)$ and $(-2, -2)$.

d. What lines are tangent to the curves at the origin?

31. Let $f(x) = x^3 - 3x^2 - 1, x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

32. Let $f(x) = x^2 - 4x - 5, x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.

33. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of df^{-1}/dx at $x = 4$.

34. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.

Inverses of Lines

35. a. Find the inverse of the function $f(x) = mx$, where m is a constant different from zero.

b. What can you conclude about the inverse of a function $y = f(x)$ whose graph is a line through the origin with a nonzero slope m?

36. Show that the graph of the inverse of $f(x) = mx + b$, where m and b are constants and $m \neq 0$, is a line with slope $1/m$ and y-intercept $-b/m$.

37. a. Find the inverse of $f(x) = x + 1$. Graph f and its inverse together. Add the line $y = x$ to your sketch, drawing it with dashes or dots for contrast.

b. Find the inverse of $f(x) = x + b$ (b constant). How is the graph of f^{-1} related to the graph of f?

c. What can you conclude about the inverses of functions whose graphs are lines parallel to the line $y = x$?

38. a. Find the inverse of $f(x) = -x + 1$. Graph the line $y = -x + 1$ together with the line $y = x$. At what angle do the lines intersect?

b. Find the inverse of $f(x) = -x + b$ (b constant). What angle does the line $y = -x + b$ make with the line $y = x$?

c. What can you conclude about the inverses of functions whose graphs are lines perpendicular to the line $y = x$?

Increasing and Decreasing Functions

39. As in Section 4.3, a function $f(x)$ increases on an interval I if for any two points x_1 and x_2 in I,

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) > f(x_1).$$

Similarly, a function decreases on I if for any two points x_1 and x_2 in I,

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) < f(x_1).$$

Show that increasing functions and decreasing functions are one-to-one. That is, show that for any x_1 and x_2 in I, $x_2 \neq x_1$ implies $f(x_2) \neq f(x_1)$.

Use the results of Exercise 39 to show that the functions in Exercises 40–44 have inverses over their domains. Find a formula for df^{-1}/dx using Theorem 1.

40. $f(x) = (1/3)x + (5/6)$ **41.** $f(x) = 27x^3$

42. $f(x) = 1 - 8x^3$ **43.** $f(x) = (1 - x)^3$

44. $f(x) = x^{5/3}$

Theory and Applications

45. If $f(x)$ is one-to-one, can anything be said about $g(x) = -f(x)$? Is it also one-to-one? Give reasons for your answer.

46. If $f(x)$ is one-to-one and $f(x)$ is never zero, can anything be said about $h(x) = 1/f(x)$? Is it also one-to-one? Give reasons for your answer.

47. Suppose that the range of g lies in the domain of f so that the composite $f \circ g$ is defined. If f and g are one-to-one, can anything be said about $f \circ g$? Give reasons for your answer.

48. If a composite $f \circ g$ is one-to-one, must g be one-to-one? Give reasons for your answer.

49. Suppose $f(x)$ is positive, continuous, and increasing over the interval $[a, b]$. By interpreting the graph of f show that

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a).$$

50. Determine conditions on the constants a, b, c, and d so that the rational function

$$f(x) = \frac{ax + b}{cx + d}$$

has an inverse.

51. If we write $g(x)$ for $f^{-1}(x)$, Equation (1) can be written as

$$g'(f(a)) = \frac{1}{f'(a)}, \quad \text{or} \quad g'(f(a)) \cdot f'(a) = 1.$$

If we then write x for a, we get

$$g'(f(x)) \cdot f'(x) = 1.$$

The latter equation may remind you of the Chain Rule, and indeed there is a connection.

Assume that f and g are differentiable functions that are inverses of one another, so that $(g \circ f)(x) = x$. Differentiate both sides of this equation with respect to x, using the Chain Rule to express $(g \circ f)'(x)$ as a product of derivatives of g and f. What do you find? (This is not a proof of Theorem 1 because we assume here the theorem's conclusion that $g = f^{-1}$ is differentiable.)

52. Equivalence of the washer and shell methods for finding volume Let f be differentiable and increasing on the interval $a \le x \le b$, with $a > 0$, and suppose that f has a differentiable inverse, f^{-1}. Revolve about the y-axis the region bounded by the graph of f and the lines $x = a$ and $y = f(b)$ to generate a solid. Then the values of the integrals given by the washer and shell methods for the volume have identical values:

$$\int_{f(a)}^{f(b)} \pi\left((f^{-1}(y))^2 - a^2\right) dy = \int_a^b 2\pi x(f(b) - f(x))\,dx.$$

To prove this equality, define

$$W(t) = \int_{f(a)}^{f(t)} \pi\left((f^{-1}(y))^2 - a^2\right) dy$$

$$S(t) = \int_a^t 2\pi x(f(t) - f(x))\,dx.$$

Then show that the functions W and S agree at a point of $[a, b]$ and have identical derivatives on $[a, b]$. As you saw in Section 4.8, Exercise 102, this will guarantee $W(t) = S(t)$ for all t in $[a, b]$. In particular, $W(b) = S(b)$. (*Source:* "Disks and Shells Revisited," by Walter Carlip, *American Mathematical Monthly*, Vol. 98, No. 2, Feb. 1991, pp. 154–156.)

COMPUTER EXPLORATIONS

In Exercises 53–60, you will explore some functions and their inverses together with their derivatives and linear approximating functions at specified points. Perform the following steps using your CAS:

a. Plot the function $y = f(x)$ together with its derivative over the given interval. Explain why you know that f is one-to-one over the interval.

b. Solve the equation $y = f(x)$ for x as a function of y, and name the resulting inverse function g.

c. Find the equation for the tangent line to f at the specified point $(x_0, f(x_0))$.

d. Find the equation for the tangent line to g at the point $(f(x_0), x_0)$ located symmetrically across the 45° line $y = x$ (which is the graph of the identity function). Use Theorem 1 to find the slope of this tangent line.

e. Plot the functions f and g, the identity, the two tangent lines, and the line segment joining the points $(x_0, f(x_0))$ and $(f(x_0), x_0)$. Discuss the symmetries you see across the main diagonal.

53. $y = \sqrt{3x - 2}, \quad \frac{2}{3} \le x \le 4, \quad x_0 = 3$

54. $y = \frac{3x + 2}{2x - 11}, \quad -2 \le x \le 2, \quad x_0 = 1/2$

55. $y = \frac{4x}{x^2 + 1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

56. $y = \frac{x^3}{x^2 + 1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

57. $y = x^3 - 3x^2 - 1, \quad 2 \le x \le 5, \quad x_0 = \frac{27}{10}$

58. $y = 2 - x - x^3, \quad -2 \le x \le 2, \quad x_0 = \frac{3}{2}$

59. $y = e^x, \quad -3 \le x \le 5, \quad x_0 = 1$

60. $y = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \quad x_0 = 1$

In Exercises 61 and 62, repeat the steps above to solve for the functions $y = f(x)$ and $x = f^{-1}(y)$ defined implicitly by the given equations over the interval.

