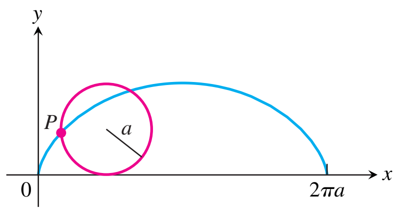
we parametrize our curves (except for the mild restrictions preventing doubling back mentioned earlier), calculate the length of the semicircle $y = \sqrt{1 - x^2}$ with these two different parametrizations:

**a.** $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi/2$

**b.** $x = \sin \pi t, \quad y = \cos \pi t, \quad -1/2 \le t \le 1/2$

**30.** Find the length of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$, shown in the accompanying figure. A **cycloid** is the curve traced out by a point $P$ on the circumference of a circle rolling along a straight line, such as the $x$-axis.




## COMPUTER EXPLORATIONS

In Exercises 31–36, use a CAS to perform the following steps for the given curve over the closed interval.

**a.** Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 6.24.)

**b.** Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.

**c.** Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ with the actual length given by the integral. How does the actual length compare with the approximations as $n$ increases? Explain your answer.

**31.** $f(x) = \sqrt{1 - x^2}, \quad -1 \le x \le 1$

**32.** $f(x) = x^{1/3} + x^{2/3}, \quad 0 \le x \le 2$

**33.** $f(x) = \sin(\pi x^2), \quad 0 \le x \le \sqrt{2}$

**34.** $f(x) = x^2 \cos x, \quad 0 \le x \le \pi$

**35.** $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad -\dfrac{1}{2} \le x \le 1$

**36.** $f(x) = x^3 - x^2, \quad -1 \le x \le 1$

**37.** $x = \dfrac{1}{3}t^3, \quad y = \dfrac{1}{2}t^2, \quad 0 \le t \le 1$

**38.** $x = 2t^3 - 16t^2 + 25t + 5, \quad y = t^2 + t - 3,$
$0 \le t \le 6$

**39.** $x = t - \cos t, \quad y = 1 + \sin t, \quad -\pi \le t \le \pi$

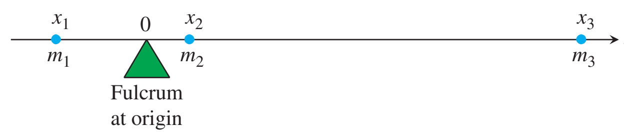
**40.** $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \le t \le \pi$

# 6.4 Moments and Centers of Mass

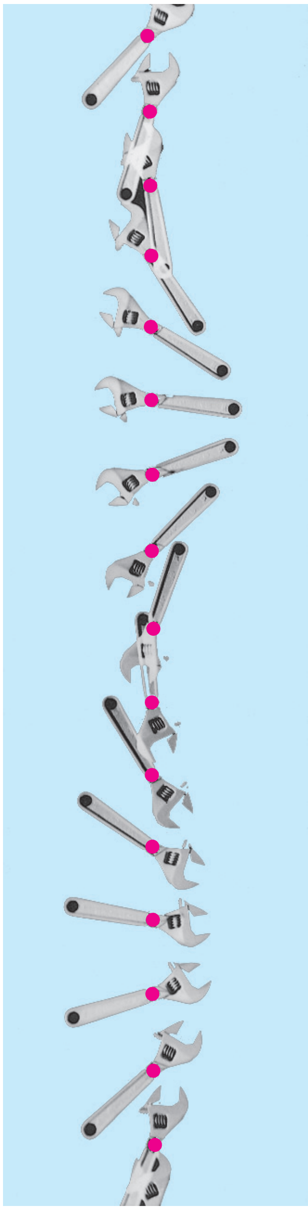
Many structures and mechanical systems behave as if their masses were concentrated at a single point, called the *center of mass* (Figure 6.29). It is important to know how to locate this point, and doing so is basically a mathematical enterprise. For the moment, we deal with one- and two-dimensional objects. Three-dimensional objects are best done with the multiple integrals of Chapter 15.

## Masses Along a Line

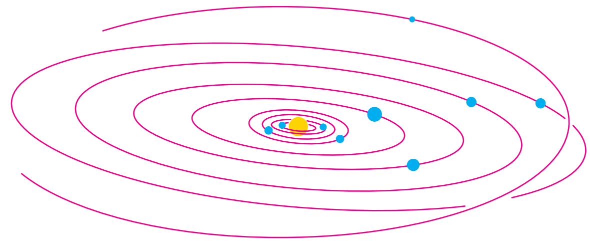
We develop our mathematical model in stages. The first stage is to imagine masses $m_1$, $m_2$, and $m_3$ on a rigid $x$-axis supported by a fulcrum at the origin.




The resulting system might balance, or it might not. It depends on how large the masses are and how they are arranged.



(a)



(b)

**FIGURE 6.29** (a) The motion of this wrench gliding on ice seems haphazard until we notice that the wrench is simply turning about its center of mass as the center glides in a straight line. (b) The planets, asteroids, and comets of our solar system revolve about their collective center of mass. (It lies inside the sun.)

Each mass $m_k$ exerts a downward force $m_k g$ (the weight of $m_k$) equal to the magnitude of the mass times the acceleration of gravity. Each of these forces has a tendency to turn the axis about the origin, the way you turn a seesaw. This turning effect, called a **torque**, is measured by multiplying the force $m_k g$ by the signed distance $x_k$ from the point of application to the origin. Masses to the left of the origin exert negative (counterclockwise) torque. Masses to the right of the origin exert positive (clockwise) torque.

The sum of the torques measures the tendency of a system to rotate about the origin. This sum is called the **system torque**.

$$\text{System torque} = m_1 g x_1 + m_2 g x_2 + m_3 g x_3 \tag{1}$$

The system will balance if and only if its torque is zero.

If we factor out the $g$ in Equation (1), we see that the system torque is

$$\underbrace{g}_{\substack{\text{a feature of the}\\ \text{environment}}} \cdot \underbrace{(m_1 x_1 + m_2 x_2 + m_3 x_3)}_{\substack{\text{a feature of}\\ \text{the system}}}$$

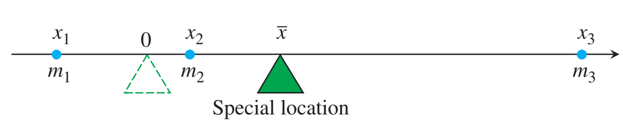
Thus, the torque is the product of the gravitational acceleration $g$, which is a feature of the environment in which the system happens to reside, and the number $(m_1 x_1 + m_2 x_2 + m_3 x_3)$, which is a feature of the system itself, a constant that stays the same no matter where the system is placed.

The number $(m_1 x_1 + m_2 x_2 + m_3 x_3)$ is called the **moment of the system about the origin**. It is the sum of the **moments** $m_1 x_1$, $m_2 x_2$, $m_3 x_3$ of the individual masses.

$$M_0 = \text{Moment of system about origin} = \sum m_k x_k .$$

(We shift to sigma notation here to allow for sums with more terms.)

We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torques add to zero.



The torque of each mass about the fulcrum in this special location is

$$\begin{aligned}\text{Torque of } m_k \text{ about } \bar{x} &= \begin{pmatrix}\text{signed distance}\\ \text{of } m_k \text{ from } \bar{x}\end{pmatrix}\begin{pmatrix}\text{downward}\\ \text{force}\end{pmatrix}\\ &= (x_k - \bar{x})m_k g.\end{aligned}$$

When we write the equation that says that the sum of these torques is zero, we get an equation we can solve for $\bar{x}$:

$$\begin{aligned}\sum (x_k - \bar{x})m_k g &= 0 && \text{Sum of the torques equals zero}\\ g\sum (x_k - \bar{x})m_k &= 0 && \text{Constant Multiple Rule for Sums}\\ \sum (m_k x_k - \bar{x}m_k) &= 0 && g \text{ divided out, } m_k \text{ distributed}\\ \sum m_k x_k - \sum \bar{x}m_k &= 0 && \text{Difference Rule for Sums}\\ \sum m_k x_k &= \bar{x}\sum m_k && \text{Rearranged, Constant Multiple Rule again}\\ \bar{x} &= \frac{\sum m_k x_k}{\sum m_k}. && \text{Solved for } \bar{x}\end{aligned}$$

This last equation tells us to find $\bar{x}$ by dividing the system's moment about the origin by the system's total mass:

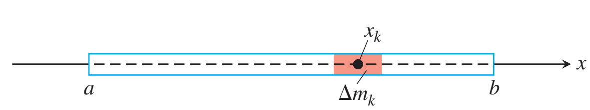
$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{\text{system moment about origin}}{\text{system mass}}.$$

The point $\bar{x}$ is called the system's **center of mass**.

## Wires and Thin Rods

In many applications, we want to know the center of mass of a rod or a thin strip of metal. In cases like these where we can model the distribution of mass with a continuous function, the summation signs in our formulas become integrals in a manner we now describe.

Imagine a long, thin strip lying along the $x$-axis from $x = a$ to $x = b$ and cut into small pieces of mass $\Delta m_k$ by a partition of the interval $[a, b]$. Choose $x_k$ to be any point in the $k$th subinterval of the partition.




The $k$th piece is $\Delta x_k$ units long and lies approximately $x_k$ units from the origin. Now observe three things.

First, the strip's center of mass $\bar{x}$ is nearly the same as that of the system of point masses we would get by attaching each mass $\Delta m_k$ to the point $x_k$:

$$\bar{x} \approx \frac{\text{system moment}}{\text{system mass}}.$$

**Density**
A material's density is its mass per unit volume. In practice, however, we tend to use units we can conveniently measure. For wires, rods, and narrow strips, we use mass per unit length. For flat sheets and plates, we use mass per unit area.

Second, the moment of each piece of the strip about the origin is approximately $x_k\,\Delta m_k$, so the system moment is approximately the sum of the $x_k\,\Delta m_k$:

$$\text{System moment} \approx \sum x_k\,\Delta m_k.$$

Third, if the density of the strip at $x_k$ is $\delta(x_k)$, expressed in terms of mass per unit length and if $\delta$ is continuous, then $\Delta m_k$ is approximately equal to $\delta(x_k)\,\Delta x_k$ (mass per unit length times length):

$$\Delta m_k \approx \delta(x_k)\,\Delta x_k.$$

Combining these three observations gives

$$\bar{x} \approx \frac{\text{system moment}}{\text{system mass}} \approx \frac{\sum x_k\,\Delta m_k}{\sum \Delta m_k} \approx \frac{\sum x_k\,\delta(x_k)\,\Delta x_k}{\sum \delta(x_k)\,\Delta x_k}. \tag{2}$$

The sum in the last numerator in Equation (2) is a Riemann sum for the continuous function $x\delta(x)$ over the closed interval $[a, b]$. The sum in the denominator is a Riemann sum for the function $\delta(x)$ over this interval. We expect the approximations in Equation (2) to improve as the strip is partitioned more finely, and we are led to the equation

$$\bar{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx}.$$

This is the formula we use to find $\bar{x}$.

**Moment, Mass, and Center of Mass of a Thin Rod or Strip Along the $x$-Axis with Density Function $\delta(x)$**

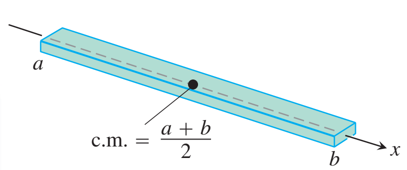
Moment about the origin: $$M_0 = \int_a^b x\delta(x)\,dx \tag{3a}$$

Mass: $$M = \int_a^b \delta(x)\,dx \tag{3b}$$

Center of mass: $$\bar{x} = \frac{M_0}{M} \tag{3c}$$

## EXAMPLE 1 Strips and Rods of Constant Density

Show that the center of mass of a straight, thin strip or rod of constant density lies halfway between its two ends.




You Try It

**FIGURE 6.30** The center of mass of a straight, thin rod or strip of constant density lies halfway between its ends (Example 1).

**Solution** We model the strip as a portion of the $x$-axis from $x = a$ to $x = b$ (Figure 6.30). Our goal is to show that $\bar{x} = (a + b)/2$, the point halfway between $a$ and $b$.

The key is the density's having a constant value. This enables us to regard the function $\delta(x)$ in the integrals in Equation (3) as a constant (call it $\delta$), with the result that

$$M_0 = \int_a^b \delta x\,dx = \delta \int_a^b x\,dx = \delta\left[\frac{1}{2}x^2\right]_a^b = \frac{\delta}{2}(b^2 - a^2)$$

$$M = \int_a^b \delta\,dx = \delta \int_a^b dx = \delta\big[x\big]_a^b = \delta(b - a)$$

$$\bar{x} = \frac{M_0}{M} = \frac{\frac{\delta}{2}(b^2 - a^2)}{\delta(b - a)}$$

$$= \frac{a + b}{2}.$$

The $\delta$'s cancel in the formula for $\bar{x}$.

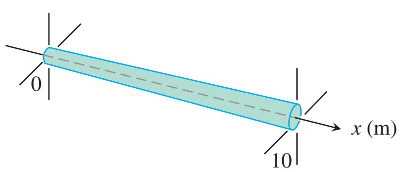



**FIGURE 6.31** We can treat a rod of variable thickness as a rod of variable density (Example 2).

## EXAMPLE 2 Variable-Density Rod

The 10-m-long rod in Figure 6.31 thickens from left to right so that its density, instead of being constant, is $\delta(x) = 1 + (x/10)$ kg/m. Find the rod's center of mass.

**Solution** The rod's moment about the origin (Equation 3a) is

$$M_0 = \int_0^{10} x\delta(x)\,dx = \int_0^{10} x\left(1 + \frac{x}{10}\right)dx = \int_0^{10}\left(x + \frac{x^2}{10}\right)dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{30}\right]_0^{10} = 50 + \frac{100}{3} = \frac{250}{3}\ \text{kg}\cdot\text{m}.$$

The units of a moment are mass $\times$ length.

The rod's mass (Equation 3b) is

$$M = \int_0^{10} \delta(x)\,dx = \int_0^{10}\left(1 + \frac{x}{10}\right)dx = \left[x + \frac{x^2}{20}\right]_0^{10} = 10 + 5 = 15\ \text{kg}.$$

The center of mass (Equation 3c) is located at the point

$$\bar{x} = \frac{M_0}{M} = \frac{250}{3}\cdot\frac{1}{15} = \frac{50}{9} \approx 5.56\ \text{m}.$$

## Masses Distributed over a Plane Region

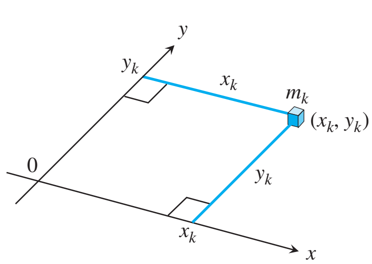



**FIGURE 6.32** Each mass $m_k$ has a moment about each axis.

Suppose that we have a finite collection of masses located in the plane, with mass $m_k$ at the point $(x_k, y_k)$ (see Figure 6.32). The mass of the system is

$$\text{System mass: } M = \sum m_k.$$

Each mass $m_k$ has a moment about each axis. Its moment about the $x$-axis is $m_k y_k$, and its moment about the $y$-axis is $m_k x_k$. The moments of the entire system about the two axes are

$$\text{Moment about } x\text{-axis:} \qquad M_x = \sum m_k y_k,$$

$$\text{Moment about } y\text{-axis:} \qquad M_y = \sum m_k x_k.$$

The $x$-coordinate of the system's center of mass is defined to be

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}. \tag{4}$$

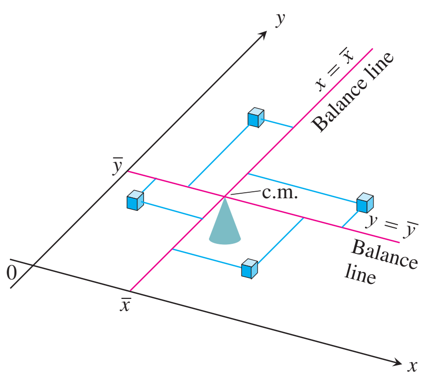



**FIGURE 6.33** A two-dimensional array of masses balances on its center of mass.

With this choice of $\bar{x}$, as in the one-dimensional case, the system balances about the line $x = \bar{x}$ (Figure 6.33).

The $y$-coordinate of the system's center of mass is defined to be

$$\bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}. \tag{5}$$

With this choice of $\bar{y}$, the system balances about the line $y = \bar{y}$ as well. The torques exerted by the masses about the line $y = \bar{y}$ cancel out. Thus, as far as balance is concerned, the system behaves as if all its mass were at the single point $(\bar{x}, \bar{y})$. We call this point the system's **center of mass**.

## Thin, Flat Plates

In many applications, we need to find the center of mass of a thin, flat plate: a disk of aluminum, say, or a triangular sheet of steel. In such cases, we assume the distribution of mass to be continuous, and the formulas we use to calculate $\bar{x}$ and $\bar{y}$ contain integrals instead of finite sums. The integrals arise in the following way.

Imagine the plate occupying a region in the $xy$-plane, cut into thin strips parallel to one of the axes (in Figure 6.34, the $y$-axis). The center of mass of a typical strip is $(\tilde{x}, \tilde{y})$. We treat the strip's mass $\Delta m$ as if it were concentrated at $(\tilde{x}, \tilde{y})$. The moment of the strip about the $y$-axis is then $\tilde{x}\,\Delta m$. The moment of the strip about the $x$-axis is $\tilde{y}\,\Delta m$. Equations (4) and (5) then become

$$\bar{x} = \frac{M_y}{M} = \frac{\sum \tilde{x}\,\Delta m}{\sum \Delta m}, \qquad \bar{y} = \frac{M_x}{M} = \frac{\sum \tilde{y}\,\Delta m}{\sum \Delta m}.$$

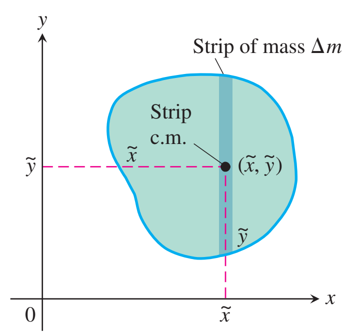



**FIGURE 6.34** A plate cut into thin strips parallel to the $y$-axis. The moment exerted by a typical strip about each axis is the moment its mass $\Delta m$ would exert if concentrated at the strip's center of mass $(\tilde{x}, \tilde{y})$.