61. $y^{1/3} - 1 = (x + 2)^3, \quad -5 \le x \le 5, \quad x_0 = -3/2$

62. $\cos y = x^{1/5}, \quad 0 \le x \le 1, \quad x_0 = 1/2$

7.2 Natural Logarithms

For any positive number a, the function value $f(x) = a^x$ is easy to define when x is an integer or rational number. When x is irrational, the meaning of a^x is not so clear. Similarly, the definition of the logarithm $\log_a x$, the inverse function of $f(x) = a^x$, is not completely obvious. In this section we use integral calculus to define the *natural logarithm* function, for which the number a is a particularly important value. This function allows us to define and analyze general exponential and logarithmic functions, $y = a^x$ and $y = \log_a x$.

Logarithms originally played important roles in arithmetic computations. Historically, considerable labor went into producing long tables of logarithms, correct to five, eight, or even more, decimal places of accuracy. Prior to the modern age of electronic calculators and computers, every engineer owned slide rules marked with logarithmic scales. Calculations with logarithms made possible the great seventeenth-century advances in offshore navigation and celestial mechanics. Today we know such calculations are done using calculators or computers, but the properties and numerous applications of logarithms are as important as ever.

Definition of the Natural Logarithm Function

One solid approach to defining and understanding logarithms begins with a study of the natural logarithm function defined as an integral through the Fundamental Theorem of Calculus. While this approach may seem indirect, it enables us to derive quickly the familiar properties of logarithmic and exponential functions. The functions we have studied so far were analyzed using the techniques of calculus, but here we do something more fundamental. We use calculus for the very definition of the logarithmic and exponential functions.

The natural logarithm of a positive number x, written as $\ln x$, is the value of an integral.

DEFINITION The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Figure 7.9). For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1. The function is not defined for $x \leq 0$. From the Zero Width Interval Rule for definite integrals, we also have

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0.$$

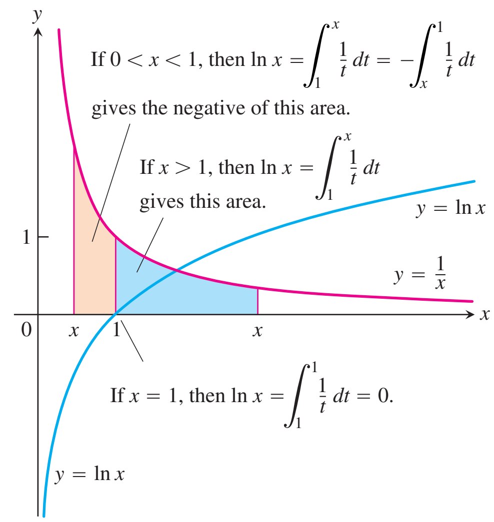


FIGURE 7.9 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

TABLE 7.1 Typical 2-place values of $\ln x$

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

Notice that we show the graph of $y = 1/x$ in Figure 7.9 but use $y = 1/t$ in the integral. Using x for everything would have us writing

$$\ln x = \int_1^x \frac{1}{x}\,dx,$$

with x meaning two different things. So we change the variable of integration to t.

By using rectangles to obtain finite approximations of the area under the graph of $y = 1/t$ and over the interval between $t = 1$ and $t = x$, as in Section 5.1, we can approximate the values of the function $\ln x$. Several values are given in Table 7.1. There is an important number whose natural logarithm equals 1.

DEFINITION The Number e

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

Geometrically, the number e corresponds to the point on the x-axis for which the area under the graph of $y = 1/t$ and above the interval $[1, e]$ is the exact area of the unit square. The area of the region shaded blue in Figure 7.9 is 1 sq unit when $x = e$.

The Derivative of $y = \ln x$

By the first part of the Fundamental Theorem of Calculus (Section 5.4),

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

For every positive value of x, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Therefore, the function $y = \ln x$ is a solution to the initial value problem $dy/dx = 1/x$, $x > 0$, with $y(1) = 0$. Notice that the derivative is always positive so the natural logarithm is an increasing function, hence it is one-to-one and invertible. Its inverse is studied in Section 7.3.

If u is a differentiable function of x whose values are positive, so that $\ln u$ is defined, then applying the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to the function $y = \ln u$ gives

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \tag{1}$$

EXAMPLE 1 Derivatives of Natural Logarithms

(a) $\dfrac{d}{dx}\ln 2x = \dfrac{1}{2x}\dfrac{d}{dx}(2x) = \dfrac{1}{2x}(2) = \dfrac{1}{x}$

(b) Equation (1) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$$ ■

Notice the remarkable occurrence in Example 1a. The function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln ax$ for any positive number a:

$$\frac{d}{dx}\ln ax = \frac{1}{ax}\cdot\frac{d}{dx}(ax) = \frac{1}{ax}(a) = \frac{1}{x}. \tag{2}$$

Since they have the same derivative, the functions $y = \ln ax$ and $y = \ln x$ differ by a constant.

HISTORICAL BIOGRAPHY

John Napier
(1550–1617)

Properties of Logarithms

Logarithms were invented by John Napier and were the single most important improvement in arithmetic calculation before the modern electronic computer. What made them so useful is that the properties of logarithms enable multiplication of positive numbers by addition of their logarithms, division of positive numbers by subtraction of their logarithms, and exponentiation of a number by multiplying its logarithm by the exponent. We summarize these properties as a series of rules in Theorem 2. For the moment, we restrict the exponent r in Rule 4 to be a rational number; you will see why when we prove the rule.

THEOREM 2 Properties of Logarithms
For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1.	*Product Rule*:	$\ln ax = \ln a + \ln x$	
2.	*Quotient Rule*:	$\ln\frac{a}{x} = \ln a - \ln x$	
3.	*Reciprocal Rule*:	$\ln\frac{1}{x} = -\ln x$	Rule 2 with $a = 1$
4.	*Power Rule*:	$\ln x^r = r\ln x$	r rational

We illustrate how these rules apply.

EXAMPLE 2 Interpreting the Properties of Logarithms

(a) $\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3$ — Product

(b) $\ln 4 - \ln 5 = \ln\frac{4}{5} = \ln 0.8$ — Quotient

(c) $\ln\frac{1}{8} = -\ln 8$ — Reciprocal

$= -\ln 2^3 = -3\ln 2$ — Power ■

EXAMPLE 3 Applying the Properties to Function Formulas

(a) $\ln 4 + \ln\sin x = \ln(4\sin x)$ — Product

(b) $\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ — Quotient

(c) $\ln \sec x = \ln \frac{1}{\cos x} = -\ln \cos x$ Reciprocal

(d) $\ln \sqrt[3]{x+1} = \ln (x+1)^{1/3} = \frac{1}{3} \ln (x+1)$ Power ■

We now give the proof of Theorem 2. The steps in the proof are similar to those used in solving problems involving logarithms.

Proof that $\ln ax = \ln a + \ln x$ The argument is unusual—and elegant. It starts by observing that $\ln ax$ and $\ln x$ have the same derivative (Equation 2). According to Corollary 2 of the Mean Value Theorem, then, the functions must differ by a constant, which means that

$$\ln ax = \ln x + C$$

for some C.

Since this last equation holds for all positive values of x, it must hold for $x = 1$. Hence,

$$\ln (a \cdot 1) = \ln 1 + C$$

$$\ln a = 0 + C \qquad \ln 1 = 0$$

$$C = \ln a.$$

By substituting we conclude,

$$\ln ax = \ln a + \ln x.$$

Proof that $\ln x^r = r \ln x$ (assuming r rational) We use the same-derivative argument again. For all positive values of x,

$$\frac{d}{dx} \ln x^r = \frac{1}{x^r} \frac{d}{dx}(x^r) \qquad \text{Eq. (1) with } u = x^r$$

$$= \frac{1}{x^r} r x^{r-1}$$

$$= r \cdot \frac{1}{x} = \frac{d}{dx}(r \ln x).$$

Here is where we need r to be rational, at least for now. We have proved the Power Rule only for rational exponents.

Since $\ln x^r$ and $r \ln x$ have the same derivative,

$$\ln x^r = r \ln x + C$$

for some constant C. Taking x to be 1 identifies C as zero, and we're done.

You are asked to prove Rule 2 in Exercise 84. Rule 3 is a special case of Rule 2, obtained by setting $a = 1$ and noting that $\ln 1 = 0$. So we have established all cases of Theorem 2. ■

We have not yet proved Rule 4 for r irrational; we will return to this case in Section 7.3. The rule does hold for all r, rational or irrational.

The Graph and Range of ln *x*

The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down.

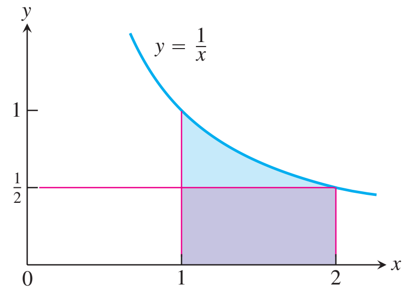


FIGURE 7.10 The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \le x \le 2$.

We can estimate the value of ln 2 by considering the area under the graph of $y = 1/x$ and above the interval [1, 2]. In Figure 7.10 a rectangle of height 1/2 over the interval [1, 2] fits under the graph. Therefore the area under the graph, which is ln 2, is greater than the area, 1/2, of the rectangle. So $\ln 2 > 1/2$. Knowing this we have,

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}$$

and

$$\ln 2^{-n} = -n \ln 2 < -n\left(\frac{1}{2}\right) = -\frac{n}{2}.$$

It follows that

$$\lim_{x\to\infty} \ln x = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} \ln x = -\infty.$$

We defined $\ln x$ for $x > 0$, so the domain of $\ln x$ is the set of positive real numbers. The above discussion and the Intermediate Value Theorem show that its range is the entire real line giving the graph of $y = \ln x$ shown in Figure 7.9.

The Integral $\int (1/u)\, du$

Equation (1) leads to the integral formula

$$\int \frac{1}{u}\, du = \ln u + C \tag{3}$$

when u is a positive differentiable function, but what if u is negative? If u is negative, then $-u$ is positive and

$$\int \frac{1}{u}\, du = \int \frac{1}{(-u)}\, d(-u) \qquad \text{Eq. (3) with } u \text{ replaced by } -u$$

$$= \ln(-u) + C. \tag{4}$$

We can combine Equations (3) and (4) into a single formula by noticing that in each case the expression on the right is $\ln |u| + C$. In Equation (3), $\ln u = \ln |u|$ because $u > 0$; in Equation (4), $\ln(-u) = \ln |u|$ because $u < 0$. Whether u is positive or negative, the integral of $(1/u)\, du$ is $\ln |u| + C$.

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\, du = \ln |u| + C. \tag{5}$$

Equation (5) applies anywhere on the domain of $1/u$, the points where $u \neq 0$.

We know that

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1 \text{ and rational}$$

Equation (5) explains what to do when n equals -1. Equation (5) says integrals of a certain *form* lead to logarithms. If $u = f(x)$, then $du = f'(x)\,dx$ and

$$\int \frac{1}{u}\,du = \int \frac{f'(x)}{f(x)}\,dx.$$

So Equation (5) gives

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

whenever $f(x)$ is a differentiable function that maintains a constant sign on the domain given for it.

EXAMPLE 4 Applying Equation (5)

(a)
$$\int_0^2 \frac{2x}{x^2 - 5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$

$u = x^2 - 5$, $du = 2x\,dx$, $u(0) = -5$, $u(2) = -1$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

(b)
$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3 + 2\sin\theta}\,d\theta = \int_1^5 \frac{2}{u}\,du$$

$u = 3 + 2\sin\theta$, $du = 2\cos\theta\,d\theta$, $u(-\pi/2) = 1$, $u(\pi/2) = 5$

$$= 2\ln|u|\Big]_1^5$$

$$= 2\ln|5| - 2\ln|1| = 2\ln 5$$

Note that $u = 3 + 2\sin\theta$ is always positive on $[-\pi/2, \pi/2]$, so Equation (5) applies. ■

The Integrals of tan x and cot x

Equation (5) tells us at last how to integrate the tangent and cotangent functions. For the tangent function,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$$

$u = \cos x > 0$ on $(-\pi/2, \pi/2)$, $du = -\sin x\,dx$

$$= -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C$$

Reciprocal Rule

$$= \ln|\sec x| + C.$$

For the cotangent,

$$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u}$$

$u = \sin x$, $du = \cos x\,dx$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

$$\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc x| + C$$

EXAMPLE 5

$$\int_0^{\pi/6} \tan 2x\,dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u\,du$$

$$= \frac{1}{2}\ln|\sec u|\Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi/3$ ■

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

You Try It

EXAMPLE 6 Using Logarithmic Differentiation

Find dy/dx if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\ln y = \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

$$= \ln\left((x^2+1)(x+3)^{1/2}\right) - \ln(x-1) \qquad \text{Rule 2}$$

$$= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) \qquad \text{Rule 1}$$

$$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). \qquad \text{Rule 3}$$

We then take derivatives of both sides with respect to x, using Equation (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$ ■

A direct computation in Example 6, using the Quotient and Product Rules, would be much longer.