As in the one-dimensional case, the sums are Riemann sums for integrals and approach these integrals as limiting values as the strips into which the plate is cut become narrower and narrower. We write these integrals symbolically as

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \quad \text{and} \quad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm}.$$

**Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the *xy*-Plane**

Moment about the $x$-axis: $\quad M_x = \int \tilde{y}\,dm$

Moment about the $y$-axis: $\quad M_y = \int \tilde{x}\,dm \qquad (6)$

Mass: $\quad M = \int dm$

Center of mass: $\quad \bar{x} = \dfrac{M_y}{M}, \quad \bar{y} = \dfrac{M_x}{M}$

To evaluate these integrals, we picture the plate in the coordinate plane and sketch a strip of mass parallel to one of the coordinates axes. We then express the strip's mass $dm$ and the coordinates $(\tilde{x}, \tilde{y})$ of the strip's center of mass in terms of $x$ or $y$. Finally, we integrate $\tilde{y}\,dm$, $\tilde{x}\,dm$, and $dm$ between limits of integration determined by the plate's location in the plane.

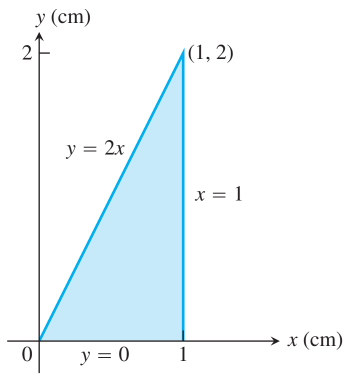



**FIGURE 6.35** The plate in Example 3.

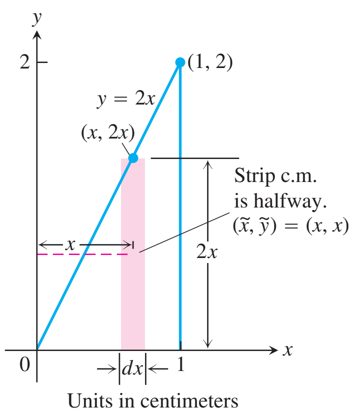



**FIGURE 6.36** Modeling the plate in Example 3 with vertical strips.

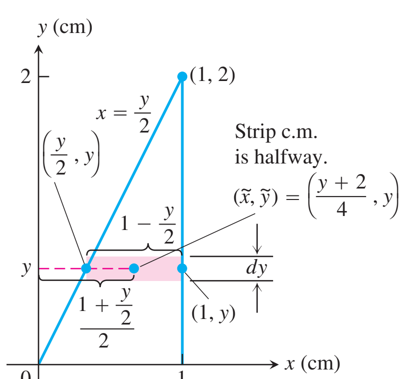



**FIGURE 6.37** Modeling the plate in Example 3 with horizontal strips.

## EXAMPLE 3 Constant-Density Plate

The triangular plate shown in Figure 6.35 has a constant density of $\delta = 3\ \text{g/cm}^2$. Find

**(a)** the plate's moment $M_y$ about the $y$-axis.

**(b)** the plate's mass $M$.

**(c)** the $x$-coordinate of the plate's center of mass (c.m.).

**Solution**

**Method 1: Vertical Strips** (Figure 6.36)

**(a)** The moment $M_y$: The typical vertical strip has

center of mass (c.m.): $(\tilde{x}, \tilde{y}) = (x, x)$
length: $2x$
width: $dx$
area: $dA = 2x\, dx$
mass: $dm = \delta\, dA = 3 \cdot 2x\, dx = 6x\, dx$
distance of c.m. from $y$-axis: $\tilde{x} = x$.

The moment of the strip about the $y$-axis is

$$\tilde{x}\, dm = x \cdot 6x\, dx = 6x^2\, dx.$$

The moment of the plate about the $y$-axis is therefore

$$M_y = \int \tilde{x}\, dm = \int_0^1 6x^2\, dx = 2x^3\Big]_0^1 = 2\ \text{g} \cdot \text{cm}.$$

**(b)** The plate's mass:

$$M = \int dm = \int_0^1 6x\, dx = 3x^2\Big]_0^1 = 3\ \text{g}.$$

**(c)** The $x$-coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2\ \text{g} \cdot \text{cm}}{3\ \text{g}} = \frac{2}{3}\ \text{cm}.$$

By a similar computation, we could find $M_x$ and $\bar{y} = M_x/M$.

**Method 2: Horizontal Strips** (Figure 6.37)

**(a)** The moment $M_y$: The $y$-coordinate of the center of mass of a typical horizontal strip is $y$ (see the figure), so

$$\tilde{y} = y.$$

The $x$-coordinate is the $x$-coordinate of the point halfway across the triangle. This makes it the average of $y/2$ (the strip's left-hand $x$-value) and 1 (the strip's right-hand $x$-value):

$$\tilde{x} = \frac{(y/2) + 1}{2} = \frac{y}{4} + \frac{1}{2} = \frac{y + 2}{4}.$$

We also have

$$\text{length:} \quad 1 - \frac{y}{2} = \frac{2-y}{2}$$

$$\text{width:} \quad dy$$

$$\text{area:} \quad dA = \frac{2-y}{2}\,dy$$

$$\text{mass:} \quad dm = \delta\, dA = 3 \cdot \frac{2-y}{2}\,dy$$

$$\text{distance of c.m. to } y\text{-axis:} \quad \tilde{x} = \frac{y+2}{4}.$$

The moment of the strip about the $y$-axis is

$$\tilde{x}\,dm = \frac{y+2}{4} \cdot 3 \cdot \frac{2-y}{2}\,dy = \frac{3}{8}(4 - y^2)\,dy.$$

The moment of the plate about the $y$-axis is

$$M_y = \int \tilde{x}\,dm = \int_0^2 \frac{3}{8}(4 - y^2)\,dy = \frac{3}{8}\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{3}{8}\left(\frac{16}{3}\right) = 2\ \text{g}\cdot\text{cm}.$$

**(b)** The plate's mass:

$$M = \int dm = \int_0^2 \frac{3}{2}(2 - y)\,dy = \frac{3}{2}\left[2y - \frac{y^2}{2}\right]_0^2 = \frac{3}{2}(4-2) = 3\ \text{g}.$$

**(c)** The $x$-coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2\ \text{g}\cdot\text{cm}}{3\ \text{g}} = \frac{2}{3}\ \text{cm}.$$

By a similar computation, we could find $M_x$ and $\bar{y}$. ■

### How to Find a Plate's Center of Mass

1. Picture the plate in the $xy$-plane.
2. Sketch a strip of mass parallel to one of the coordinate axes and find its dimensions.
3. Find the strip's mass $dm$ and center of mass $(\tilde{x}, \tilde{y})$.
4. Integrate $\tilde{y}\,dm$, $\tilde{x}\,dm$, and $dm$ to find $M_x$, $M_y$, and $M$.
5. Divide the moments by the mass to calculate $\bar{x}$ and $\bar{y}$.

If the distribution of mass in a thin, flat plate has an axis of symmetry, the center of mass will lie on this axis. If there are two axes of symmetry, the center of mass will lie at their intersection. These facts often help to simplify our work.

You Try It

## EXAMPLE 4 Constant-Density Plate

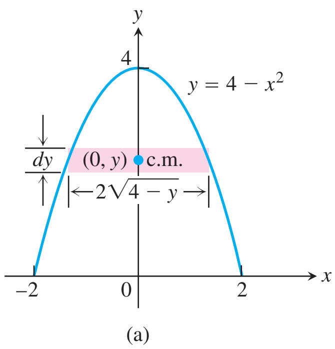
Find the center of mass of a thin plate of constant density $\delta$ covering the region bounded above by the parabola $y = 4 - x^2$ and below by the $x$-axis (Figure 6.38).

**Solution** Since the plate is symmetric about the $y$-axis and its density is constant, the distribution of mass is symmetric about the $y$-axis and the center of mass lies on the $y$-axis. Thus, $\bar{x} = 0$. It remains to find $\bar{y} = M_x/M$.

A trial calculation with horizontal strips (Figure 6.38a) leads to an inconvenient integration

$$M_x = \int_0^4 2\delta y\sqrt{4 - y}\,dy.$$

We therefore model the distribution of mass with vertical strips instead (Figure 6.38b).



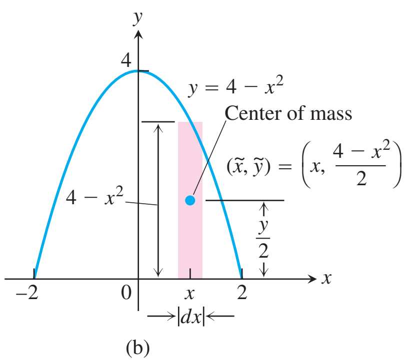




**FIGURE 6.38** Modeling the plate in Example 4 with (a) horizontal strips leads to an inconvenient integration, so we model with (b) vertical strips instead.

The typical vertical strip has

$$\text{center of mass (c.m.):} \quad (\tilde{x}, \tilde{y}) = \left(x, \frac{4 - x^2}{2}\right)$$

$$\text{length:} \quad 4 - x^2$$

$$\text{width:} \quad dx$$

$$\text{area:} \quad dA = (4 - x^2)\,dx$$

$$\text{mass:} \quad dm = \delta\,dA = \delta(4 - x^2)\,dx$$

$$\text{distance from c.m. to } x\text{-axis:} \quad \tilde{y} = \frac{4 - x^2}{2}.$$

The moment of the strip about the $x$-axis is

$$\tilde{y}\,dm = \frac{4 - x^2}{2} \cdot \delta(4 - x^2)\,dx = \frac{\delta}{2}(4 - x^2)^2\,dx.$$

The moment of the plate about the $x$-axis is

$$M_x = \int \tilde{y}\,dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\,dx$$

$$= \frac{\delta}{2}\int_{-2}^{2}(16 - 8x^2 + x^4)\,dx = \frac{256}{15}\delta. \qquad (7)$$

The mass of the plate is

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\,dx = \frac{32}{3}\delta. \qquad (8)$$

Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{(256/15)\,\delta}{(32/3)\,\delta} = \frac{8}{5}.$$

The plate's center of mass is the point

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right).$$

■


**EXAMPLE 5** Variable-Density Plate

Find the center of mass of the plate in Example 4 if the density at the point $(x, y)$ is $\delta = 2x^2$, twice the square of the distance from the point to the $y$-axis.

**Solution** The mass distribution is still symmetric about the $y$-axis, so $\bar{x} = 0$. With $\delta = 2x^2$, Equations (7) and (8) become

$$M_x = \int \tilde{y}\, dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\, dx = \int_{-2}^{2} x^2(4 - x^2)^2\, dx$$

$$= \int_{-2}^{2} (16x^2 - 8x^4 + x^6)\, dx = \frac{2048}{105} \qquad (7')$$

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\, dx = \int_{-2}^{2} 2x^2(4 - x^2)\, dx$$

$$= \int_{-2}^{2} (8x^2 - 2x^4)\, dx = \frac{256}{15}. \qquad (8')$$

Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{2048}{105} \cdot \frac{15}{256} = \frac{8}{7}.$$

The plate's new center of mass is

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{7}\right).$$

## EXAMPLE 6 Constant-Density Wire

Find the center of mass of a wire of constant density $\delta$ shaped like a semicircle of radius $a$.

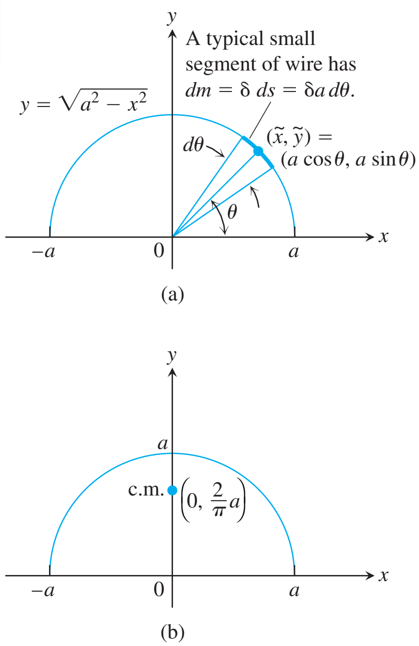



**FIGURE 6.39** The semicircular wire in Example 6. (a) The dimensions and variables used in finding the center of mass. (b) The center of mass does not lie on the wire.

**Solution** We model the wire with the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.39). The distribution of mass is symmetric about the $y$-axis, so $\bar{x} = 0$. To find $\bar{y}$, we imagine the wire divided into short segments. The typical segment (Figure 6.39a) has

length: $ds = a\, d\theta$

mass: $dm = \delta\, ds = \delta a\, d\theta$ Mass per unit length times length

distance of c.m. to $x$-axis: $\tilde{y} = a\sin\theta$.

Hence,

$$\bar{y} = \frac{\int \tilde{y}\, dm}{\int dm} = \frac{\int_0^{\pi} a\sin\theta \cdot \delta a\, d\theta}{\int_0^{\pi} \delta a\, d\theta} = \frac{\delta a^2\left[-\cos\theta\right]_0^{\pi}}{\delta a\pi} = \frac{2}{\pi}a.$$

The center of mass lies on the axis of symmetry at the point $(0, 2a/\pi)$, about two-thirds of the way up from the origin (Figure 6.39b).

## Centroids

When the density function is constant, it cancels out of the numerator and denominator of the formulas for $\bar{x}$ and $\bar{y}$. This happened in nearly every example in this section. As far as $\bar{x}$ and $\bar{y}$ were concerned, $\delta$ might as well have been 1. Thus, when the density is constant, the location of the center of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases, engineers may call the center of mass the **centroid** of the shape, as in "Find the centroid of a triangle or a solid cone." To do so, just set $\delta$ equal to 1 and proceed to find $\bar{x}$ and $\bar{y}$ as before, by dividing moments by masses.

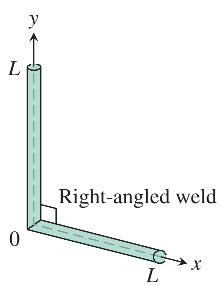
# EXERCISES 6.4

## Thin Rods

**1.** An 80-lb child and a 100-lb child are balancing on a seesaw. The 80-lb child is 5 ft from the fulcrum. How far from the fulcrum is the 100-lb child?

**2.** The ends of a log are placed on two scales. One scale reads 100 kg and the other 200 kg. Where is the log's center of mass?

**3.** The ends of two thin steel rods of equal length are welded together to make a right-angled frame. Locate the frame's center of mass. (*Hint:* Where is the center of mass of each rod?)




**4.** You weld the ends of two steel rods into a right-angled frame. One rod is twice the length of the other. Where is the frame's center of mass? (*Hint:* Where is the center of mass of each rod?)

Exercises 5–12 give density functions of thin rods lying along various intervals of the $x$-axis. Use Equations (3a) through (3c) to find each rod's moment about the origin, mass, and center of mass.

**5.** $\delta(x) = 4, \quad 0 \le x \le 2$

**6.** $\delta(x) = 4, \quad 1 \le x \le 3$

**7.** $\delta(x) = 1 + (x/3), \quad 0 \le x \le 3$

**8.** $\delta(x) = 2 - (x/4), \quad 0 \le x \le 4$

**9.** $\delta(x) = 1 + \left(1/\sqrt{x}\right), \quad 1 \le x \le 4$

**10.** $\delta(x) = 3(x^{-3/2} + x^{-5/2}), \quad 0.25 \le x \le 1$

**11.** $\delta(x) = \begin{cases} 2 - x, & 0 \le x < 1 \\ x, & 1 \le x \le 2 \end{cases}$

**12.** $\delta(x) = \begin{cases} x + 1, & 0 \le x < 1 \\ 2, & 1 \le x \le 2 \end{cases}$

## Thin Plates with Constant Density

In Exercises 13–24, find the center of mass of a thin plate of constant density $\delta$ covering the given region.

**13.** The region bounded by the parabola $y = x^2$ and the line $y = 4$

**14.** The region bounded by the parabola $y = 25 - x^2$ and the $x$-axis

**15.** The region bounded by the parabola $y = x - x^2$ and the line $y = -x$

**16.** The region enclosed by the parabolas $y = x^2 - 3$ and $y = -2x^2$

**17.** The region bounded by the $y$-axis and the curve $x = y - y^3$, $0 \le y \le 1$

**18.** The region bounded by the parabola $x = y^2 - y$ and the line $y = x$

**19.** The region bounded by the $x$-axis and the curve $y = \cos x$, $-\pi/2 \le x \le \pi/2$

**20.** The region between the $x$-axis and the curve $y = \sec^2 x$, $-\pi/4 \le x \le \pi/4$

**21.** The region bounded by the parabolas $y = 2x^2 - 4x$ and $y = 2x - x^2$

**22.** **a.** The region cut from the first quadrant by the circle $x^2 + y^2 = 9$

**b.** The region bounded by the $x$-axis and the semicircle $y = \sqrt{9 - x^2}$

Compare your answer in part (b) with the answer in part (a).

**23.** The "triangular" region in the first quadrant between the circle $x^2 + y^2 = 9$ and the lines $x = 3$ and $y = 3$. (*Hint:* Use geometry to find the area.)

**24.** The region bounded above by the curve $y = 1/x^3$, below by the curve $y = -1/x^3$, and on the left and right by the lines $x = 1$ and $x = a > 1$. Also, find $\lim_{a\to\infty} \bar{x}$.

## Thin Plates with Varying Density

**25.** Find the center of mass of a thin plate covering the region between the $x$-axis and the curve $y = 2/x^2$, $1 \le x \le 2$, if the plate's density at the point $(x, y)$ is $\delta(x) = x^2$.

**26.** Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$ if the plate's density at the point $(x, y)$ is $\delta(x) = 12x$.

**27.** The region bounded by the curves $y = \pm 4/\sqrt{x}$ and the lines $x = 1$ and $x = 4$ is revolved about the $y$-axis to generate a solid.

**a.** Find the volume of the solid.

**b.** Find the center of mass of a thin plate covering the region if the plate's density at the point $(x, y)$ is $\delta(x) = 1/x$.

**c.** Sketch the plate and show the center of mass in your sketch.

**28.** The region between the curve $y = 2/x$ and the $x$-axis from $x = 1$ to $x = 4$ is revolved about the $x$-axis to generate a solid.

**a.** Find the volume of the solid.

**b.** Find the center of mass of a thin plate covering the region if the plate's density at the point $(x, y)$ is $\delta(x) = \sqrt{x}$.

**c.** Sketch the plate and show the center of mass in your sketch.

## Centroids of Triangles

**29.** **The centroid of a triangle lies at the intersection of the triangle's medians** (*Figure 6.40a*) You may recall that the point inside a triangle that lies one-third of the way from each side toward the opposite vertex is the point where the triangle's three medians intersect. Show that the centroid lies at the intersection of the medians by showing that it too lies one-third of the way from each side toward the opposite vertex. To do so, take the following steps.

**i.** Stand one side of the triangle on the $x$-axis as in Figure 6.40b. Express $dm$ in terms of $L$ and $dy$.

**ii.** Use similar triangles to show that $L = (b/h)(h - y)$. Substitute this expression for $L$ in your formula for $dm$.

**iii.** Show that $\bar{y} = h/3$.

**iv.** Extend the argument to the other sides.

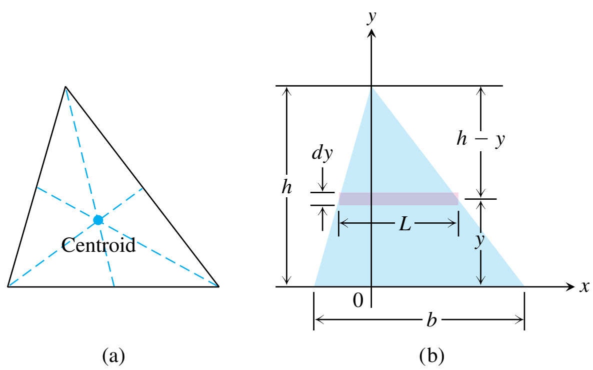



**FIGURE 6.40** The triangle in Exercise 29. (a) The centroid. (b) The dimensions and variables to use in locating the center of mass.

Use the result in Exercise 29 to find the centroids of the triangles whose vertices appear in Exercises 30–34. Assume $a, b > 0$.

**30.** $(-1, 0), (1, 0), (0, 3)$

**31.** $(0, 0), (1, 0), (0, 1)$

**32.** $(0, 0), (a, 0), (0, a)$

**33.** $(0, 0), (a, 0), (0, b)$

**34.** $(0, 0), (a, 0), (a/2, b)$

## Thin Wires

Exercises

**35. Constant density** Find the moment about the $x$-axis of a wire of constant density that lies along the curve $y = \sqrt{x}$ from $x = 0$ to $x = 2$.

**36. Constant density** Find the moment about the $x$-axis of a wire of constant density that lies along the curve $y = x^3$ from $x = 0$ to $x = 1$.

**37. Variable density** Suppose that the density of the wire in Example 6 is $\delta = k \sin \theta$ ($k$ constant). Find the center of mass.

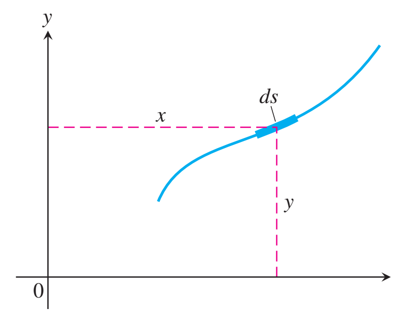
**38. Variable density** Suppose that the density of the wire in Example 6 is $\delta = 1 + k|\cos \theta|$ ($k$ constant). Find the center of mass.

## Engineering Formulas

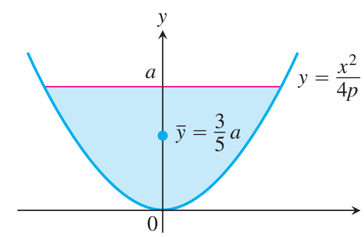
Verify the statements and formulas in Exercises 39–42.

**39.** The coordinates of the centroid of a differentiable plane curve are

$$\bar{x} = \frac{\int x\, ds}{\text{length}}, \qquad \bar{y} = \frac{\int y\, ds}{\text{length}}.$$

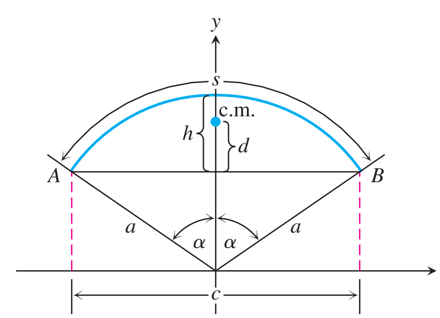



**40.** Whatever the value of $p > 0$ in the equation $y = x^2/(4p)$, the $y$-coordinate of the centroid of the parabolic segment shown here is $\bar{y} = (3/5)a$.




**41.** For wires and thin rods of constant density shaped like circular arcs centered at the origin and symmetric about the $y$-axis, the $y$-coordinate of the center of mass is

$$\bar{y} = \frac{a \sin \alpha}{\alpha} = \frac{ac}{s}.$$




**42.** (*Continuation of Exercise 41.*)

**a.** Show that when $\alpha$ is small, the distance $d$ from the centroid to chord $AB$ is about $2h/3$ (in the notation of the figure here) by taking the following steps.

**i.** Show that

$$\frac{d}{h} = \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha}. \tag{9}$$

T **ii.** Graph

$$f(\alpha) = \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha}$$

and use the trace feature to show that $\lim_{\alpha \to 0^+} f(\alpha) \approx 2/3$.

**b.** The error (difference between $d$ and $2h/3$) is small even for angles greater that 45°. See for yourself by evaluating the right-hand side of Equation (9) for $\alpha = 0.2, 0.4, 0.6, 0.8$, and $1.0$ rad.

# 6.5 Areas of Surfaces of Revolution and the Theorems of Pappus

When you jump rope, the rope sweeps out a surface in the space around you called a *surface of revolution*. The "area" of this surface depends on the length of the rope and the distance of each of its segments from the axis of revolution. In this section we define areas of surfaces of revolution. More complicated surfaces are treated in Chapter 16.

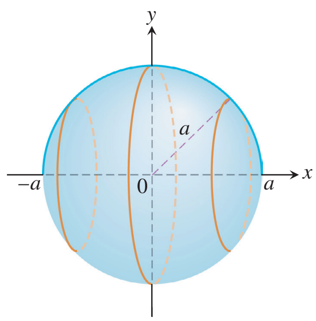



**FIGURE 6.41** Rotating the semicircle $y = \sqrt{a^2 - x^2}$ of radius $a$ with center at the origin generates a spherical surface with area $4\pi a^2$.

## Defining Surface Area

We want our definition of the area of a surface of revolution to be consistent with known results from classical geometry for the surface areas of spheres, circular cylinders, and cones. So if the jump rope discussed in the introduction takes the shape of a semicircle with radius $a$ rotated about the $x$-axis (Figure 6.41), it generates a sphere with surface area $4\pi a^2$.

Before considering general curves, we begin by rotating horizontal and slanted line segments about the $x$-axis. If we rotate the horizontal line segment $AB$ having length $\Delta x$ about the $x$-axis (Figure 6.42a), we generate a cylinder with surface area $2\pi y \Delta x$. This area is the same as that of a rectangle with side lengths $\Delta x$ and $2\pi y$ (Figure 6.42b). The length $2\pi y$ is the circumference of the circle of radius $y$ generated by rotating the point $(x, y)$ on the line $AB$ about the $x$-axis.

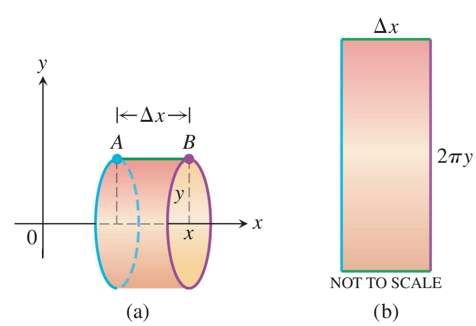



**FIGURE 6.42** (a) A cylindrical surface generated by rotating the horizontal line segment $AB$ of length $\Delta x$ about the $x$-axis has area $2\pi y \Delta x$. (b) The cut and rolled out cylindrical surface as a rectangle.

Suppose the line segment $AB$ has length $\Delta s$ and is slanted rather than horizontal. Now when $AB$ is rotated about the $x$-axis, it generates a frustum of a cone (Figure 6.43a). From classical geometry, the surface area of this frustum is $2\pi y^* \Delta s$, where $y^* = (y_1 + y_2)/2$ is the average height of the slanted segment $AB$ above the $x$-axis. This surface area is the same as that of a rectangle with side lengths $\Delta s$ and $2\pi y^*$ (Figure 6.43b).

Let's build on these geometric principles to define the area of a surface swept out by revolving more general curves about the $x$-axis. Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative continuous function $y = f(x)$, $a \le x \le b$, about the $x$-axis. We partition the closed interval $[a, b]$ in the usual way and use the points in the partition to subdivide the graph into short arcs. Figure 6.44 shows a typical arc $PQ$ and the band it sweeps out as part of the graph of $f$.

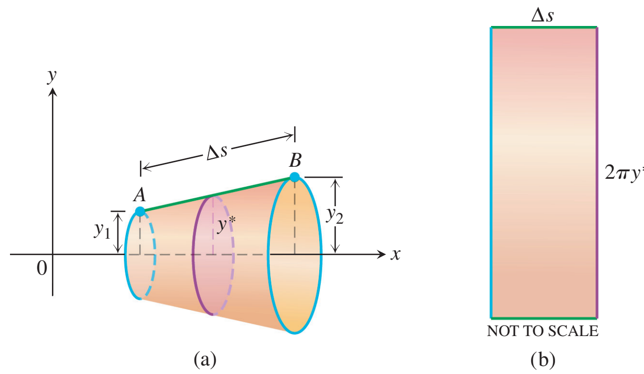



**FIGURE 6.43** (a) The frustum of a cone generated by rotating the slanted line segment $AB$ of length $\Delta s$ about the $x$-axis has area $2\pi y^* \Delta s$. (b) The area of the rectangle for $y^* = \dfrac{y_1 + y_2}{2}$, the average height of $AB$ above the $x$-axis.

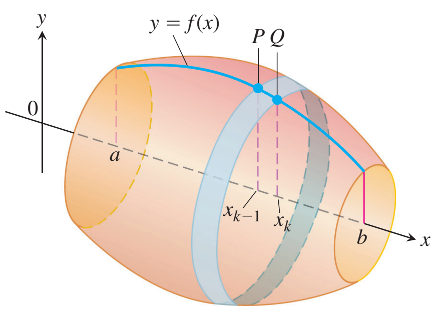



**FIGURE 6.44** The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \le x \le b$, about the $x$-axis. The surface is a union of bands like the one swept out by the arc $PQ$.

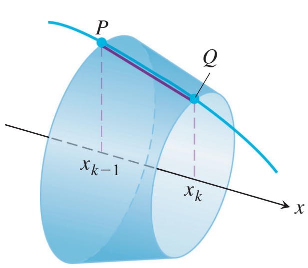



**FIGURE 6.45** The line segment joining $P$ and $Q$ sweeps out a frustum of a cone.

As the arc $PQ$ revolves about the $x$-axis, the line segment joining $P$ and $Q$ sweeps out a frustum of a cone whose axis lies along the $x$-axis (Figure 6.45). The surface area of this frustum approximates the surface area of the band swept out by the arc $PQ$. The surface area of the frustum of the cone shown in Figure 6.45 is $2\pi y^* L$, where $y^*$ is the average height of the line segment joining $P$ and $Q$, and $L$ is its length (just as before). Since $f \ge 0$, from Figure 6.46 we see that the average height of the line segment is $y^* = (f(x_{k-1}) + f(x_k))/2$, and the slant length is $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. Therefore,

$$\text{Frustum surface area} = 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc $PQ$, is approximated by the frustum area sum

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \tag{1}$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. Moreover, if the function $f$ is differentiable, then by the Mean Value Theorem, there is a point $(c_k, f(c_k))$ on the curve between $P$ and $Q$ where the tangent is parallel to the segment $PQ$ (Figure 6.47). At this point,

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k},$$

$$\Delta y_k = f'(c_k)\,\Delta x_k.$$

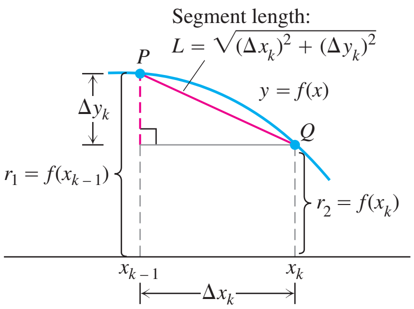



**FIGURE 6.46** Dimensions associated with the arc and line segment $PQ$.

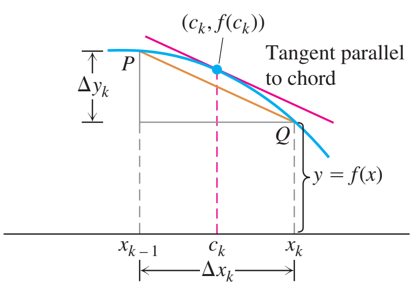



**FIGURE 6.47** If $f$ is smooth, the Mean Value Theorem guarantees the existence of a point $c_k$ where the tangent is parallel to segment $PQ$.

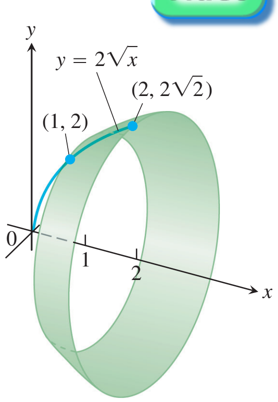




**FIGURE 6.48** In Example 1 we calculate the area of this surface.

With this substitution for $\Delta y_k$, the sums in Equation (1) take the form

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (f'(c_k)\,\Delta x_k)^2}$$

$$= \sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{1 + (f'(c_k))^2}\,\Delta x_k. \qquad (2)$$

These sums are not the Riemann sums of any function because the points $x_{k-1}$, $x_k$, and $c_k$ are not the same. However, a theorem from advanced calculus assures us that as the norm of the partition of $[a, b]$ goes to zero, the sums in Equation (2) converge to the integral

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

We therefore define this integral to be the area of the surface swept out by the graph of $f$ from $a$ to $b$.

**DEFINITION Surface Area for Revolution About the $x$-Axis**

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the $x$-axis is

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx. \qquad (3)$$

The square root in Equation (3) is the same one that appears in the formula for the length of the generating curve in Equation (2) of Section 6.3.

## EXAMPLE 1 Applying the Surface Area Formula

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the $x$-axis (Figure 6.48).

**Solution** We evaluate the formula

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x + 1}{x}} = \frac{\sqrt{x + 1}}{\sqrt{x}}.$$

With these substitutions,

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\,\frac{\sqrt{x+1}}{\sqrt{x}}\,dx = 4\pi \int_1^2 \sqrt{x+1}\,dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$

## Revolution About the $y$-Axis

For revolution about the $y$-axis, we interchange $x$ and $y$ in Equation (3).

**Surface Area for Revolution About the $y$-Axis**
If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the $y$-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\,dy. \qquad (4)$$

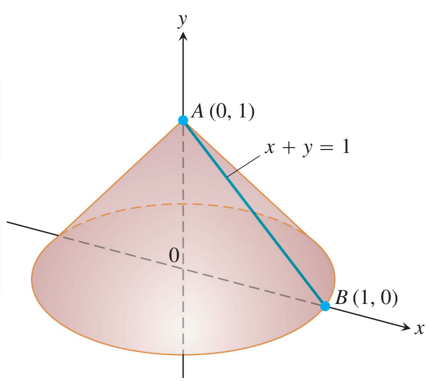



**FIGURE 6.49** Revolving line segment $AB$ about the $y$-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

**EXAMPLE 2** Finding Area for Revolution about the $y$-Axis

The line segment $x = 1 - y, 0 \leq y \leq 1$, is revolved about the $y$-axis to generate the cone in Figure 6.49. Find its lateral surface area (which excludes the base area).

**Solution** Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Equation (4) gives the same result, we take

$$c = 0, \qquad d = 1, \qquad x = 1 - y, \qquad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 2\pi(1 - y)\sqrt{2}\,dy$$

$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right)$$

$$= \pi\sqrt{2}.$$

The results agree, as they should.

## Parametrized Curves

Regardless of the coordinate axis of revolution, the square roots appearing in Equations (3) and (4) are the same ones that appear in the formulas for arc length in Section 6.3. If the curve is parametrized by the equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where $f$ and $g$ are continuously differentiable on $[a, b]$, then the corresponding square root appearing in the arc length formula is

$$\sqrt{[f'(t)]^2 + [g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

This observation leads to the following formulas for area of surfaces of revolution for smooth parametrized curves.

### Surface Area of Revolution for Parametrized Curves

If a smooth curve $x = f(t), y = g(t), a \leq t \leq b$, is traversed exactly once as $t$ increases from $a$ to $b$, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

**1.** Revolution about the $x$-axis $(y \geq 0)$:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{5}$$

**2.** Revolution about the $y$-axis $(x \geq 0)$:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{6}$$

As with length, we can calculate surface area from any convenient parametrization that meets the stated criteria.