EXERCISES 7.2

Using the Properties of Logarithms

1. Express the following logarithms in terms of ln 2 and ln 3.
   a. $\ln 0.75$ **b.** $\ln(4/9)$ **c.** $\ln(1/2)$
   d. $\ln\sqrt[3]{9}$ **e.** $\ln 3\sqrt{2}$ **f.** $\ln\sqrt{13.5}$

2. Express the following logarithms in terms of ln 5 and ln 7.
   a. $\ln(1/125)$ **b.** $\ln 9.8$ **c.** $\ln 7\sqrt{7}$
   d. $\ln 1225$ **e.** $\ln 0.056$
   f. $(\ln 35 + \ln(1/7))/(\ln 25)$

Use the properties of logarithms to simplify the expressions in Exercises 3 and 4.

3. **a.** $\ln\sin\theta - \ln\left(\dfrac{\sin\theta}{5}\right)$ **b.** $\ln(3x^2 - 9x) + \ln\left(\dfrac{1}{3x}\right)$
   c. $\dfrac{1}{2}\ln(4t^4) - \ln 2$

4. **a.** $\ln\sec\theta + \ln\cos\theta$ **b.** $\ln(8x + 4) - 2\ln 2$
   c. $3\ln\sqrt[3]{t^2 - 1} - \ln(t + 1)$

Derivatives of Logarithms

In Exercises 5–36, find the derivative of y with respect to x, t, or θ, as appropriate.

5. $y = \ln 3x$
6. $y = \ln kx$, k constant
7. $y = \ln(t^2)$
8. $y = \ln(t^{3/2})$
9. $y = \ln\dfrac{3}{x}$
10. $y = \ln\dfrac{10}{x}$
11. $y = \ln(\theta + 1)$
12. $y = \ln(2\theta + 2)$
13. $y = \ln x^3$
14. $y = (\ln x)^3$
15. $y = t(\ln t)^2$
16. $y = t\sqrt{\ln t}$
17. $y = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16}$
18. $y = \dfrac{x^3}{3}\ln x - \dfrac{x^3}{9}$
19. $y = \dfrac{\ln t}{t}$
20. $y = \dfrac{1 + \ln t}{t}$
21. $y = \dfrac{\ln x}{1 + \ln x}$
22. $y = \dfrac{x\ln x}{1 + \ln x}$
23. $y = \ln(\ln x)$
24. $y = \ln(\ln(\ln x))$
25. $y = \theta(\sin(\ln\theta) + \cos(\ln\theta))$
26. $y = \ln(\sec\theta + \tan\theta)$
27. $y = \ln\dfrac{1}{x\sqrt{x + 1}}$
28. $y = \dfrac{1}{2}\ln\dfrac{1 + x}{1 - x}$
29. $y = \dfrac{1 + \ln t}{1 - \ln t}$
30. $y = \sqrt{\ln\sqrt{t}}$
31. $y = \ln(\sec(\ln\theta))$
32. $y = \ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$
33. $y = \ln\left(\dfrac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$
34. $y = \ln\sqrt{\dfrac{(x + 1)^5}{(x + 2)^{20}}}$
35. $y = \displaystyle\int_{x^2/2}^{x^2} \ln\sqrt{t}\,dt$
36. $y = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\,dt$

Integration

Evaluate the integrals in Exercises 37–54.

37. $\displaystyle\int_{-3}^{-2} \frac{dx}{x}$
38. $\displaystyle\int_{-1}^{0} \frac{3\,dx}{3x - 2}$
39. $\displaystyle\int \frac{2y\,dy}{y^2 - 25}$
40. $\displaystyle\int \frac{8r\,dr}{4r^2 - 5}$
41. $\displaystyle\int_0^{\pi} \frac{\sin t}{2 - \cos t}\,dt$
42. $\displaystyle\int_0^{\pi/3} \frac{4\sin\theta}{1 - 4\cos\theta}\,d\theta$
43. $\displaystyle\int_1^2 \frac{2\ln x}{x}\,dx$
44. $\displaystyle\int_2^4 \frac{dx}{x\ln x}$
45. $\displaystyle\int_2^4 \frac{dx}{x(\ln x)^2}$
46. $\displaystyle\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$
47. $\displaystyle\int \frac{3\sec^2 t}{6 + 3\tan t}\,dt$
48. $\displaystyle\int \frac{\sec y\tan y}{2 + \sec y}\,dy$
49. $\displaystyle\int_0^{\pi/2} \tan\frac{x}{2}\,dx$
50. $\displaystyle\int_{\pi/4}^{\pi/2} \cot t\,dt$
51. $\displaystyle\int_{\pi/2}^{\pi} 2\cot\frac{\theta}{3}\,d\theta$
52. $\displaystyle\int_0^{\pi/12} 6\tan 3x\,dx$
53. $\displaystyle\int \frac{dx}{2\sqrt{x} + 2x}$
54. $\displaystyle\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}}$

Logarithmic Differentiation

In Exercises 55–68, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

55. $y = \sqrt{x(x+1)}$

56. $y = \sqrt{(x^2+1)(x-1)^2}$

57. $y = \sqrt{\dfrac{t}{t+1}}$

58. $y = \sqrt{\dfrac{1}{t(t+1)}}$

59. $y = \sqrt{\theta + 3}\,\sin\theta$

60. $y = (\tan\theta)\sqrt{2\theta + 1}$

61. $y = t(t+1)(t+2)$

62. $y = \dfrac{1}{t(t+1)(t+2)}$

63. $y = \dfrac{\theta + 5}{\theta\cos\theta}$

64. $y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$

65. $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

66. $y = \sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$

67. $y = \sqrt[3]{\dfrac{x(x-2)}{x^2+1}}$

68. $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

Theory and Applications

69. Locate and identify the absolute extreme values of

a. $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,

b. $\cos(\ln x)$ on $[1/2, 2]$.

70. a. Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.

b. Using part (a), show that $\ln x < x$ if $x > 1$.

71. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

72. Find the area between the curve $y = \tan x$ and the x-axis from $x = -\pi/4$ to $x = \pi/3$.

73. The region in the first quadrant bounded by the coordinate axes, the line $y = 3$, and the curve $x = 2/\sqrt{y+1}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

74. The region between the curve $y = \sqrt{\cot x}$ and the x-axis from $x = \pi/6$ to $x = \pi/2$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

75. The region between the curve $y = 1/x^2$ and the x-axis from $x = 1/2$ to $x = 2$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

76. In Section 6.2, Exercise 6, we revolved about the y-axis the region between the curve $y = 9x/\sqrt{x^3+9}$ and the x-axis from $x = 0$ to $x = 3$ to generate a solid of volume 36π. What volume do you get if you revolve the region about the x-axis instead? (See Section 6.2, Exercise 6, for a graph.)