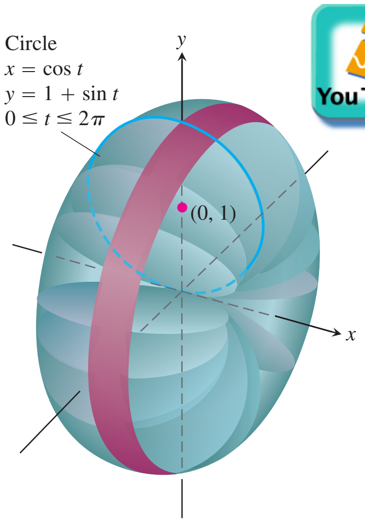



**FIGURE 6.50** In Example 3 we calculate the area of the surface of revolution swept out by this parametrized curve.

**EXAMPLE 3** Applying Surface Area Formula

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the $xy$-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \leq t \leq 2\pi.$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the $x$-axis (Figure 6.50).

**Solution** We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Eq. (5) for revolution about the $x$-axis; $y = 1 + \sin t > 0$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}_{1}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt$$

$$= 2\pi\Big[t - \cos t\Big]_0^{2\pi} = 4\pi^2.$$ ■

## The Differential Form

The equations

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{and} \qquad S = \int_c^d 2\pi x \sqrt{\left(\frac{dx}{dy}\right)^2}\, dy$$

are often written in terms of the arc length differential $ds = \sqrt{dx^2 + dy^2}$ as

$$S = \int_a^b 2\pi y\, ds \qquad \text{and} \qquad S = \int_c^d 2\pi x\, ds.$$

In the first of these, $y$ is the distance from the $x$-axis to an element of arc length $ds$. In the second, $x$ is the distance from the $y$-axis to an element of arc length $ds$. Both integrals have the form

$$\boxed{S = \int 2\pi(\text{radius})(\text{band width}) = \int 2\pi\rho\, ds \qquad (7)}$$

where $\rho$ is the radius from the axis of revolution to an element of arc length $ds$ (Figure 6.51).

In any particular problem, you would then express the radius function $\rho$ and the arc length differential $ds$ in terms of a common variable and supply limits of integration for that variable.

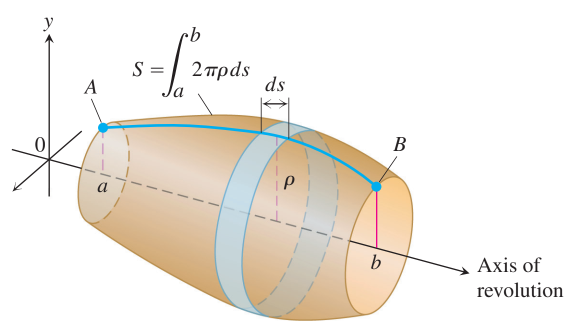



**FIGURE 6.51** The area of the surface swept out by revolving arc $AB$ about the axis shown here is $\int_a^b 2\pi\rho\, ds$. The exact expression depends on the formulas for $\rho$ and $ds$.

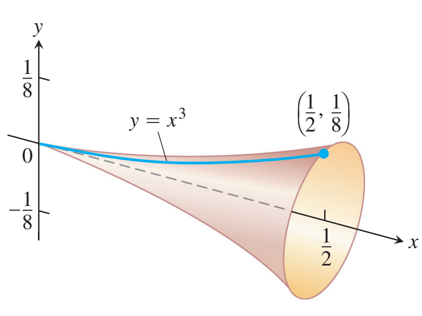



**FIGURE 6.52** The surface generated by revolving the curve $y = x^3$, $0 \le x \le 1/2$, about the $x$-axis could be the design for a champagne glass (Example 4).

## EXAMPLE 4 Using the Differential Form for Surface Areas

Find the area of the surface generated by revolving the curve $y = x^3$, $0 \le x \le 1/2$, about the $x$-axis (Figure 6.52).

**Solution** We start with the short differential form:

$$\begin{aligned} S &= \int 2\pi\rho\, ds \\ &= \int 2\pi y\, ds && \text{For revolution about the } x\text{-axis, the radius function is } \rho = y > 0 \text{ on } 0 \le x \le 1/2. \\ &= \int 2\pi y \sqrt{dx^2 + dy^2}. && ds = \sqrt{dx^2 + dy^2} \end{aligned}$$

We then decide whether to express $dy$ in terms of $dx$ or $dx$ in terms of $dy$. The original form of the equation, $y = x^3$, makes it easier to express $dy$ in terms of $dx$, so we continue the calculation with

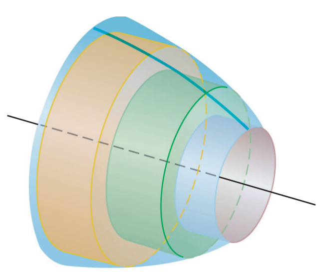
$$y = x^3, \qquad dy = 3x^2\,dx, \qquad \text{and} \qquad \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + (3x^2\,dx)^2}$$
$$= \sqrt{1 + 9x^4}\,dx.$$

With these substitutions, $x$ becomes the variable of integration and

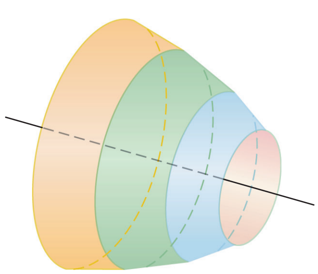
$$\begin{aligned} S &= \int_{x=0}^{x=1/2} 2\pi y\sqrt{dx^2 + dy^2} \\ &= \int_0^{1/2} 2\pi x^3\sqrt{1 + 9x^4}\,dx \\ &= 2\pi\left(\frac{1}{36}\right)\left(\frac{2}{3}\right)(1 + 9x^4)^{3/2}\Big]_0^{1/2} \\ &= \frac{\pi}{27}\left[\left(1 + \frac{9}{16}\right)^{3/2} - 1\right] \\ &= \frac{\pi}{27}\left[\left(\frac{25}{16}\right)^{3/2} - 1\right] = \frac{\pi}{27}\left(\frac{125}{64} - 1\right) \\ &= \frac{61\pi}{1728}. \end{aligned}$$

Substitute $u = 1 + 9x^4$, $du/36 = x^3\,dx$; integrate, and substitute back.

## Cylindrical Versus Conical Bands



(a)



(b)

**FIGURE 6.53** Why not use (a) cylindrical bands instead of (b) conical bands to approximate surface area?

Why not find the surface area by approximating with cylindrical bands instead of conical bands, as suggested in Figure 6.53? The Riemann sums we get this way converge just as nicely as the ones based on conical bands, and the resulting integral is simpler. For revolution about the $x$-axis in this case, the radius in Equation (7) is $\rho = y$ and the band width is $ds = dx$. This leads to the integral formula

$$S = \int_a^b 2\pi f(x)\,dx \tag{8}$$

rather than the defining Equation (3). The problem with this new formula is that it fails to give results consistent with the surface area formulas from classical geometry, and that was one of our stated goals at the outset. Just because we end up with a nice-looking integral from a Riemann sum derivation does not mean it will calculate what we intend. (See Exercise 40.)

**CAUTION** Do not use Equation (8) to calculate surface area. It does *not* give the correct result.

## The Theorems of Pappus

In the third century, an Alexandrian Greek named Pappus discovered two formulas that relate centroids to surfaces and solids of revolution. The formulas provide shortcuts to a number of otherwise lengthy calculations.

**THEOREM 1 Pappus's Theorem for Volumes**

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If $\rho$ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi \rho A. \tag{9}$$

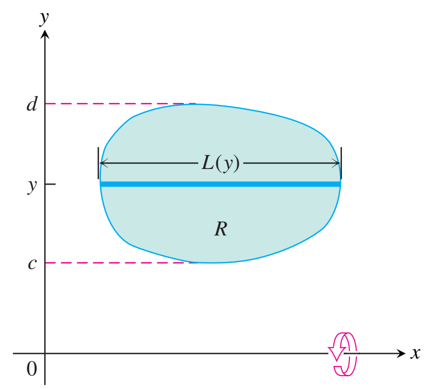



**FIGURE 6.54** The region $R$ is to be revolved (once) about the $x$-axis to generate a solid. A 1700-year-old theorem says that the solid's volume can be calculated by multiplying the region's area by the distance traveled by its centroid during the revolution.

**Proof** We draw the axis of revolution as the $x$-axis with the region $R$ in the first quadrant (Figure 6.54). We let $L(y)$ denote the length of the cross-section of $R$ perpendicular to the $y$-axis at $y$. We assume $L(y)$ to be continuous.

By the method of cylindrical shells, the volume of the solid generated by revolving the region about the $x$-axis is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\, dy = 2\pi \int_c^d y\, L(y)\, dy. \tag{10}$$

The $y$-coordinate of $R$'s centroid is

$$\bar{y} = \frac{\int_c^d \tilde{y}\, dA}{A} = \frac{\int_c^d y\, L(y)\, dy}{A}, \qquad \tilde{y} = y,\ dA = L(y)dy$$

so that

$$\int_c^d y\, L(y)\, dy = A\bar{y}.$$

Substituting $A\bar{y}$ for the last integral in Equation (10) gives $V = 2\pi\bar{y}A$. With $\rho$ equal to $\bar{y}$, we have $V = 2\pi\rho A$. ■

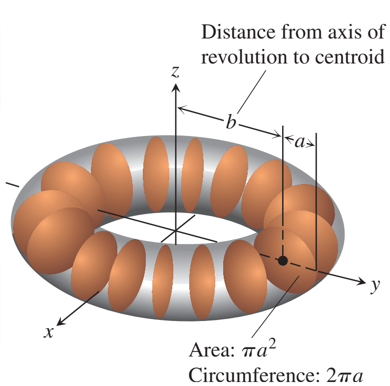



**FIGURE 6.55** With Pappus's first theorem, we can find the volume of a torus without having to integrate (Example 5).

**EXAMPLE 5** Volume of a Torus

The volume of the torus (doughnut) generated by revolving a circular disk of radius $a$ about an axis in its plane at a distance $b \geq a$ from its center (Figure 6.55) is

$$V = 2\pi(b)(\pi a^2) = 2\pi^2 b a^2. \quad ■$$

**EXAMPLE 6** Locate the Centroid of a Semicircular Region

**Solution** We model the region as the region between the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.56) and the $x$-axis and imagine revolving the region about the $x$-axis to generate a solid sphere. By symmetry, the $x$-coordinate of the centroid is $\bar{x} = 0$. With $\bar{y} = \rho$ in Equation (9), we have

$$\bar{y} = \frac{V}{2\pi A} = \frac{(4/3)\pi a^3}{2\pi(1/2)\pi a^2} = \frac{4}{3\pi}a. \quad ■$$

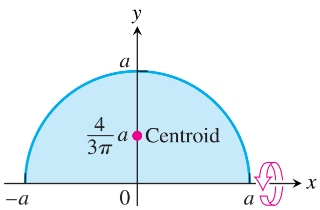



**FIGURE 6.56** With Pappus's first theorem, we can locate the centroid of a semicircular region without having to integrate (Example 6).

**THEOREM 2 Pappus's Theorem for Surface Areas**

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length of the arc times the distance traveled by the arc's centroid during the revolution. If $\rho$ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \qquad (11)$$

The proof we give assumes that we can model the axis of revolution as the $x$-axis and the arc as the graph of a continuously differentiable function of $x$.

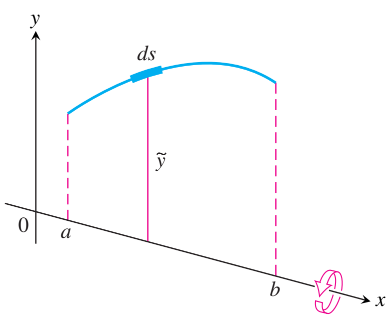



**FIGURE 6.57** Figure for proving Pappus's area theorem.

**Proof** We draw the axis of revolution as the $x$-axis with the arc extending from $x = a$ to $x = b$ in the first quadrant (Figure 6.57). The area of the surface generated by the arc is

$$S = \int_{x=a}^{x=b} 2\pi y \, ds = 2\pi \int_{x=a}^{x=b} y \, ds. \qquad (12)$$

The $y$-coordinate of the arc's centroid is

$$\bar{y} = \frac{\int_{x=a}^{x=b} \tilde{y} \, ds}{\int_{x=a}^{x=b} ds} = \frac{\int_{x=a}^{x=b} y \, ds}{L}.$$

$L = \int ds$ is the arc's length and $\tilde{y} = y$.

Hence

$$\int_{x=a}^{x=b} y \, ds = \bar{y}L.$$

Substituting $\bar{y}L$ for the last integral in Equation (12) gives $S = 2\pi\bar{y}L$. With $\rho$ equal to $\bar{y}$, we have $S = 2\pi\rho L$. ■

**EXAMPLE 7** Surface Area of a Torus

The surface area of the torus in Example 5 is

$$S = 2\pi(b)(2\pi a) = 4\pi^2 ba.$$ ■

# EXERCISES 6.5

## Finding Integrals for Surface Area

In Exercises 1–8:

**a.** Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.

T **b.** Graph the curve to see what it looks like. If you can, graph the surface, too.

T **c.** Use your grapher's or computer's integral evaluator to find the surface's area numerically.

Exercises

1. $y = \tan x, \quad 0 \le x \le \pi/4; \quad x\text{-axis}$

2. $y = x^2, \quad 0 \le x \le 2; \quad x\text{-axis}$

3. $xy = 1, \quad 1 \le y \le 2; \quad y\text{-axis}$

4. $x = \sin y, \quad 0 \le y \le \pi; \quad y\text{-axis}$

5. $x^{1/2} + y^{1/2} = 3$ from $(4, 1)$ to $(1, 4)$; $x$-axis

6. $y + 2\sqrt{y} = x, \quad 1 \le y \le 2; \quad y\text{-axis}$

7. $x = \displaystyle\int_0^y \tan t\, dt, \quad 0 \le y \le \pi/3; \quad y\text{-axis}$

8. $y = \displaystyle\int_1^x \sqrt{t^2 - 1}\, dt, \quad 1 \le x \le \sqrt{5}; \quad x\text{-axis}$

## Finding Surface Areas

Exercises

**9.** Find the lateral (side) surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$, about the $x$-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

**10.** Find the lateral surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$ about the $y$-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

**11.** Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the $x$-axis. Check your result with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

**12.** Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the $y$-axis. Check your result with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

Find the areas of the surfaces generated by revolving the curves in Exercises 13–22 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

**13.** $y = x^3/9, \quad 0 \le x \le 2; \quad x$-axis

**14.** $y = \sqrt{x}, \quad 3/4 \le x \le 15/4; \quad x$-axis

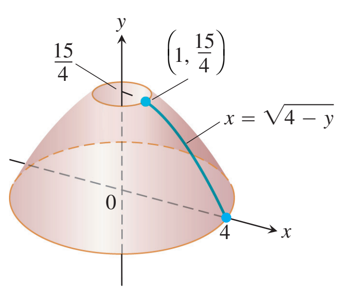
**15.** $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x$-axis

**16.** $y = \sqrt{x + 1}, \quad 1 \le x \le 5; \quad x$-axis

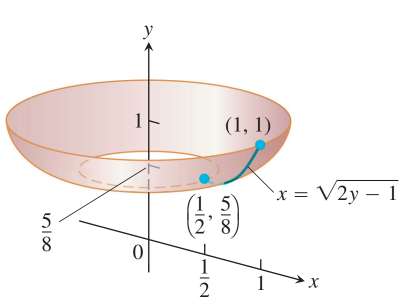
**17.** $x = y^3/3, \quad 0 \le y \le 1; \quad y$-axis

**18.** $x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \le y \le 3; \quad y$-axis

**19.** $x = 2\sqrt{4 - y}, \quad 0 \le y \le 15/4; \quad y$-axis




**20.** $x = \sqrt{2y - 1}, \quad 5/8 \le y \le 1; \quad y$-axis




**21.** $x = (y^4/4) + 1/(8y^2), \quad 1 \le y \le 2; \quad x$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of $dy$, and evaluate the integral $S = \int 2\pi y\, ds$ with appropriate limits.)

**22.** $y = (1/3)(x^2 + 2)^{3/2}, \quad 0 \le x \le \sqrt{2}; \quad y$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of $dx$, and evaluate the integral $S = \int 2\pi x\, ds$ with appropriate limits.)

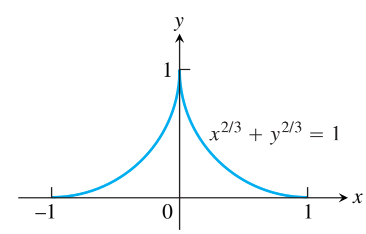
**23. Testing the new definition** Show that the surface area of a sphere of radius $a$ is still $4\pi a^2$ by using Equation (3) to find the area of the surface generated by revolving the curve $y = \sqrt{a^2 - x^2}, -a \le x \le a$, about the $x$-axis.

**24. Testing the new definition** The lateral (side) surface area of a cone of height $h$ and base radius $r$ should be $\pi r\sqrt{r^2 + h^2}$, the semiperimeter of the base times the slant height. Show that this is still the case by finding the area of the surface generated by revolving the line segment $y = (r/h)x, 0 \le x \le h$, about the $x$-axis.

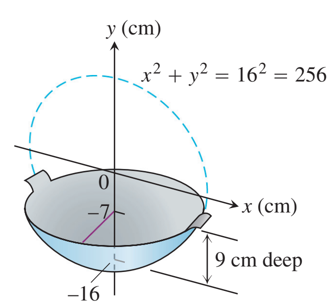
**25.** Write an integral for the area of the surface generated by revolving the curve $y = \cos x, -\pi/2 \le x \le \pi/2$, about the $x$-axis. In Section 8.5 we will see how to evaluate such integrals.

Exercises

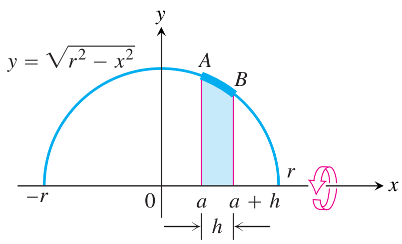
**26. The surface of an astroid** Find the area of the surface generated by revolving about the $x$-axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$ shown here. (*Hint:* Revolve the first-quadrant portion $y = (1 - x^{2/3})^{3/2}, 0 \le x \le 1$, about the $x$-axis and double your result.)



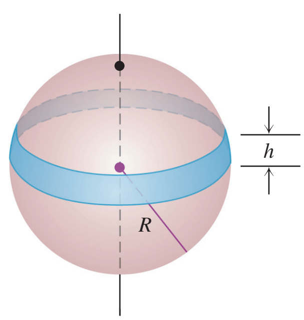

T **27. Enameling woks** Your company decided to put out a deluxe version of the successful wok you designed in Section 6.1, Exercise 55. The plan is to coat it inside with white enamel and outside with blue enamel. Each enamel will be sprayed on 0.5 mm thick before baking. (See diagram here.) Your manufacturing department wants to know how much enamel to have on hand for a production run of 5000 woks. What do you tell them? (Neglect waste and unused material and give your answer in liters. Remember that $1 \text{ cm}^3 = 1 \text{ mL}$, so $1 \text{ L} = 1000 \text{ cm}^3$.)



**28. Slicing bread** Did you know that if you cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the $x$-axis to generate a sphere. Let $AB$ be an arc of the semicircle that lies above an interval of length $h$ on the $x$-axis. Show that the area swept out by $AB$ does not depend on the location of the interval. (It does depend on the length of the interval.)



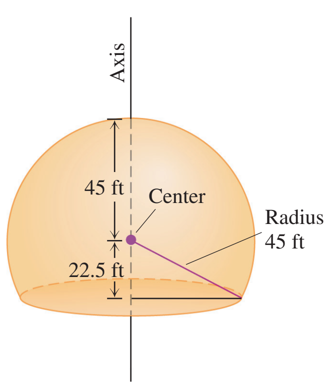

**29.** The shaded band shown here is cut from a sphere of radius $R$ by parallel planes $h$ units apart. Show that the surface area of the band is $2\pi Rh$.




**30.** Here is a schematic drawing of the 90-ft dome used by the U.S. National Weather Service to house radar in Bozeman, Montana.

**a.** How much outside surface is there to paint (not counting the bottom)?

T **b.** Express the answer to the nearest square foot.




**31. Surfaces generated by curves that cross the axis of revolution** The surface area formula in Equation (3) was developed under the assumption that the function $f$ whose graph generated the surface was nonnegative over the interval $[a, b]$. For curves that cross the axis of revolution, we replace Equation (3) with the absolute value formula

$$S = \int 2\pi\rho\, ds = \int 2\pi |f(x)|\, ds. \qquad (13)$$

Use Equation (13) to find the surface area of the double cone generated by revolving the line segment $y = x$, $-1 \le x \le 2$, about the $x$-axis.

**32.** (*Continuation of Exercise 31.*) Find the area of the surface generated by revolving the curve $y = x^3/9$, $-\sqrt{3} \le x \le \sqrt{3}$, about the $x$-axis. What do you think will happen if you drop the absolute value bars from Equation (13) and attempt to find the surface area with the formula $S = \int 2\pi f(x)\, ds$ instead? Try it.

## Parametrizations

Find the areas of the surfaces generated by revolving the curves in Exercises 33–35 about the indicated axes.

**33.** $x = \cos t$, $y = 2 + \sin t$, $0 \le t \le 2\pi$; $x$-axis

**34.** $x = (2/3)t^{3/2}$, $y = 2\sqrt{t}$, $0 \le t \le \sqrt{3}$; $y$-axis

**35.** $x = t + \sqrt{2}$, $y = (t^2/2) + \sqrt{2}t$, $-\sqrt{2} \le t \le \sqrt{2}$; $y$-axis

**36.** Set up, but do not evaluate, an integral that represents the area of the surface obtained by rotating the curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \le t \le 2\pi$, about the $x$-axis.

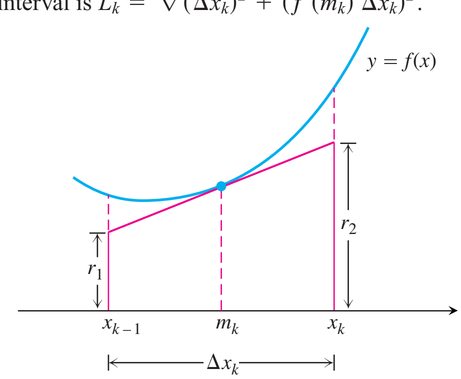
**37. A cone frustum** The line segment joining the points (0, 1) and (2, 2) is revolved about the $x$-axis to generate a frustum of a cone. Find the surface area of the frustum using the parametrization $x = 2t$, $y = t + 1$, $0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi(r_1 + r_2)$(slant height).

**38. A cone** The line segment joining the origin to the point $(h, r)$ is revolved about the $x$-axis to generate a cone of height $h$ and base radius $r$. Find the cone's surface area with the parametric equations $x = ht$, $y = rt$, $0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi r$(slant height).

**39. An alternative derivation of the surface area formula** Assume $f$ is smooth on $[a, b]$ and partition $[a, b]$ in the usual way. In the $k$th subinterval $[x_{k-1}, x_k]$ construct the tangent line to the curve at the midpoint $m_k = (x_{k-1} + x_k)/2$, as in the figure here.

**a.** Show that $r_1 = f(m_k) - f'(m_k)\dfrac{\Delta x_k}{2}$ and $r_2 = f(m_k) + f'(m_k)\dfrac{\Delta x_k}{2}$.

**b.** Show that the length $L_k$ of the tangent line segment in the $k$th subinterval is $L_k = \sqrt{(\Delta x_k)^2 + (f'(m_k)\,\Delta x_k)^2}$.

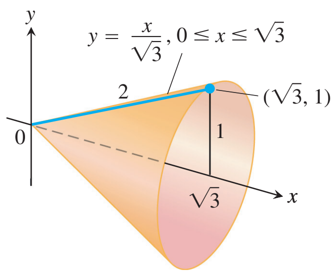


**c.** Show that the lateral surface area of the frustum of the cone swept out by the tangent line segment as it revolves about the $x$-axis is $2\pi f(m_k)\sqrt{1 + (f'(m_k))^2}\,\Delta x_k$.

**d.** Show that the area of the surface generated by revolving $y = f(x)$ about the $x$-axis over $[a, b]$ is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \begin{pmatrix}\text{lateral surface area}\\ \text{of } k\text{th frustum}\end{pmatrix} = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

**40. Modeling surface area** The lateral surface area of the cone swept out by revolving the line segment $y = x/\sqrt{3},\ 0 \le x \le \sqrt{3}$, about the $x$-axis should be $(1/2)(\text{base circumference})(\text{slant height}) = (1/2)(2\pi)(2) = 2\pi$. What do you get if you use Equation (8) with $f(x) = x/\sqrt{3}$?




## The Theorems of Pappus

**41.** The square region with vertices (0, 2), (2, 0), (4, 2), and (2, 4) is revolved about the $x$-axis to generate a solid. Find the volume and surface area of the solid.

**42.** Use a theorem of Pappus to find the volume generated by revolving about the line $x = 5$ the triangular region bounded by the coordinate axes and the line $2x + y = 6$. (As you saw in Exercise 29 of Section 6.4, the centroid of a triangle lies at the intersection of the medians, one-third of the way from the midpoint of each side toward the opposite vertex.)

**43.** Find the volume of the torus generated by revolving the circle $(x - 2)^2 + y^2 = 1$ about the $y$-axis.

**44.** Use the theorems of Pappus to find the lateral surface area and the volume of a right circular cone.

**45.** Use the Second Theorem of Pappus and the fact that the surface area of a sphere of radius $a$ is $4\pi a^2$ to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.

**46.** As found in Exercise 45, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface swept out by revolving the semicircle about the line $y = a$.

**47.** The area of the region $R$ enclosed by the semiellipse $y = (b/a)\sqrt{a^2 - x^2}$ and the $x$-axis is $(1/2)\pi ab$ and the volume of the ellipsoid generated by revolving $R$ about the $x$-axis is $(4/3)\pi ab^2$. Find the centroid of $R$. Notice that the location is independent of $a$.

**48.** As found in Example 6, the centroid of the region enclosed by the $x$-axis and the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 4a/3\pi)$. Find the volume of the solid generated by revolving this region about the line $y = -a$.

**49.** The region of Exercise 48 is revolved about the line $y = x - a$ to generate a solid. Find the volume of the solid.

**50.** As found in Exercise 45, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface generated by revolving the semicircle about the line $y = x - a$.

**51.** Find the moment about the $x$-axis of the semicircular region in Example 6. If you use results already known, you will not need to integrate.

# 6.6 Work

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on a body and the body's subsequent displacement. This section shows how to calculate work. The applications run from compressing railroad car springs and emptying subterranean tanks to forcing electrons together and lifting satellites into orbit.

## Work Done by a Constant Force

When a body moves a distance $d$ along a straight line as a result of being acted on by a force of constant magnitude $F$ in the direction of motion, we define the **work** $W$ done by the force on the body with the formula

$$W = Fd \qquad \text{(Constant-force formula for work).} \tag{1}$$

**Joules**
The joule, abbreviated J and pronounced "jewel," is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}).$$

In symbols, $1\text{ J} = 1\text{ N} \cdot \text{m}$.

From Equation (1) we see that the unit of work in any system is the unit of force multiplied by the unit of distance. In SI units (SI stands for *Système International*, or International System), the unit of force is a newton, the unit of distance is a meter, and the unit of work is a newton-meter $(\text{N} \cdot \text{m})$. This combination appears so often it has a special name, the **joule**. In the British system, the unit of work is the foot-pound, a unit frequently used by engineers.

### EXAMPLE 1 Jacking Up a Car

If you jack up the side of a 2000-lb car 1.25 ft to change a tire (you have to apply a constant vertical force of about 1000 lb) you will perform $1000 \times 1.25 = 1250$ ft-lb of work on the car. In SI units, you have applied a force of 4448 N through a distance of 0.381 m to do $4448 \times 0.381 \approx 1695$ J of work. ■

## Work Done by a Variable Force Along a Line

If the force you apply varies along the way, as it will if you are compressing a spring, the formula $W = Fd$ has to be replaced by an integral formula that takes the variation in $F$ into account.

Suppose that the force performing the work acts along a line that we take to be the $x$-axis and that its magnitude $F$ is a continuous function of the position. We want to find the work done over the interval from $x = a$ to $x = b$. We partition $[a, b]$ in the usual way and choose an arbitrary point $c_k$ in each subinterval $[x_{k-1}, x_k]$. If the subinterval is short enough, $F$, being continuous, will not vary much from $x_{k-1}$ to $x_k$. The amount of work done across the interval will be about $F(c_k)$ times the distance $\Delta x_k$, the same as it would be if $F$ were constant and we could apply Equation (1). The total work done from $a$ to $b$ is therefore approximated by the Riemann sum

$$\text{Work} \approx \sum_{k=1}^{n} F(c_k)\, \Delta x_k.$$

We expect the approximation to improve as the norm of the partition goes to zero, so we define the work done by the force from $a$ to $b$ to be the integral of $F$ from $a$ to $b$.

**DEFINITION Work**

The **work** done by a variable force $F(x)$ directed along the $x$-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\, dx. \qquad (2)$$

The units of the integral are joules if $F$ is in newtons and $x$ is in meters, and foot-pounds if $F$ is in pounds and $x$ in feet. So, the work done by a force of $F(x) = 1/x^2$ newtons along the $x$-axis from $x = 1$ m to $x = 10$ m is

$$W = \int_1^{10} \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg]_1^{10} = -\frac{1}{10} + 1 = 0.9 \text{ J}.$$

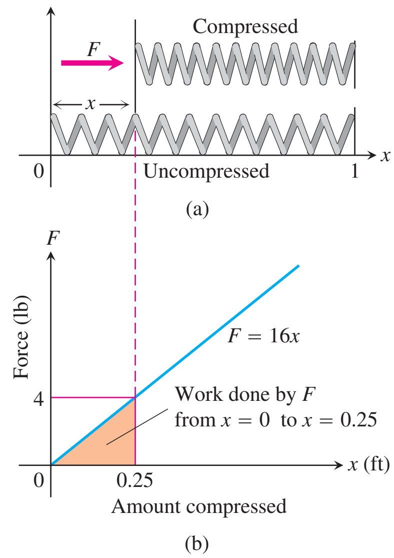



**FIGURE 6.58** The force $F$ needed to hold a spring under compression increases linearly as the spring is compressed (Example 2).

### Hooke's Law for Springs: $F = kx$

**Hooke's Law** says that the force it takes to stretch or compress a spring $x$ length units from its natural (unstressed) length is proportional to $x$. In symbols,

$$F = kx. \tag{3}$$

The constant $k$, measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or **spring constant**) of the spring. Hooke's Law, Equation (3), gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.

### EXAMPLE 2 Compressing a Spring

Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.

**Solution** We picture the uncompressed spring laid out along the $x$-axis with its movable end at the origin and its fixed end at $x = 1$ ft (Figure 6.58). This enables us to describe the force required to compress the spring from 0 to $x$ with the formula $F = 16x$. To compress the spring from 0 to 0.25 ft, the force must increase from

$$F(0) = 16 \cdot 0 = 0 \text{ lb} \qquad \text{to} \qquad F(0.25) = 16 \cdot 0.25 = 4 \text{ lb}.$$

The work done by $F$ over this interval is

$$W = \int_0^{0.25} 16x\,dx = 8x^2\Big]_0^{0.25} = 0.5 \text{ ft-lb}.$$

Eq. (2) with $a = 0, b = 0.25, F(x) = 16x$

### EXAMPLE 3 Stretching a Spring

A spring has a natural length of 1 m. A force of 24 N stretches the spring to a length of 1.8 m.

**(a)** Find the force constant $k$.

**(b)** How much work will it take to stretch the spring 2 m beyond its natural length?

**(c)** How far will a 45-N force stretch the spring?

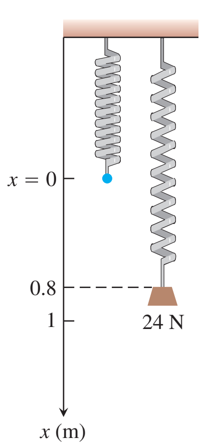



**FIGURE 6.59** A 24-N weight stretches this spring 0.8 m beyond its unstressed length (Example 3).

**Solution**

**(a)** *The force constant.* We find the force constant from Equation (3). A force of 24 N stretches the spring 0.8 m, so

$$24 = k(0.8)$$
$$k = 24/0.8 = 30 \text{ N/m}.$$

Eq. (3) with $F = 24, x = 0.8$

**(b)** *The work to stretch the spring 2 m.* We imagine the unstressed spring hanging along the $x$-axis with its free end at $x = 0$ (Figure 6.59). The force required to stretch the spring $x$ m beyond its natural length is the force required to pull the free end of the spring $x$ units from the origin. Hooke's Law with $k = 30$ says that this force is

$$F(x) = 30x.$$

The work done by $F$ on the spring from $x = 0$ m to $x = 2$ m is

$$W = \int_0^2 30x\,dx = 15x^2\Big]_0^2 = 60\text{ J}.$$

**(c)** *How far will a* 45-N *force stretch the spring?* We substitute $F = 45$ in the equation $F = 30x$ to find

$$45 = 30x, \qquad \text{or} \qquad x = 1.5\text{ m}.$$

A 45-N force will stretch the spring 1.5 m. No calculus is required to find this. ■

The work integral is useful to calculate the work done in lifting objects whose weights vary with their elevation.

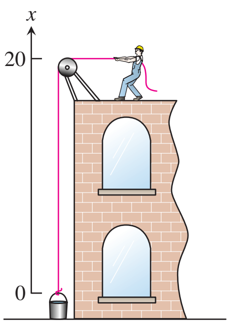



**FIGURE 6.60** Lifting the bucket in Example 4.

## EXAMPLE 4 Lifting a Rope and Bucket

A 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed (Figure 6.60). The rope weighs 0.08 lb/ft. How much work was spent lifting the bucket and rope?

**Solution** The bucket has constant weight so the work done lifting it alone is weight × distance $= 5 \cdot 20 = 100$ ft-lb.

The weight of the rope varies with the bucket's elevation, because less of it is freely hanging. When the bucket is $x$ ft off the ground, the remaining proportion of the rope still being lifted weighs $(0.08) \cdot (20 - x)$ lb. So the work in lifting the rope is

$$\begin{aligned}\text{Work on rope} &= \int_0^{20} (0.08)(20 - x)\,dx = \int_0^{20} (1.6 - 0.08x)\,dx \\ &= \Big[1.6x - 0.04x^2\Big]_0^{20} = 32 - 16 = 16\text{ ft-lb}.\end{aligned}$$

The total work for the bucket and rope combined is

$$100 + 16 = 116\text{ ft-lb}.$$ ■

## Pumping Liquids from Containers

How much work does it take to pump all or part of the liquid from a container? To find out, we imagine lifting the liquid out one thin horizontal slab at a time and applying the equation $W = Fd$ to each slab. We then evaluate the integral this leads to as the slabs become thinner and more numerous. The integral we get each time depends on the weight of the liquid and the dimensions of the container, but the way we find the integral is always the same. The next examples show what to do.