77. Find the lengths of the following curves.

a. $y = (x^2/8) - \ln x, \quad 4 \le x \le 8$

b. $x = (y/4)^2 - 2\ln(y/4), \quad 4 \le y \le 12$

78. Find a curve through the point (1, 0) whose length from $x = 1$ to $x = 2$ is

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\,dx.$$

T **79. a.** Find the centroid of the region between the curve $y = 1/x$ and the x-axis from $x = 1$ to $x = 2$. Give the coordinates to two decimal places.

b. Sketch the region and show the centroid in your sketch.

80. a. Find the center of mass of a thin plate of constant density covering the region between the curve $y = 1/\sqrt{x}$ and the x-axis from $x = 1$ to $x = 16$.

b. Find the center of mass if, instead of being constant, the density function is $\delta(x) = 4/\sqrt{x}$.

Solve the initial value problems in Exercises 81 and 82.

81. $\dfrac{dy}{dx} = 1 + \dfrac{1}{x}, \quad y(1) = 3$

82. $\dfrac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1$

T **83. The linearization of $\ln(1+x)$ at $x = 0$** Instead of approximating $\ln x$ near $x = 1$, we approximate $\ln(1+x)$ near $x = 0$. We get a simpler formula this way.

a. Derive the linearization $\ln(1+x) \approx x$ at $x = 0$.

b. Estimate to five decimal places the error involved in replacing $\ln(1+x)$ by x on the interval [0, 0.1].

c. Graph $\ln(1+x)$ and x together for $0 \le x \le 0.5$. Use different colors, if available. At what points does the approximation of $\ln(1+x)$ seem best? Least good? By reading coordinates from the graphs, find as good an upper bound for the error as your grapher will allow.

84. Use the same-derivative argument, as was done to prove Rules 1 and 4 of Theorem 2, to prove the Quotient Rule property of logarithms.

Grapher Explorations

85. Graph $\ln x$, $\ln 2x$, $\ln 4x$, $\ln 8x$, and $\ln 16x$ (as many as you can) together for $0 < x \le 10$. What is going on? Explain.

86. Graph $y = \ln|\sin x|$ in the window $0 \le x \le 22$, $-2 \le y \le 0$. Explain what you see. How could you change the formula to turn the arches upside down?

87. a. Graph $y = \sin x$ and the curves $y = \ln(a + \sin x)$ for $a = 2$, 4, 8, 20, and 50 together for $0 \le x \le 23$.

b. Why do the curves flatten as a increases? (*Hint:* Find an a-dependent upper bound for $|y'|$.)

88. Does the graph of $y = \sqrt{x} - \ln x$, $x > 0$, have an inflection point? Try to answer the question **(a)** by graphing, **(b)** by using calculus.

7.3 The Exponential Function

Having developed the theory of the function $\ln x$, we introduce the exponential function $\exp x = e^x$ as the inverse of $\ln x$. We study its properties and compute its derivative and integral. Knowing its derivative, we prove the power rule to differentiate x^n when n is *any* real number, rational or irrational.

The Inverse of ln *x* and the Number *e*

The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. As you can see in Figure 7.11,

$$\lim_{x \to \infty} \ln^{-1} x = \infty \qquad \text{and} \qquad \lim_{x \to -\infty} \ln^{-1} x = 0.$$

The function $\ln^{-1} x$ is also denoted by $\exp x$.

In Section 7.2 we defined the number e by the equation $\ln(e) = 1$, so $e = \ln^{-1}(1) = \exp(1)$. Although e is not a rational number, later in this section we see one way to express it as a limit. In Chapter 11, we will calculate its value with a computer to as many places of accuracy as we want with a different formula (Section 11.9, Example 6). To 15 places,

$$e = 2.718281828459045.$$

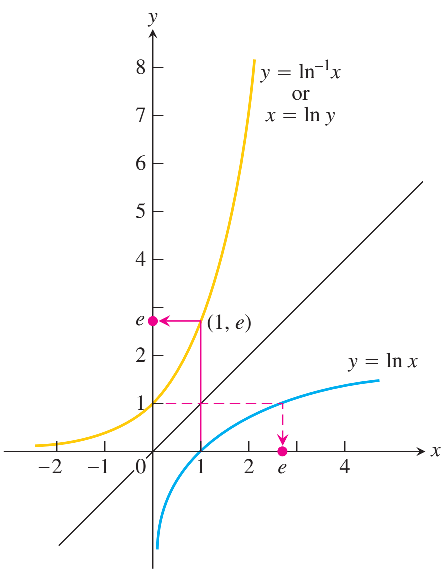


FIGURE 7.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

The Function $y = e^x$

We can raise the number e to a rational power r in the usual way:

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e},$$

and so on. Since e is positive, e^r is positive too. Thus, e^r has a logarithm. When we take the logarithm, we find that

$$\ln e^r = r \ln e = r \cdot 1 = r.$$

Since $\ln x$ is one-to-one and $\ln(\ln^{-1} r) = r$, this equation tells us that

$$e^r = \ln^{-1} r = \exp r \qquad \text{for } r \text{ rational}. \tag{1}$$

We have not yet found a way to give an obvious meaning to e^x for x irrational. But $\ln^{-1} x$ has meaning for any x, rational or irrational. So Equation (1) provides a way to extend the definition of e^x to irrational values of x. The function $\ln^{-1} x$ is defined for all x, so we use it to assign a value to e^x at every point where e^x had no previous definition.

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

DEFINITION The Natural Exponential Function
For every real number x, $e^x = \ln^{-1} x = \exp x$.

For the first time we have a precise meaning for an irrational exponent. Usually the exponential function is denoted by e^x rather than $\exp x$. Since $\ln x$ and e^x are inverses of one another, we have

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \tag{2}$$

$$\ln(e^x) = x \qquad (\text{all } x) \tag{3}$$

Transcendental Numbers and Transcendental Functions

Numbers that are solutions of polynomial equations with rational coefficients are called **algebraic**: -2 is algebraic because it satisfies the equation $x + 2 = 0$, and $\sqrt{3}$ is algebraic because it satisfies the equation $x^2 - 3 = 0$. Numbers that are not algebraic are called **transcendental**, like e and π. In 1873, Charles Hermite proved the transcendence of e in the sense that we describe. In 1882, C.L.F. Lindemann proved the transcendence of π.