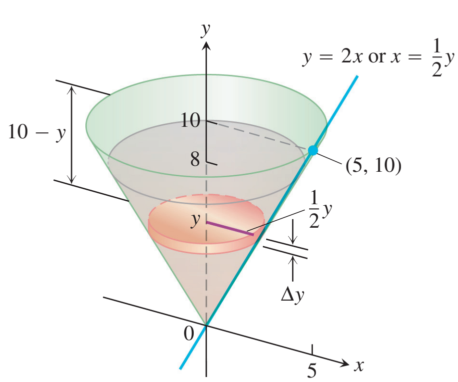



**FIGURE 6.61** The olive oil and tank in Example 5.

## EXAMPLE 5 Pumping Oil from a Conical Tank

The conical tank in Figure 6.61 is filled to within 2 ft of the top with olive oil weighing 57 lb/ft$^3$. How much work does it take to pump the oil to the rim of the tank?

**Solution** We imagine the oil divided into thin slabs by planes perpendicular to the $y$-axis at the points of a partition of the interval [0, 8].

The typical slab between the planes at $y$ and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{1}{2}y\right)^2 \Delta y = \frac{\pi}{4}y^2\,\Delta y\text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight,

$$F(y) = 57\,\Delta V = \frac{57\pi}{4}y^2\,\Delta y \text{ lb}.$$

Weight = weight per unit volume × volume

The distance through which $F(y)$ must act to lift this slab to the level of the rim of the cone is about $(10 - y)$ ft, so the work done lifting the slab is about

$$\Delta W = \frac{57\pi}{4}(10 - y)y^2\,\Delta y \text{ ft-lb}.$$

Assuming there are $n$ slabs associated with the partition of $[0, 8]$, and that $y = y_k$ denotes the plane associated with the $k$th slab of thickness $\Delta y_k$, we can approximate the work done lifting all of the slabs with the Riemann sum

$$W \approx \sum_{k=1}^{n} \frac{57\pi}{4}(10 - y_k){y_k}^2\,\Delta y_k \text{ ft-lb}.$$

The work of pumping the oil to the rim is the limit of these sums as the norm of the partition goes to zero.

$$\begin{aligned} W &= \int_0^8 \frac{57\pi}{4}(10 - y)y^2\,dy \\ &= \frac{57\pi}{4}\int_0^8 (10y^2 - y^3)\,dy \\ &= \frac{57\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 \approx 30{,}561 \text{ ft-lb}. \end{aligned}$$

■

## EXAMPLE 6 Pumping Water from a Glory Hole

A glory hole is a vertical drain pipe that keeps the water behind a dam from getting too high. The top of the glory hole for a dam is 14 ft below the top of the dam and 375 ft above the bottom (Figure 6.62). The hole needs to be pumped out from time to time to permit the removal of seasonal debris.

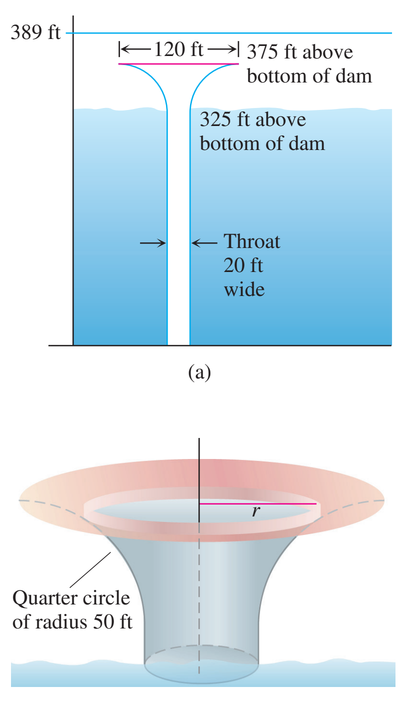



**FIGURE 6.62** (a) Cross-section of the glory hole for a dam and (b) the top of the glory hole (Example 6).

From the cross-section in Figure 6.62a, we see that the glory hole is a funnel-shaped drain. The throat of the funnel is 20 ft wide and the head is 120 ft across. The outside boundary of the head cross-section are quarter circles formed with 50-ft radii, shown in Figure 6.62b. The glory hole is formed by rotating a cross-section around its center. Consequently, all horizontal cross-sections are circular disks throughout the entire glory hole. We calculate the work required to pump water from

**(a)** the throat of the hole.

**(b)** the funnel portion.

### Solution

**(a)** *Pumping from the throat.* A typical slab in the throat between the planes at $y$ and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi(10)^2\,\Delta y \text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight (about 62.4 lb/ft$^3$ for water),

$$F(y) = 62.4\,\Delta V = 6240\pi\,\Delta y \text{ lb}.$$

The distance through which $F(y)$ must act to lift this slab to the top of the hole is $(375 - y)$ ft, so the work done lifting the slab is

$$\Delta W = 6240\pi(375 - y)\,\Delta y \text{ ft-lb}.$$

We can approximate the work done in pumping the water from the throat by summing the work done lifting all the slabs individually, and then taking the limit of this Riemann sum as the norm of the partition goes to zero. This gives the integral

$$\begin{aligned} W &= \int_0^{325} 6240\pi(375 - y)\,dy \\ &= 6240\pi\left[375y - \frac{y^2}{2}\right]_0^{325} \\ &\approx 1{,}353{,}869{,}354 \text{ ft-lb}. \end{aligned}$$

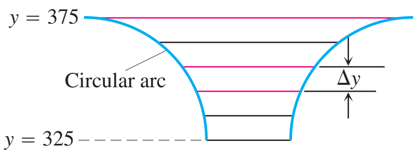



**FIGURE 6.63** The glory hole funnel portion.

**(b)** *Pumping from the funnel*. To compute the work necessary to pump water from the funnel portion of the glory hole, from $y = 325$ to $y = 375$, we need to compute $\Delta V$ for approximating elements in the funnel as shown in Figure 6.63. As can be seen from the figure, the radii of the slabs vary with height $y$.

In Exercises 33 and 34, you are asked to complete the analysis to determine the total work required to pump the water and to find the power of the pumps necessary to pump out the glory hole. ■

# EXERCISES 6.6

## Springs

1. **Spring constant** It took 1800 J of work to stretch a spring from its natural length of 2 m to a length of 5 m. Find the spring's force constant.

2. **Stretching a spring** A spring has a natural length of 10 in. An 800-lb force stretches the spring to 14 in.

   **a.** Find the force constant.

   **b.** How much work is done in stretching the spring from 10 in. to 12 in.?

   **c.** How far beyond its natural length will a 1600-lb force stretch the spring?

3. **Stretching a rubber band** A force of 2 N will stretch a rubber band 2 cm (0.02 m). Assuming that Hooke's Law applies, how far will a 4-N force stretch the rubber band? How much work does it take to stretch the rubber band this far?

4. **Stretching a spring** If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?

5. **Subway car springs** It takes a force of 21,714 lb to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 8 in. to its fully compressed height of 5 in.

   **a.** What is the assembly's force constant?

   **b.** How much work does it take to compress the assembly the first half inch? the second half inch? Answer to the nearest in.-lb.

   (Data courtesy of Bombardier, Inc., Mass Transit Division, for spring assemblies in subway cars delivered to the New York City Transit Authority from 1985 to 1987.)

6. **Bathroom scale** A bathroom scale is compressed $1/16$ in. when a 150-lb person stands on it. Assuming that the scale behaves like a spring that obeys Hooke's Law, how much does someone who compresses the scale $1/8$ in. weigh? How much work is done compressing the scale $1/8$ in.?

## Work Done By a Variable Force

7. **Lifting a rope** A mountain climber is about to haul up a 50 m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?

8. **Leaky sandbag** A bag of sand originally weighing 144 lb was lifted at a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been

lifted to 18 ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

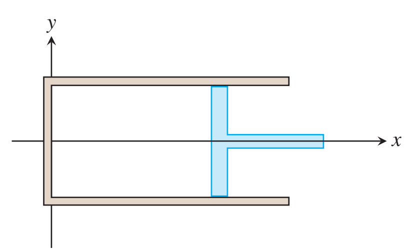
9. **Lifting an elevator cable** An electric elevator with a motor at the top has a multistrand cable weighing 4.5 lb/ft. When the car is at the first floor, 180 ft of cable are paid out, and effectively 0 ft are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

10. **Force of attraction** When a particle of mass $m$ is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is $k/x^2$. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time it reaches $x = a$, $0 < a < b$.

11. **Compressing gas** Suppose that the gas in a circular cylinder of cross-sectional area $A$ is being compressed by a piston. If $p$ is the pressure of the gas in pounds per square inch and $V$ is the volume in cubic inches, show that the work done in compressing the gas from state $(p_1, V_1)$ to state $(p_2, V_2)$ is given by the equation

$$\text{Work} = \int_{(p_1, V_1)}^{(p_2, V_2)} p\, dV.$$

(*Hint:* In the coordinates suggested in the figure here, $dV = A\, dx$. The force against the piston is $pA$.)




12. (*Continuation of Exercise 11.*) Use the integral in Exercise 11 to find the work done in compressing the gas from $V_1 = 243$ in.$^3$ to $V_2 = 32$ in.$^3$ if $p_1 = 50$ lb/in.$^3$ and $p$ and $V$ obey the gas law $pV^{1.4} = \text{constant}$ (for adiabatic processes).

13. **Leaky bucket** Assume the bucket in Example 4 is leaking. It starts with 2 gal of water (16 lb) and leaks at a constant rate. It finishes draining just as it reaches the top. How much work was spent lifting the water alone? (*Hint:* Do not include the rope and bucket, and find the proportion of water left at elevation $x$ ft.)

14. (*Continuation of Exercise 13.*) The workers in Example 4 and Exercise 13 changed to a larger bucket that held 5 gal (40 lb) of water, but the new bucket had an even larger leak so that it, too, was empty by the time it reached the top. Assuming that the water leaked out at a steady rate, how much work was done lifting the water alone? (Do not include the rope and bucket.)

## Pumping Liquids from Containers

### The Weight of Water

Because of Earth's rotation and variations in its gravitational field, the weight of a cubic foot of water at sea level can vary from about 62.26 lb at the equator to as much as 62.59 lb near the poles, a variation of about 0.5%. A cubic foot that weighs about 62.4 lb in Melbourne and New York City will weigh 62.5 lb in Juneau and Stockholm. Although 62.4 is a typical figure and common textbook value, there is considerable variation.

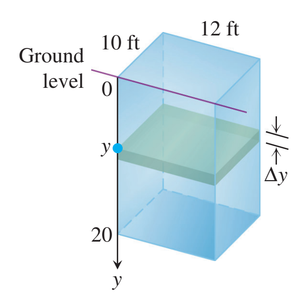
15. **Pumping water** The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.4 lb/ft$^3$.

    a. How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?

    b. If the water is pumped to ground level with a (5/11)-horsepower (hp) motor (work output 250 ft-lb/sec), how long will it take to empty the full tank (to the nearest minute)?

    c. Show that the pump in part (b) will lower the water level 10 ft (halfway) during the first 25 min of pumping.

    d. **The weight of water** What are the answers to parts (a) and (b) in a location where water weighs 62.26 lb/ft$^3$? 62.59 lb/ft$^3$?



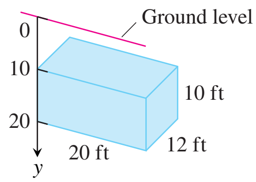

16. **Emptying a cistern** The rectangular cistern (storage tank for rainwater) shown below has its top 10 ft below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.

    a. How much work will it take to empty the cistern?

    b. How long will it take a 1/2 hp pump, rated at 275 ft-lb/sec, to pump the tank dry?

    c. How long will it take the pump in part (b) to empty the tank halfway? (It will be less than half the time required to empty the tank completely.)

    d. **The weight of water** What are the answers to parts (a) through (c) in a location where water weighs 62.26 lb/ft$^3$? 62.59 lb/ft$^3$?

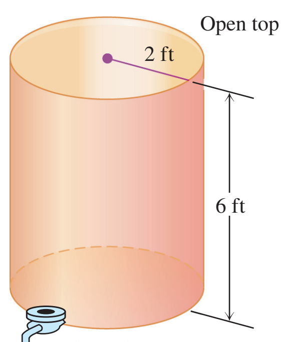



**17. Pumping oil** How much work would it take to pump oil from the tank in Example 5 to the level of the top of the tank if the tank were completely full?

**18. Pumping a half-full tank** Suppose that, instead of being full, the tank in Example 5 is only half full. How much work does it take to pump the remaining oil to a level 4 ft above the top of the tank?

**19. Emptying a tank** A vertical right circular cylindrical tank measures 30 ft high and 20 ft in diameter. It is full of kerosene weighing 51.2 lb/ft$^3$. How much work does it take to pump the kerosene to the level of the top of the tank?

**20.** The cylindrical tank shown here can be filled by pumping water from a lake 15 ft below the bottom of the tank. There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reasons for your answer.

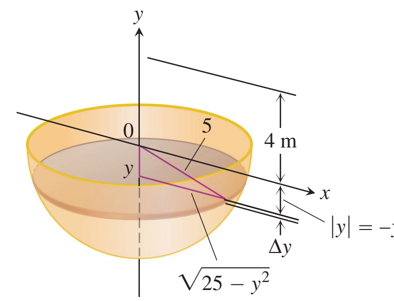



**21. a. Pumping milk** Suppose that the conical container in Example 5 contains milk (weighing 64.5 lb/ft$^3$) instead of olive oil. How much work will it take to pump the contents to the rim?

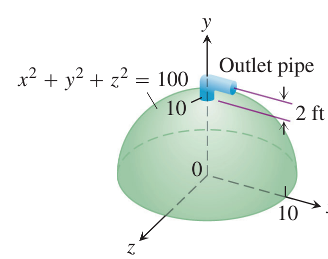
**b. Pumping oil** How much work will it take to pump the oil in Example 5 to a level 3 ft above the cone's rim?

**22. Pumping seawater** To design the interior surface of a huge stainless-steel tank, you revolve the curve $y = x^2, 0 \le x \le 4$, about the $y$-axis. The container, with dimensions in meters, is to be filled with seawater, which weighs 10,000 N/m$^3$. How much work will it take to empty the tank by pumping the water to the tank's top?

**23. Emptying a water reservoir** We model pumping from spherical containers the way we do from other containers, with the axis of integration along the vertical axis of the sphere. Use the figure here to find how much work it takes to empty a full hemispherical water reservoir of radius 5 m by pumping the water to a height of 4 m above the top of the reservoir. Water weighs 9800 N/m$^3$.




**24.** You are in charge of the evacuation and repair of the storage tank shown here. The tank is a hemisphere of radius 10 ft and is full of benzene weighing 56 lb/ft$^3$. A firm you contacted says it can empty the tank for $1/2$¢ per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 ft above the top of the tank. If you have \$5000 budgeted for the job, can you afford to hire the firm?




## Work and Kinetic Energy

**25. Kinetic energy** If a variable force of magnitude $F(x)$ moves a body of mass $m$ along the $x$-axis from $x_1$ to $x_2$, the body's velocity $v$ can be written as $dx/dt$ (where $t$ represents time). Use Newton's second law of motion $F = m(dv/dt)$ and the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

to show that the net work done by the force in moving the body from $x_1$ to $x_2$ is

$$W = \int_{x_1}^{x_2} F(x)\,dx = \frac{1}{2}mv_2{}^2 - \frac{1}{2}mv_1{}^2,$$

where $v_1$ and $v_2$ are the body's velocities at $x_1$ and $x_2$. In physics, the expression $(1/2)mv^2$ is called the *kinetic energy* of a body of mass $m$ moving with velocity $v$. Therefore, *the work done by the force equals the change in the body's kinetic energy*, and we can find the work by calculating this change.

In Exercises 26–32, use the result of Exercise 25.

**26. Tennis** A 2-oz tennis ball was served at 160 ft/sec (about 109 mph). How much work was done on the ball to make it go this fast? (To find the ball's mass from its weight, express the weight in pounds and divide by 32 ft/sec$^2$, the acceleration of gravity.)

**27. Baseball** How many foot-pounds of work does it take to throw a baseball 90 mph? A baseball weighs 5 oz, or 0.3125 lb.

**28. Golf** A 1.6-oz golf ball is driven off the tee at a speed of 280 ft/sec (about 191 mph). How many foot-pounds of work are done on the ball getting it into the air?

**29. Tennis** During the match in which Pete Sampras won the 1990 U.S. Open men's tennis championship, Sampras hit a serve that was clocked at a phenomenal 124 mph. How much work did Sampras have to do on the 2-oz ball to get it to that speed?

**30. Football** A quarterback threw a 14.5-oz football 88 ft/sec (60 mph). How many foot-pounds of work were done on the ball to get it to this speed?

**31. Softball** How much work has to be performed on a 6.5-oz softball to pitch it 132 ft/sec (90 mph)?

**32. A ball bearing** A 2-oz steel ball bearing is placed on a vertical spring whose force constant is $k = 18$ lb/ft. The spring is compressed 2 in. and released. About how high does the ball bearing go?

**33. Pumping the funnel of the glory hole** (*Continuation of Example 6.*)

**a.** Find the radius of the cross-section (funnel portion) of the glory hole in Example 6 as a function of the height $y$ above the floor of the dam (from $y = 325$ to $y = 375$).

**b.** Find $\Delta V$ for the funnel section of the glory hole (from $y = 325$ to $y = 375$).

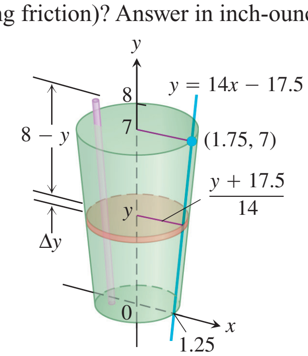
**c.** Find the work necessary to pump out the funnel section by formulating and evaluating the appropriate definite integral.

**34. Pumping water from a glory hole** (*Continuation of Exercise 33.*)

**a.** Find the total work necessary to pump out the glory hole, by adding the work necessary to pump both the throat and funnel sections.