Today, we call a function $y = f(x)$ algebraic if it satisfies an equation of the form

$$P_n y^n + \cdots + P_1 y + P_0 = 0$$

in which the P's are polynomials in x with rational coefficients. The function $y = 1/\sqrt{x+1}$ is algebraic because it satisfies the equation $(x+1)y^2 - 1 = 0$. Here the polynomials are $P_2 = x + 1$, $P_1 = 0$, and $P_0 = -1$. Functions that are not algebraic are called transcendental.

The domain of $\ln x$ is $(0, \infty)$ and its range is $(-\infty, \infty)$. So the domain of e^x is $(-\infty, \infty)$ and its range is $(0, \infty)$.

EXAMPLE 1 Using the Inverse Equations

(a) $\ln e^2 = 2$

(b) $\ln e^{-1} = -1$

(c) $\ln \sqrt{e} = \dfrac{1}{2}$

(d) $\ln e^{\sin x} = \sin x$

(e) $e^{\ln 2} = 2$

(f) $e^{\ln(x^2+1)} = x^2 + 1$

(g) $e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$ One way

(h) $e^{3\ln 2} = (e^{\ln 2})^3 = 2^3 = 8$ Another way ■

EXAMPLE 2 Solving for an Exponent

Find k if $e^{2k} = 10$.

Solution Take the natural logarithm of both sides:

$$\begin{aligned} e^{2k} &= 10 \\ \ln e^{2k} &= \ln 10 \\ 2k &= \ln 10 && \text{Eq. (3)} \\ k &= \frac{1}{2}\ln 10. \end{aligned}$$

■

The General Exponential Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can think of a^x as $(e^{\ln a})^x = e^{x \ln a}$. We therefore make the following definition.

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

When $a = e$, the definition gives $a^x = e^{x \ln a} = e^{x \ln e} = e^{x \cdot 1} = e^x$.

HISTORICAL BIOGRAPHY

Siméon Denis Poisson
(1781–1840)

EXAMPLE 3 Evaluating Exponential Functions

(a) $2^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$

(b) $2^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$ ■

We study the calculus of general exponential functions and their inverses in the next section. Here we need the definition in order to discuss the laws of exponents for e^x.

Laws of Exponents

Even though e^x is defined in a seemingly roundabout way as $\ln^{-1} x$, it obeys the familiar laws of exponents from algebra. Theorem 3 shows us that these laws are consequences of the definitions of $\ln x$ and e^x.

THEOREM 3 **Laws of Exponents for e^x**

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Proof of Law 1 Let

$$y_1 = e^{x_1} \qquad \text{and} \qquad y_2 = e^{x_2}. \tag{4}$$

Then

$$\begin{aligned}
x_1 &= \ln y_1 \quad \text{and} \quad x_2 = \ln y_2 && \text{Take logs of both sides of Eqs. (4).}\\
x_1 + x_2 &= \ln y_1 + \ln y_2 \\
&= \ln y_1 y_2 && \text{Product Rule for logarithms}\\
e^{x_1 + x_2} &= e^{\ln y_1 y_2} && \text{Exponentiate.}\\
&= y_1 y_2 && e^{\ln u} = u\\
&= e^{x_1} e^{x_2}.
\end{aligned}$$

■

The proof of Law 4 is similar. Laws 2 and 3 follow from Law 1 (Exercise 78).

EXAMPLE 4 Applying the Exponent Laws

(a) $e^{x + \ln 2} = e^x \cdot e^{\ln 2} = 2e^x$ — Law 1

(b) $e^{-\ln x} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$ — Law 2

(c) $\dfrac{e^{2x}}{e} = e^{2x-1}$ — Law 3

(d) $(e^3)^x = e^{3x} = (e^x)^3$ — Law 4 ■

Theorem 3 is also valid for a^x, the exponential function with base a. For example,

$$\begin{aligned} a^{x_1} \cdot a^{x_2} &= e^{x_1 \ln a} \cdot e^{x_2 \ln a} && \text{Definition of } a^x \\ &= e^{x_1 \ln a + x_2 \ln a} && \text{Law 1} \\ &= e^{(x_1+x_2)\ln a} && \text{Factor } \ln a \\ &= a^{x_1+x_2}. && \text{Definition of } a^x \end{aligned}$$

The Derivative and Integral of e^x

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero (Theorem 1). We calculate its derivative using Theorem 1 and our knowledge of the derivative of $\ln x$. Let

$$f(x) = \ln x \qquad \text{and} \qquad y = e^x = \ln^{-1} x = f^{-1}(x).$$

Then,

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}(e^x) &= \frac{d}{dx} \ln^{-1} x \\ &= \frac{d}{dx} f^{-1}(x) \\ &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\ &= \frac{1}{f'(e^x)} && f^{-1}(x) = e^x \\ &= \frac{1}{\left(\frac{1}{e^x}\right)} && f'(z) = \tfrac{1}{z} \text{ with } z = e^x \\ &= e^x. \end{aligned}$$

That is, for $y = e^x$, we find that $dy/dx = e^x$ so the natural exponential function e^x is its own derivative. We will see in Section 7.5 that the only functions that behave this way are constant multiples of e^x. In summary,

$$\frac{d}{dx} e^x = e^x \tag{5}$$

EXAMPLE 5 Differentiating an Exponential

$$\begin{aligned} \frac{d}{dx}(5e^x) &= 5 \frac{d}{dx} e^x \\ &= 5e^x \end{aligned}$$

■

The Chain Rule extends Equation (5) in the usual way to a more general form.

If u is any differentiable function of x, then

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}. \tag{6}$$

EXAMPLE 6 Applying the Chain Rule with Exponentials

(a) $\dfrac{d}{dx} e^{-x} = e^{-x} \dfrac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (6) with $u = -x$

(b) $\dfrac{d}{dx} e^{\sin x} = e^{\sin x} \dfrac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ Eq. (6) with $u = \sin x$ ■

The integral equivalent of Equation (6) is

$$\int e^u \, du = e^u + C.$$

EXAMPLE 7 Integrating Exponentials

(a)
$$\begin{aligned}
\int_0^{\ln 2} e^{3x} \, dx &= \int_0^{\ln 8} e^u \cdot \frac{1}{3} \, du \qquad u = 3x, \ \frac{1}{3} du = dx, \ u(0) = 0, \ u(\ln 2) = 3 \ln 2 = \ln 2^3 = \ln 8 \\
&= \frac{1}{3} \int_0^{\ln 8} e^u \, du \\
&= \frac{1}{3} e^u \Big]_0^{\ln 8} \\
&= \frac{1}{3}(8 - 1) = \frac{7}{3}
\end{aligned}$$

(b)
$$\begin{aligned}
\int_0^{\pi/2} e^{\sin x} \cos x \, dx &= e^{\sin x} \Big]_0^{\pi/2} \qquad \text{Antiderivative from Example 6} \\
&= e^1 - e^0 = e - 1
\end{aligned}$$
■

EXAMPLE 8 Solving an Initial Value Problem

Solve the initial value problem

$$e^y \frac{dy}{dx} = 2x, \qquad x > \sqrt{3}; \qquad y(2) = 0.$$

Solution We integrate both sides of the differential equation with respect to x to obtain

$$e^y = x^2 + C.$$

We use the initial condition $y(2) = 0$ to determine C:

$$\begin{aligned} C &= e^0 - (2)^2 \\ &= 1 - 4 = -3. \end{aligned}$$

This completes the formula for e^y:

$$e^y = x^2 - 3.$$

To find y, we take logarithms of both sides:

$$\begin{aligned} \ln e^y &= \ln(x^2 - 3) \\ y &= \ln(x^2 - 3). \end{aligned}$$

Notice that the solution is valid for $x > \sqrt{3}$.

Let's check the solution in the original equation.

$$\begin{aligned} e^y \frac{dy}{dx} &= e^y \frac{d}{dx} \ln(x^2 - 3) \\ &= e^y \frac{2x}{x^2 - 3} && \text{Derivative of } \ln(x^2 - 3) \\ &= e^{\ln(x^2-3)} \frac{2x}{x^2 - 3} && y = \ln(x^2 - 3) \\ &= (x^2 - 3) \frac{2x}{x^2 - 3} && e^{\ln y} = y \\ &= 2x. \end{aligned}$$

The solution checks. ■

The Number e Expressed as a Limit

We have defined the number e as the number for which $\ln e = 1$, or the value $\exp(1)$. We see that e is an important constant for the logarithmic and exponential functions, but what is its numerical value? The next theorem shows one way to calculate e as a limit.

THEOREM 4 The Number e as a Limit

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1 + x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1 + x) && \ln 1 = 0 \\ &= \lim_{x \to 0} \ln(1 + x)^{1/x} = \ln\left[\lim_{x \to 0} (1 + x)^{1/x}\right] && \ln \text{ is continuous.} \end{aligned}$$

Because $f'(1) = 1$, we have

$$\ln\left[\lim_{x\to 0}(1 + x)^{1/x}\right] = 1$$

Therefore,

$$\lim_{x\to 0}(1 + x)^{1/x} = e \qquad \ln e = 1 \text{ and } \ln \text{ is one-to-one}$$

■

By substituting $y = 1/x$, we can also express the limit in Theorem 4 as

$$e = \lim_{y\to\infty}\left(1 + \frac{1}{y}\right)^y. \tag{7}$$

At the beginning of the section we noted that $e = 2.718281828459045$ to 15 decimal places.

The Power Rule (General Form)

We can now define x^n for any $x > 0$ and any real number n as $x^n = e^{n\ln x}$. Therefore, the n in the equation $\ln x^n = n\ln x$ no longer needs to be rational—it can be any number as long as $x > 0$:

$$\ln x^n = \ln(e^{n\ln x}) = n\ln x \qquad \ln e^u = u, \text{ any } u$$

Together, the law $a^x/a^y = a^{x-y}$ and the definition $x^n = e^{n\ln x}$ enable us to establish the Power Rule for differentiation in its final form. Differentiating x^n with respect to x gives

$$\begin{aligned}
\frac{d}{dx}x^n &= \frac{d}{dx}e^{n\ln x} && \text{Definition of } x^n,\ x > 0\\
&= e^{n\ln x}\cdot\frac{d}{dx}(n\ln x) && \text{Chain Rule for } e^u\\
&= x^n\cdot\frac{n}{x} && \text{The definition again}\\
&= nx^{n-1}.
\end{aligned}$$

In short, as long as $x > 0$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

The Chain Rule extends this equation to the Power Rule's general form.

Power Rule (General Form)

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

EXAMPLE 9 Using the Power Rule with Irrational Powers

(a) $\dfrac{d}{dx}x^{\sqrt{2}} = \sqrt{2}x^{\sqrt{2}-1} \qquad (x > 0)$

(b) $\dfrac{d}{dx}(2 + \sin 3x)^{\pi} = \pi(2 + \sin 3x)^{\pi-1}(\cos 3x) \cdot 3$

$= 3\pi(2 + \sin 3x)^{\pi-1}(\cos 3x).$ ■

EXERCISES 7.3

Algebraic Calculations with the Exponential and Logarithm

Find simpler expressions for the quantities in Exercises 1–4.

1. **a.** $e^{\ln 7.2}$ **b.** $e^{-\ln x^2}$ **c.** $e^{\ln x - \ln y}$

2. **a.** $e^{\ln (x^2+y^2)}$ **b.** $e^{-\ln 0.3}$ **c.** $e^{\ln \pi x - \ln 2}$

3. **a.** $2 \ln \sqrt{e}$ **b.** $\ln (\ln e^e)$ **c.** $\ln (e^{-x^2-y^2})$

4. **a.** $\ln (e^{\sec \theta})$ **b.** $\ln (e^{(e^x)})$ **c.** $\ln (e^{2 \ln x})$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 5–10, solve for y in terms of t or x, as appropriate.

5. $\ln y = 2t + 4$

6. $\ln y = -t + 5$

7. $\ln (y - 40) = 5t$

8. $\ln (1 - 2y) = t$

9. $\ln (y - 1) - \ln 2 = x + \ln x$

10. $\ln (y^2 - 1) - \ln (y + 1) = \ln (\sin x)$

In Exercises 11 and 12, solve for k.

11. **a.** $e^{2k} = 4$ **b.** $100e^{10k} = 200$ **c.** $e^{k/1000} = a$

12. **a.** $e^{5k} = \frac{1}{4}$ **b.** $80e^k = 1$ **c.** $e^{(\ln 0.8)k} = 0.8$

In Exercises 13–16, solve for t.

13. **a.** $e^{-0.3t} = 27$ **b.** $e^{kt} = \frac{1}{2}$ **c.** $e^{(\ln 0.2)t} = 0.4$

14. **a.** $e^{-0.01t} = 1000$ **b.** $e^{kt} = \frac{1}{10}$ **c.** $e^{(\ln 2)t} = \frac{1}{2}$

15. $e^{\sqrt{t}} = x^2$

16. $e^{(x^2)}e^{(2x+1)} = e^t$

Derivatives

In Exercises 17–36, find the derivative of y with respect to x, t, or θ, as appropriate.