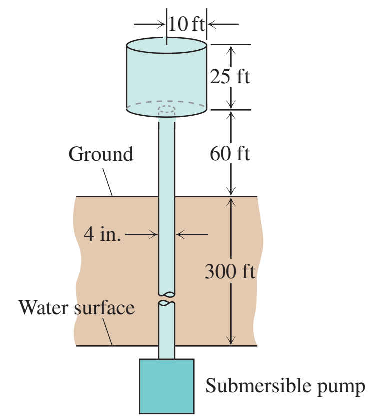
**b.** Your answer to part (a) is in foot-pounds. A more useful form is horsepower-hours, since motors are rated in horsepower. To convert from foot-pounds to horsepower-hours, divide by $1.98 \times 10^6$. How many hours would it take a 1000-horsepower motor to pump out the glory hole, assuming that the motor was fully efficient?

**35. Drinking a milkshake** The truncated conical container shown here is full of strawberry milkshake that weighs 4/9 oz/in.$^3$ As you can see, the container is 7 in. deep, 2.5 in. across at the base, and 3.5 in. across at the top (a standard size at Brigham's in Boston). The straw sticks up an inch above the top. About how much work does it take to suck up the milkshake through the straw (neglecting friction)? Answer in inch-ounces.



Dimensions in inches

**36. Water tower** Your town has decided to drill a well to increase its water supply. As the town engineer, you have determined that a water tower will be necessary to provide the pressure needed for distribution, and you have designed the system shown here. The water is to be pumped from a 300 ft well through a vertical 4 in. pipe into the base of a cylindrical tank 20 ft in diameter and 25 ft high. The base of the tank will be 60 ft aboveground. The pump is a 3 hp pump, rated at 1650 ft · lb/sec. To the nearest hour, how long will it take to fill the tank the first time? (Include the time it takes to fill the pipe.) Assume that water weighs 62.4 lb/ft$^3$.




**37. Putting a satellite in orbit** The strength of Earth's gravitational field varies with the distance $r$ from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass $m$ during and after launch is

$$F(r) = \frac{mMG}{r^2}.$$

Here, $M = 5.975 \times 10^{24}$ kg is Earth's mass, $G = 6.6720 \times 10^{-11}\ \mathrm{N} \cdot \mathrm{m}^2\,\mathrm{kg}^{-2}$ is the universal gravitational constant, and $r$ is measured in meters. The work it takes to lift a 1000-kg satellite from Earth's surface to a circular orbit 35,780 km above Earth's center is therefore given by the integral

$$\text{Work} = \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{1000MG}{r^2}\,dr \text{ joules}.$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spent lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

**38. Forcing electrons together** Two electrons $r$ meters apart repel each other with a force of

$$F = \frac{23 \times 10^{-29}}{r^2} \text{ newtons}.$$

**a.** Suppose one electron is held fixed at the point (1, 0) on the $x$-axis (units in meters). How much work does it take to move a second electron along the $x$-axis from the point $(-1, 0)$ to the origin?

**b.** Suppose an electron is held fixed at each of the points $(-1, 0)$ and (1, 0). How much work does it take to move a third electron along the $x$-axis from (5, 0) to (3, 0)?

# 6.7 Fluid Pressures and Forces

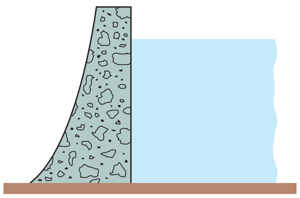


**FIGURE 6.64** To withstand the increasing pressure, dams are built thicker as they go down.

We make dams thicker at the bottom than at the top (Figure 6.64) because the pressure against them increases with depth. The pressure at any point on a dam depends only on how far below the surface the point is and not on how much the surface of the dam happens to be tilted at that point. The pressure, in pounds per square foot at a point $h$ feet below the surface, is always $62.4h$. The number 62.4 is the weight-density of water in pounds per cubic foot. The pressure $h$ feet below the surface of any fluid is the fluid's *weight-density* times $h$.

**The Pressure-Depth Equation**

In a fluid that is standing still, the pressure $p$ at depth $h$ is the fluid's weight-density $w$ times $h$:

$$p = wh. \tag{1}$$

In this section we use the equation $p = wh$ to derive a formula for the total force exerted by a fluid against all or part of a vertical or horizontal containing wall.

**Weight-density**

A fluid's weight-density is its weight per unit volume. Typical values ($\text{lb/ft}^3$) are

| | |
|---|---|
| Gasoline | 42 |
| Mercury | 849 |
| Milk | 64.5 |
| Molasses | 100 |
| Olive oil | 57 |
| Seawater | 64 |
| Water | 62.4 |

## The Constant-Depth Formula for Fluid Force

In a container of fluid with a flat horizontal base, the total force exerted by the fluid against the base can be calculated by multiplying the area of the base by the pressure at the base. We can do this because total force equals force per unit area (pressure) times area. (See Figure 6.65.) If $F$, $p$, and $A$ are the total force, pressure, and area, then

$$\begin{aligned} F &= \text{total force} = \text{force per unit area} \times \text{area} \\ &= \text{pressure} \times \text{area} = pA \\ &= whA. \end{aligned}$$

$p = wh$ from Eq. (1)

**Fluid Force on a Constant-Depth Surface**

$$F = pA = whA \tag{2}$$

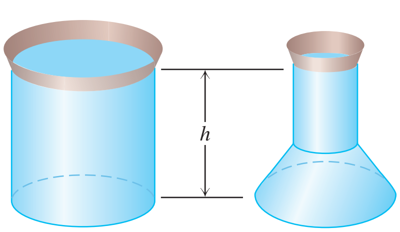



**FIGURE 6.65** These containers are filled with water to the same depth and have the same base area. The total force is therefore the same on the bottom of each container. The containers' shapes do not matter here.

For example, the weight-density of water is $62.4\ \text{lb/ft}^3$, so the fluid force at the bottom of a $10\ \text{ft} \times 20\ \text{ft}$ rectangular swimming pool 3 ft deep is

$$\begin{aligned} F = whA &= (62.4\ \text{lb/ft}^3)(3\ \text{ft})(10 \cdot 20\ \text{ft}^2) \\ &= 37{,}440\ \text{lb}. \end{aligned}$$

For a flat plate submerged *horizontally*, like the bottom of the swimming pool just discussed, the downward force acting on its upper face due to liquid pressure is given by Equation (2). If the plate is submerged *vertically*, however, then the pressure against it will be different at different depths and Equation (2) no longer is usable in that form (because $h$ varies). By dividing the plate into many narrow horizontal bands or strips, we can create a Riemann sum whose limit is the fluid force against the side of the submerged vertical plate. Here is the procedure.

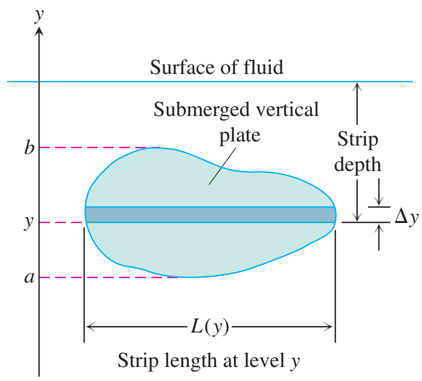



**FIGURE 6.66** The force exerted by a fluid against one side of a thin, flat horizontal strip is about $\Delta F =$ pressure $\times$ area $= w \times (\text{strip depth}) \times L(y)\,\Delta y$.

## The Variable-Depth Formula

Suppose we want to know the force exerted by a fluid against one side of a vertical plate submerged in a fluid of weight-density $w$. To find it, we model the plate as a region extending from $y = a$ to $y = b$ in the $xy$-plane (Figure 6.66). We partition $[a, b]$ in the usual way and imagine the region to be cut into thin horizontal strips by planes perpendicular to the $y$-axis at the partition points. The typical strip from $y$ to $y + \Delta y$ is $\Delta y$ units wide by $L(y)$ units long. We assume $L(y)$ to be a continuous function of $y$.

The pressure varies across the strip from top to bottom. If the strip is narrow enough, however, the pressure will remain close to its bottom-edge value of $w \times (\text{strip depth})$. The force exerted by the fluid against one side of the strip will be about

$$\begin{aligned}\Delta F &= (\text{pressure along bottom edge}) \times (\text{area}) \\ &= w \cdot (\text{strip depth}) \cdot L(y)\,\Delta y.\end{aligned}$$

Assume there are $n$ strips associated with the partition of $a \le y \le b$ and that $y_k$ is the bottom edge of the $k$th strip having length $L(y_k)$ and width $\Delta y_k$. The force against the entire plate is approximated by summing the forces against each strip, giving the Riemann sum

$$F \approx \sum_{k=1}^{n} (w \cdot (\text{strip depth})_k \cdot L(y_k))\,\Delta y_k. \tag{3}$$

The sum in Equation (3) is a Riemann sum for a continuous function on $[a, b]$, and we expect the approximations to improve as the norm of the partition goes to zero. The force against the plate is the limit of these sums.

### The Integral for Fluid Force Against a Vertical Flat Plate

Suppose that a plate submerged vertically in fluid of weight-density $w$ runs from $y = a$ to $y = b$ on the $y$-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level $y$. Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\,dy. \tag{4}$$

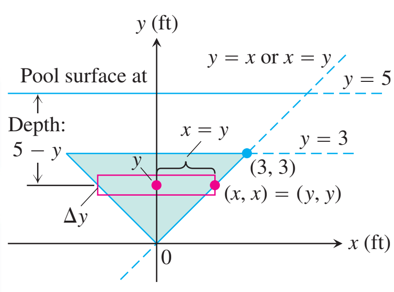



**FIGURE 6.67** To find the force on one side of the submerged plate in Example 1, we can use a coordinate system like the one here.

## EXAMPLE 1 Applying the Integral for Fluid Force

A flat isosceles right triangular plate with base 6 ft and height 3 ft is submerged vertically, base up, 2 ft below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.

**Solution** We establish a coordinate system to work in by placing the origin at the plate's bottom vertex and running the $y$-axis upward along the plate's axis of symmetry (Figure 6.67). The surface of the pool lies along the line $y = 5$ and the plate's top edge along the line $y = 3$. The plate's right-hand edge lies along the line $y = x$, with the upper right vertex at $(3, 3)$. The length of a thin strip at level $y$ is

$$L(y) = 2x = 2y.$$

The depth of the strip beneath the surface is $(5 - y)$. The force exerted by the water against one side of the plate is therefore

$$\begin{aligned} F &= \int_a^b w \cdot \begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix} \cdot L(y)\, dy \qquad \text{Eq. (4)} \\ &= \int_0^3 62.4(5 - y)2y\, dy \\ &= 124.8 \int_0^3 (5y - y^2)\, dy \\ &= 124.8\left[\frac{5}{2}y^2 - \frac{y^3}{3}\right]_0^3 = 1684.8 \text{ lb}. \end{aligned}$$

## Fluid Forces and Centroids

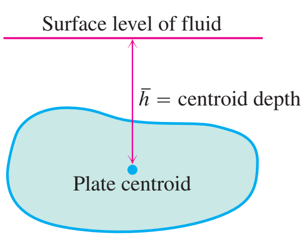



**FIGURE 6.68** The force against one side of the plate is $w \cdot \overline{h} \cdot$ plate area.

If we know the location of the centroid of a submerged flat vertical plate (Figure 6.68), we can take a shortcut to find the force against one side of the plate. From Equation (4),

$$\begin{aligned} F &= \int_a^b w \times (\text{strip depth}) \times L(y)\, dy \\ &= w \int_a^b (\text{strip depth}) \times L(y)\, dy \\ &= w \times (\text{moment about surface level line of region occupied by plate}) \\ &= w \times (\text{depth of plate's centroid}) \times (\text{area of plate}). \end{aligned}$$

**Fluid Forces and Centroids**

The force of a fluid of weight-density $w$ against one side of a submerged flat vertical plate is the product of $w$, the distance $\overline{h}$ from the plate's centroid to the fluid surface, and the plate's area:

$$F = w\overline{h}A. \tag{5}$$

Video Video You Try It

**EXAMPLE 2** Finding Fluid Force Using Equation (5)

Use Equation (5) to find the force in Example 1.

**Solution** The centroid of the triangle (Figure 6.67) lies on the $y$-axis, one-third of the way from the base to the vertex, so $\overline{h} = 3$. The triangle's area is

$$\begin{aligned} A &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(6)(3) = 9. \end{aligned}$$

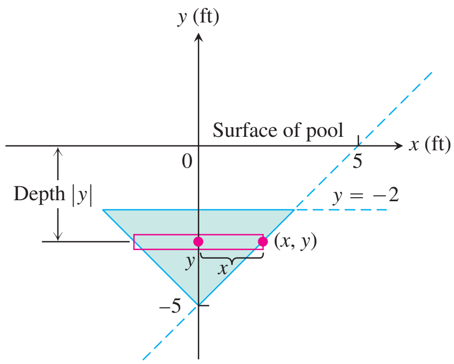
Hence,

$$\begin{aligned} F &= w\overline{h}A = (62.4)(3)(9) \\ &= 1684.8 \text{ lb}. \end{aligned}$$

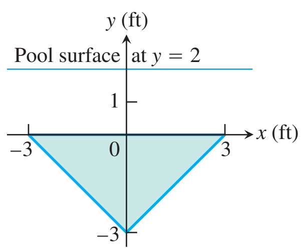
# EXERCISES 6.7

The weight-densities of the fluids in the following exercises can be found in the table on page 456.

**1. Triangular plate** Calculate the fluid force on one side of the plate in Example 1 using the coordinate system shown here.




**2. Triangular plate** Calculate the fluid force on one side of the plate in Example 1 using the coordinate system shown here.



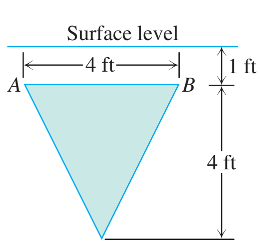

**3. Lowered triangular plate** The plate in Example 1 is lowered another 2 ft into the water. What is the fluid force on one side of the plate now?

**4. Raised triangular plate** The plate in Example 1 is raised to put its top edge at the surface of the pool. What is the fluid force on one side of the plate now?

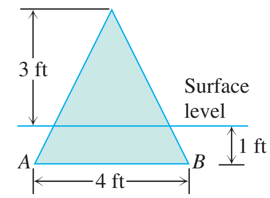
**5. Triangular plate** The isosceles triangular plate shown here is submerged vertically 1 ft below the surface of a freshwater lake.

**a.** Find the fluid force against one face of the plate.

**b.** What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?




**6. Rotated triangular plate** The plate in Exercise 5 is revolved 180° about line $AB$ so that part of the plate sticks out of the lake, as shown here. What force does the water exert on one face of the plate now?




**7. New England Aquarium** The viewing portion of the rectangular glass window in a typical fish tank at the New England Aquarium in Boston is 63 in. wide and runs from 0.5 in. below the water's surface to 33.5 in. below the surface. Find the fluid force against this portion of the window. The weight-density of seawater is 64 $\text{lb/ft}^3$. (In case you were wondering, the glass is $3/4$ in. thick and the tank walls extend 4 in. above the water to keep the fish from jumping out.)

**8. Fish tank** A horizontal rectangular freshwater fish tank with base 2 ft $\times$ 4 ft and height 2 ft (interior dimensions) is filled to within 2 in. of the top.

**a.** Find the fluid force against each side and end of the tank.

**b.** If the tank is sealed and stood on end (without spilling), so that one of the square ends is the base, what does that do to the fluid forces on the rectangular sides?

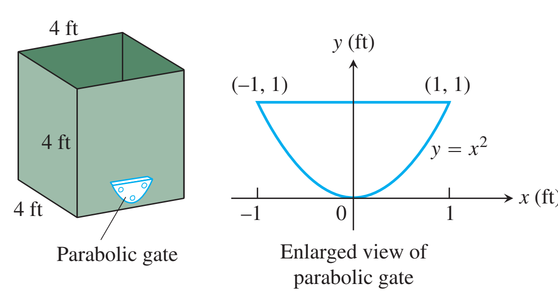
**9. Semicircular plate** A semicircular plate 2 ft in diameter sticks straight down into freshwater with the diameter along the surface. Find the force exerted by the water on one side of the plate.

**10. Milk truck** A tank truck hauls milk in a 6-ft-diameter horizontal right circular cylindrical tank. How much force does the milk exert on each end of the tank when the tank is half full?

**11.** The cubical metal tank shown here has a parabolic gate, held in place by bolts and designed to withstand a fluid force of 160 lb without rupturing. The liquid you plan to store has a weight-density of 50 $\text{lb/ft}^3$.

**a.** What is the fluid force on the gate when the liquid is 2 ft deep?

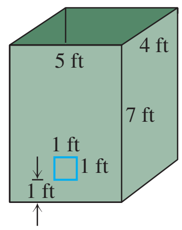
**b.** What is the maximum height to which the container can be filled without exceeding its design limitation?



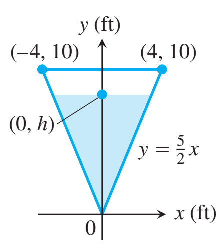
**12.** The rectangular tank shown here has a 1 ft × 1 ft square window 1 ft above the base. The window is designed to withstand a fluid force of 312 lb without cracking.

**a.** What fluid force will the window have to withstand if the tank is filled with water to a depth of 3 ft?

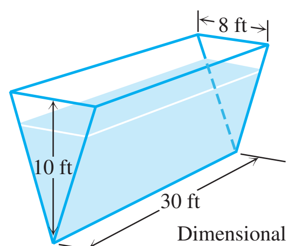
**b.** To what level can the tank be filled with water without exceeding the window's design limitation?




**13.** The end plates of the trough shown here were designed to withstand a fluid force of 6667 lb. How many cubic feet of water can the tank hold without exceeding this limitation? Round down to the nearest cubic foot.