17. $y = e^{-5x}$

18. $y = e^{2x/3}$

19. $y = e^{5-7x}$

20. $y = e^{(4\sqrt{x}+x^2)}$

21. $y = xe^x - e^x$

22. $y = (1 + 2x)e^{-2x}$

23. $y = (x^2 - 2x + 2)e^x$

24. $y = (9x^2 - 6x + 2)e^{3x}$

25. $y = e^{\theta}(\sin \theta + \cos \theta)$

26. $y = \ln (3\theta e^{-\theta})$

27. $y = \cos (e^{-\theta^2})$

28. $y = \theta^3 e^{-2\theta} \cos 5\theta$

29. $y = \ln (3te^{-t})$

30. $y = \ln (2e^{-t} \sin t)$

31. $y = \ln \left(\dfrac{e^{\theta}}{1 + e^{\theta}}\right)$

32. $y = \ln \left(\dfrac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right)$

33. $y = e^{(\cos t + \ln t)}$

34. $y = e^{\sin t}(\ln t^2 + 1)$

35. $y = \displaystyle\int_0^{\ln x} \sin e^t \, dt$

36. $y = \displaystyle\int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

In Exercises 37–40, find dy/dx.

37. $\ln y = e^y \sin x$

38. $\ln xy = e^{x+y}$

39. $e^{2x} = \sin (x + 3y)$

40. $\tan y = e^x + \ln x$

Integrals

Evaluate the integrals in Exercises 41–62.

41. $\displaystyle\int (e^{3x} + 5e^{-x}) \, dx$

42. $\displaystyle\int (2e^x - 3e^{-2x}) \, dx$

43. $\displaystyle\int_{\ln 2}^{\ln 3} e^x \, dx$

44. $\displaystyle\int_{-\ln 2}^{0} e^{-x} \, dx$

45. $\displaystyle\int 8e^{(x+1)} \, dx$

46. $\displaystyle\int 2e^{(2x-1)} \, dx$

47. $\displaystyle\int_{\ln 4}^{\ln 9} e^{x/2} \, dx$

48. $\displaystyle\int_0^{\ln 16} e^{x/4} \, dx$

49. $\displaystyle\int \frac{e^{\sqrt{r}}}{\sqrt{r}} \, dr$

50. $\displaystyle\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} \, dr$

51. $\displaystyle\int 2t \, e^{-t^2} \, dt$

52. $\displaystyle\int t^3 e^{(t^4)} \, dt$

53. $\displaystyle\int \frac{e^{1/x}}{x^2} \, dx$

54. $\displaystyle\int \frac{e^{-1/x^2}}{x^3} \, dx$

55. $\displaystyle\int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta \, d\theta$

56. $\displaystyle\int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta \, d\theta$

57. $\displaystyle\int e^{\sec \pi t} \sec \pi t \tan \pi t \, dt$

58. $\displaystyle\int e^{\csc (\pi + t)} \csc (\pi + t) \cot (\pi + t) \, dt$

59. $\displaystyle\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v\, dv$

60. $\displaystyle\int_0^{\sqrt{\ln \pi}} 2x\, e^{x^2} \cos(e^{x^2})\, dx$

61. $\displaystyle\int \frac{e^r}{1 + e^r}\, dr$

62. $\displaystyle\int \frac{dx}{1 + e^x}$

Initial Value Problems

Solve the initial value problems in Exercises 63–66.

Exercises

63. $\dfrac{dy}{dt} = e^t \sin(e^t - 2), \quad y(\ln 2) = 0$

64. $\dfrac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t}), \quad y(\ln 4) = 2/\pi$

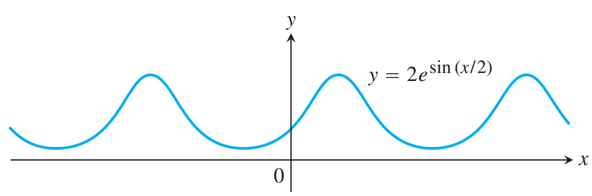
65. $\dfrac{d^2y}{dx^2} = 2e^{-x}, \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0$

66. $\dfrac{d^2y}{dt^2} = 1 - e^{2t}, \quad y(1) = -1 \quad \text{and} \quad y'(1) = 0$

Theory and Applications

67. Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

68. Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?



69. Find the absolute maximum value of $f(x) = x^2 \ln(1/x)$ and say where it is assumed.

T **70.** Graph $f(x) = (x - 3)^2 e^x$ and its first derivative together. Comment on the behavior of f in relation to the signs and values of f'. Identify significant points on the graphs with calculus, as necessary.

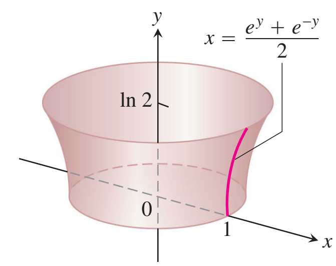
71. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.

72. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2\ln 2$.

73. Find a curve through the origin in the xy-plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\, dx.$$

74. Find the area of the surface generated by revolving the curve $x = (e^y + e^{-y})/2,\ 0 \le y \le \ln 2$, about the y-axis.



75. **a.** Show that $\int \ln x\, dx = x \ln x - x + C$.

b. Find the average value of $\ln x$ over $[1, e]$.

76. Find the average value of $f(x) = 1/x$ on $[1, 2]$.

77. **The linearization of e^x at $x = 0$**

a. Derive the linear approximation $e^x \approx 1 + x$ at $x = 0$.

T **b.** Estimate to five decimal places the magnitude of the error involved in replacing e^x by $1 + x$ on the interval $[0, 0.2]$.

T **c.** Graph e^x and $1 + x$ together for $-2 \le x \le 2$. Use different colors, if available. On what intervals does the approximation appear to overestimate e^x? Underestimate e^x?

78. **Laws of Exponents**

a. Starting with the equation $e^{x_1}e^{x_2} = e^{x_1 + x_2}$, derived in the text, show that $e^{-x} = 1/e^x$ for any real number x. Then show that $e^{x_1}/e^{x_2} = e^{x_1 - x_2}$ for any numbers x_1 and x_2.

b. Show that $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$ for any numbers x_1 and x_2.

T **79.** **A decimal representation of e** Find e to as many decimal places as your calculator allows by solving the equation $\ln x = 1$.

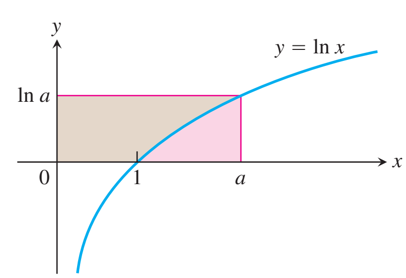
T **80.** **The inverse relation between e^x and $\ln x$** Find out how good your calculator is at evaluating the composites

$$e^{\ln x} \quad \text{and} \quad \ln(e^x).$$

81. Show that for any number $a > 1$

$$\int_1^a \ln x\, dx + \int_0^{\ln a} e^y\, dy = a \ln a.$$

(See accompanying figure.)



82. **The geometric, logarithmic, and arithmetic mean inequality**

a. Show that the graph of e^x is concave up over every interval of x-values.