End view of trough

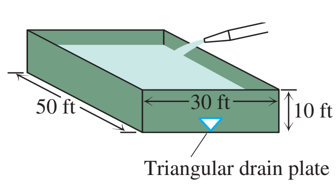


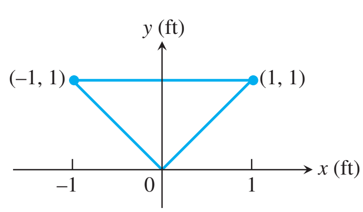
Dimensional view of trough

**14.** Water is running into the rectangular swimming pool shown here at the rate of 1000 $\text{ft}^3/\text{h}$.

**a.** Find the fluid force against the triangular drain plate after 9 h of filling.

**b.** The drain plate is designed to withstand a fluid force of 520 lb. How high can you fill the pool without exceeding this limitation?



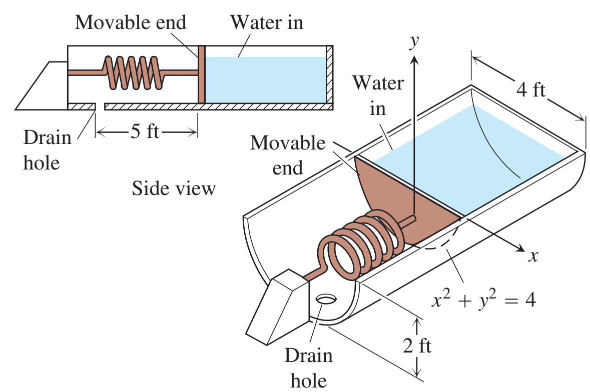



Enlarged view of drain plate

**15.** A vertical rectangular plate $a$ units long by $b$ units wide is submerged in a fluid of weight-density $w$ with its long edges parallel to the fluid's surface. Find the average value of the pressure along the vertical dimension of the plate. Explain your answer.

**16.** (*Continuation of Exercise 15.*) Show that the force exerted by the fluid on one side of the plate is the average value of the pressure (found in Exercise 15) times the area of the plate.

**17.** Water pours into the tank here at the rate of 4 $\text{ft}^3/\text{min}$. The tank's cross-sections are 4-ft-diameter semicircles. One end of the tank is movable, but moving it to increase the volume compresses a spring. The spring constant is $k = 100$ lb/ft. If the end of the tank moves 5 ft against the spring, the water will drain out of a safety hole in the bottom at the rate of 5 $\text{ft}^3/\text{min}$. Will the movable end reach the hole before the tank overflows?




**18. Watering trough** The vertical ends of a watering trough are squares 3 ft on a side.

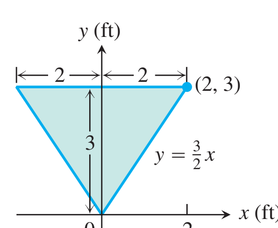
**a.** Find the fluid force against the ends when the trough is full.

**b.** How many inches do you have to lower the water level in the trough to reduce the fluid force by 25%?

**19. Milk carton** A rectangular milk carton measures 3.75 in. × 3.75 in. at the base and is 7.75 in. tall. Find the force of the milk on one side when the carton is full.

**20. Olive oil can** A standard olive oil can measures 5.75 in. × 3.5 in. at the base and is 10 in. tall. Find the fluid force against the base and each side when the can is full.

**21. Watering trough** The vertical ends of a watering trough are isosceles triangles like the one shown here (dimensions in feet).

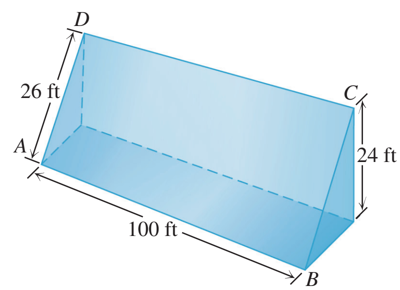



**a.** Find the fluid force against the ends when the trough is full.

**b.** How many inches do you have to lower the water level in the trough to cut the fluid force on the ends in half? (Answer to the nearest half-inch.)

**c.** Does it matter how long the trough is? Give reasons for your answer.

**22.** The face of a dam is a rectangle, *ABCD*, of dimensions $AB = CD = 100$ ft, $AD = BC = 26$ ft. Instead of being vertical, the plane *ABCD* is inclined as indicated in the accompanying figure, so that the top of the dam is 24 ft higher than the bottom. Find the force due to water pressure on the dam when the surface of the water is level with the top of the dam.



## Chapter 6 Questions to Guide Your Review

1. How do you define and calculate the volumes of solids by the method of slicing? Give an example.
2. How are the disk and washer methods for calculating volumes derived from the method of slicing? Give examples of volume calculations by these methods.
3. Describe the method of cylindrical shells. Give an example.
4. How do you define the length of a smooth parametrized curve $x = f(t), y = g(t), a \leq t \leq b$? What does smoothness have to do with length? What else do you need to know about the parametrization to find the curve's length? Give examples.
5. How do you find the length of the graph of a smooth function over a closed interval? Give an example. What about functions that do not have continuous first derivatives?
6. What is a center of mass?
7. How do you locate the center of mass of a straight, narrow rod or strip of material? Give an example. If the density of the material is constant, you can tell right away where the center of mass is. Where is it?
8. How do you locate the center of mass of a thin flat plate of material? Give an example.
9. How do you define and calculate the area of the surface swept out by revolving the graph of a smooth function $y = f(x)$, $a \leq x \leq b$, about the $x$-axis? Give an example.
10. Under what conditions can you find the area of the surface generated by revolving a curve $x = f(t), y = g(t), a \leq t \leq b$, about the $x$-axis? The $y$-axis? Give examples.
11. What do Pappus's two theorems say? Give examples of how they are used to calculate surface areas and volumes and to locate centroids.
12. How do you define and calculate the work done by a variable force directed along a portion of the $x$-axis? How do you calculate the work it takes to pump a liquid from a tank? Give examples.
13. How do you calculate the force exerted by a liquid against a portion of a vertical wall? Give an example.

# Chapter 6 Practice Exercises

## Volumes

Find the volumes of the solids in Exercises 1–16.

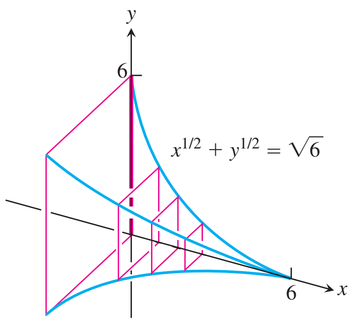
1. The solid lies between planes perpendicular to the $x$-axis at $x = 0$ and $x = 1$. The cross-sections perpendicular to the $x$-axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$.

2. The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross-sections of

the solid perpendicular to the $x$-axis are equilateral triangles whose bases stretch from the line to the curve.

**3.** The solid lies between planes perpendicular to the $x$-axis at $x = \pi/4$ and $x = 5\pi/4$. The cross-sections between these planes are circular disks whose diameters run from the curve $y = 2\cos x$ to the curve $y = 2\sin x$.

**4.** The solid lies between planes perpendicular to the $x$-axis at $x = 0$ and $x = 6$. The cross-sections between these planes are squares whose bases run from the $x$-axis up to the curve $x^{1/2} + y^{1/2} = \sqrt{6}$.




**5.** The solid lies between planes perpendicular to the $x$-axis at $x = 0$ and $x = 4$. The cross-sections of the solid perpendicular to the $x$-axis between these planes are circular disks whose diameters run from the curve $x^2 = 4y$ to the curve $y^2 = 4x$.

**6.** The base of the solid is the region bounded by the parabola $y^2 = 4x$ and the line $x = 1$ in the $xy$-plane. Each cross-section perpendicular to the $x$-axis is an equilateral triangle with one edge in the plane. (The triangles all lie on the same side of the plane.)

**7.** Find the volume of the solid generated by revolving the region bounded by the $x$-axis, the curve $y = 3x^4$, and the lines $x = 1$ and $x = -1$ about **(a)** the $x$-axis; **(b)** the $y$-axis; **(c)** the line $x = 1$; **(d)** the line $y = 3$.

**8.** Find the volume of the solid generated by revolving the "triangular" region bounded by the curve $y = 4/x^3$ and the lines $x = 1$ and $y = 1/2$ about **(a)** the $x$-axis; **(b)** the $y$-axis; **(c)** the line $x = 2$; **(d)** the line $y = 4$.

**9.** Find the volume of the solid generated by revolving the region bounded on the left by the parabola $x = y^2 + 1$ and on the right by the line $x = 5$ about **(a)** the $x$-axis; **(b)** the $y$-axis; **(c)** the line $x = 5$.

**10.** Find the volume of the solid generated by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about **(a)** the $x$-axis; **(b)** the $y$-axis; **(c)** the line $x = 4$; **(d)** the line $y = 4$.

**11.** Find the volume of the solid generated by revolving the "triangular" region bounded by the $x$-axis, the line $x = \pi/3$, and the curve $y = \tan x$ in the first quadrant about the $x$-axis.

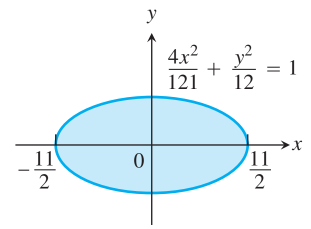
**12.** Find the volume of the solid generated by revolving the region bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$, and $y = 2$ about the line $y = 2$.

**13.** Find the volume of the solid generated by revolving the region between the $x$-axis and the curve $y = x^2 - 2x$ about **(a)** the $x$-axis; **(b)** the line $y = -1$; **(c)** the line $x = 2$; **(d)** the line $y = 2$.

**14.** Find the volume of the solid generated by revolving about the $x$-axis the region bounded by $y = 2\tan x$, $y = 0$, $x = -\pi/4$, and $x = \pi/4$. (The region lies in the first and third quadrants and resembles a skewed bowtie.)

**15. Volume of a solid sphere hole** A round hole of radius $\sqrt{3}$ ft is bored through the center of a solid sphere of a radius 2 ft. Find the volume of material removed from the sphere.

**16. Volume of a football** The profile of a football resembles the ellipse shown here. Find the football's volume to the nearest cubic inch.




## Lengths of Curves

Find the lengths of the curves in Exercises 17–23.

**17.** $y = x^{1/2} - (1/3)x^{3/2}, \quad 1 \le x \le 4$

**18.** $x = y^{2/3}, \quad 1 \le y \le 8$

**19.** $y = (5/12)x^{6/5} - (5/8)x^{4/5}, \quad 1 \le x \le 32$

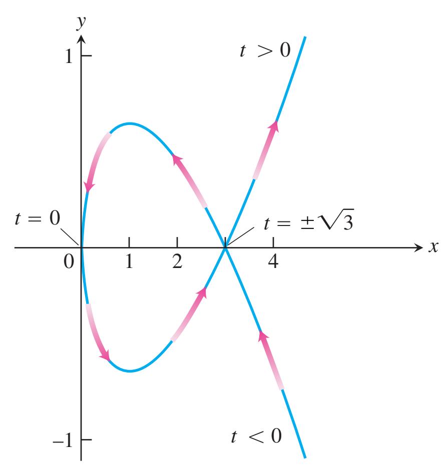
**20.** $x = (y^3/12) + (1/y), \quad 1 \le y \le 2$

**21.** $x = 5\cos t - \cos 5t, \quad y = 5\sin t - \sin 5t, \quad 0 \le t \le \pi/2$

**22.** $x = t^3 - 6t^2, \quad y = t^3 + 6t^2, \quad 0 \le t \le 1$

**23.** $x = 3\cos\theta, \quad y = 3\sin\theta, \quad 0 \le \theta \le \dfrac{3\pi}{2}$

**24.** Find the length of the enclosed loop $x = t^2$, $y = (t^3/3) - t$ shown here. The loop starts at $t = -\sqrt{3}$ and ends at $t = \sqrt{3}$.




## Centroids and Centers of Mass

**25.** Find the centroid of a thin, flat plate covering the region enclosed by the parabolas $y = 2x^2$ and $y = 3 - x^2$.

**26.** Find the centroid of a thin, flat plate covering the region enclosed by the $x$-axis, the lines $x = 2$ and $x = -2$, and the parabola $y = x^2$.

**27.** Find the centroid of a thin, flat plate covering the "triangular" region in the first quadrant bounded by the $y$-axis, the parabola $y = x^2/4$, and the line $y = 4$.

**28.** Find the centroid of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$.

**29.** Find the center of mass of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$ if the density function is $\delta(y) = 1 + y$. (Use horizontal strips.)

**30. a.** Find the center of mass of a thin plate of constant density covering the region between the curve $y = 3/x^{3/2}$ and the $x$-axis from $x = 1$ to $x = 9$.

**b.** Find the plate's center of mass if, instead of being constant, the density is $\delta(x) = x$. (Use vertical strips.)

## Areas of Surfaces of Revolution

In Exercises 31–36, find the areas of the surfaces generated by revolving the curves about the given axes.

**31.** $y = \sqrt{2x + 1}, \quad 0 \le x \le 3; \quad x\text{-axis}$

**32.** $y = x^3/3, \quad 0 \le x \le 1; \quad x\text{-axis}$

**33.** $x = \sqrt{4y - y^2}, \quad 1 \le y \le 2; \quad y\text{-axis}$

**34.** $x = \sqrt{y}, \quad 2 \le y \le 6; \quad y\text{-axis}$

**35.** $x = t^2/2, \quad y = 2t, \quad 0 \le t \le \sqrt{5}; \quad x\text{-axis}$

**36.** $x = t^2 + 1/(2t), \quad y = 4\sqrt{t}, \quad 1/\sqrt{2} \le t \le 1; \quad y\text{-axis}$

## Work

**37. Lifting equipment** A rock climber is about to haul up 100 N (about 22.5 lb) of equipment that has been hanging beneath her on 40 m of rope that weighs 0.8 newton per meter. How much work will it take? (*Hint:* Solve for the rope and equipment separately, then add.)

**38. Leaky tank truck** You drove an 800-gal tank truck of water from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-ft elevation change in 50 min. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weights 8 lb/U.S. gal.

**39. Stretching a spring** If a force of 20 lb is required to hold a spring 1 ft beyond its unstressed length, how much work does it take to stretch the spring this far? An additional foot?

**40. Garage door spring** A force of 200 N will stretch a garage door spring 0.8 m beyond its unstressed length. How far will a 300-N force stretch the spring? How much work does it take to stretch the spring this far from its unstressed length?

**41. Pumping a reservoir** A reservoir shaped like a right circular cone, point down, 20 ft across the top and 8 ft deep, is full of water. How much work does it take to pump the water to a level 6 ft above the top?

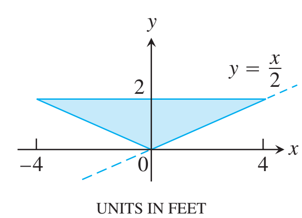
**42. Pumping a reservoir** (*Continuation of Exercise 41.*) The reservoir is filled to a depth of 5 ft, and the water is to be pumped to the same level as the top. How much work does it take?

**43. Pumping a conical tank** A right circular conical tank, point down, with top radius 5 ft and height 10 ft is filled with a liquid whose weight-density is 60 lb/ft$^3$. How much work does it take to pump the liquid to a point 2 ft above the tank? If the pump is driven by a motor rated at 275 ft-lb/sec (1/2 hp), how long will it take to empty the tank?

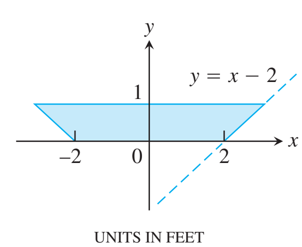
**44. Pumping a cylindrical tank** A storage tank is a right circular cylinder 20 ft long and 8 ft in diameter with its axis horizontal. If the tank is half full of olive oil weighing 57 lb/ft$^3$, find the work done in emptying it through a pipe that runs from the bottom of the tank to an outlet that is 6 ft above the top of the tank.

## Fluid Force

**45. Trough of water** The vertical triangular plate shown here is the end plate of a trough full of water $(w = 62.4)$. What is the fluid force against the plate?



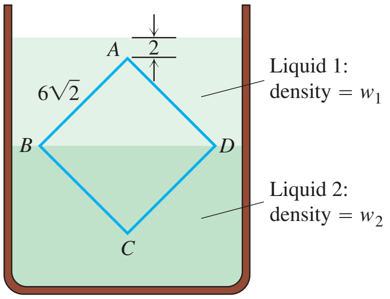

**46. Trough of maple syrup** The vertical trapezoid plate shown here is the end plate of a trough full of maple syrup weighing 75 lb/ft$^3$. What is the force exerted by the syrup against the end plate of the trough when the syrup is 10 in. deep?




**47. Force on a parabolic gate** A flat vertical gate in the face of a dam is shaped like the parabolic region between the curve $y = 4x^2$ and the line $y = 4$, with measurements in feet. The top of the gate lies 5 ft below the surface of the water. Find the force exerted by the water against the gate $(w = 62.4)$.

T **48.** You plan to store mercury ($w = 849$ lb/ft$^3$) in a vertical rectangular tank with a 1 ft square base side whose interior side wall can withstand a total fluid force of 40,000 lb. About how many cubic feet of mercury can you store in the tank at any one time?

**49.** The container profiled in the accompanying figure is filled with two nonmixing liquids of weight-density $w_1$ and $w_2$. Find the fluid force on one side of the vertical square plate $ABCD$. The points $B$ and $D$ lie in the boundary layer and the square is $6\sqrt{2}$ ft on a side.




**50.** The isosceles trapezoidal plate shown here is submerged vertically in water ($w = 62.4$) with its upper edge 4 ft below the surface. Find the fluid force on one side of the plate in two different ways:

**a.** By evaluating an integral.

**b.** By dividing the plate into a parallelogram and an isosceles triangle, locating their centroids, and using the equation $F = w\bar{h}A$ from Section 6.7.