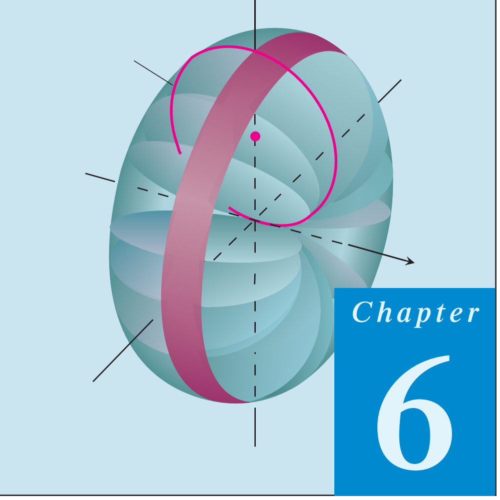


Chapter 6

Applications of Definite Integrals

OVERVIEW In Chapter 5 we discovered the connection between Riemann sums

$$S_P = \sum_{k=1}^{n} f(c_k)\, \Delta x_k$$

associated with a partition P of the finite closed interval $[a, b]$ and the process of integration. We found that for a continuous function f on $[a, b]$, the limit of S_P as the norm of the partition $\|P\|$ approaches zero is the number

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is any antiderivative of f. We applied this to the problems of computing the area between the x-axis and the graph of $y = f(x)$ for $a \leq x \leq b$, and to finding the area between two curves.

In this chapter we extend the applications to finding volumes, lengths of plane curves, centers of mass, areas of surfaces of revolution, work, and fluid forces against planar walls. We define all these as limits of Riemann sums of continuous functions on closed intervals—that is, as definite integrals which can be evaluated using the Fundamental Theorem of Calculus.

6.1 Volumes by Slicing and Rotation About an Axis

In this section we define volumes of solids whose cross-sections are plane regions. A **cross-section** of a solid S is the plane region formed by intersecting S with a plane (Figure 6.1).

Suppose we want to find the volume of a solid S like the one in Figure 6.1. We begin by extending the definition of a cylinder from classical geometry to cylindrical solids with arbitrary bases (Figure 6.2). If the cylindrical solid has a known base area A and height h, then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

This equation forms the basis for defining the volumes of many solids that are not cylindrical by the *method of slicing*.

If the cross-section of the solid S at each point x in the interval $[a, b]$ is a region $R(x)$ of area $A(x)$, and A is a continuous function of x, we can define and calculate the volume of the solid S as a definite integral in the following way.

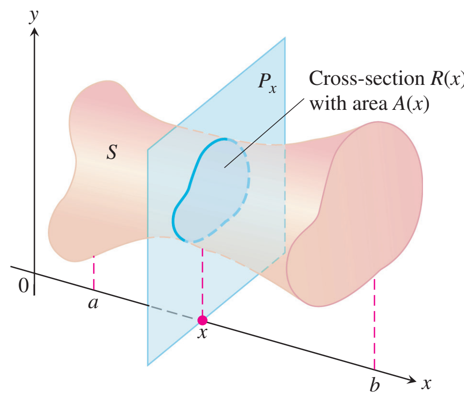


FIGURE 6.1 A cross-section of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

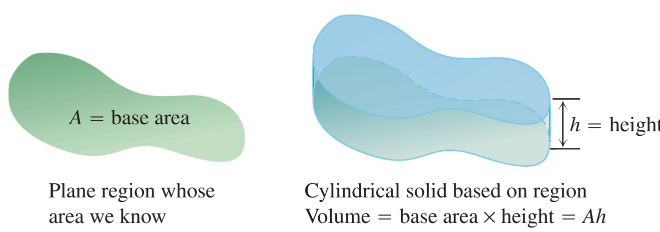


FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

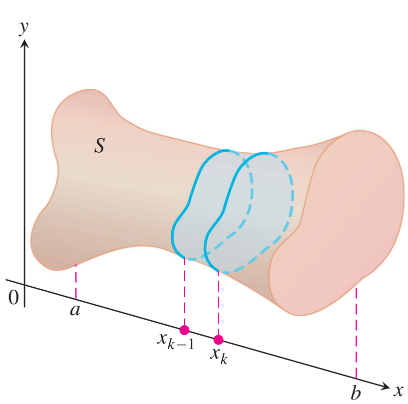


FIGURE 6.3 A typical thin slab in the solid S.

We partition $[a, b]$ into subintervals of width (length) Δx_k and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points $a = x_0 < x_1 < \cdots < x_n = b$. The planes P_{x_k}, perpendicular to the x-axis at the partition points, slice S into thin "slabs" (like thin slices of a loaf of bread). A typical slab is shown in Figure 6.3. We approximate the slab between the plane at x_{k-1} and the plane at x_k by a cylindrical solid with base area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.4). The volume V_k of this cylindrical solid is $A(x_k) \cdot \Delta x_k$, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k)\,\Delta x_k.$$

The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. We expect the approximations from these sums to improve as the norm of the partition of $[a, b]$ goes to zero, so we define their limiting definite integral to be the volume of the solid S.

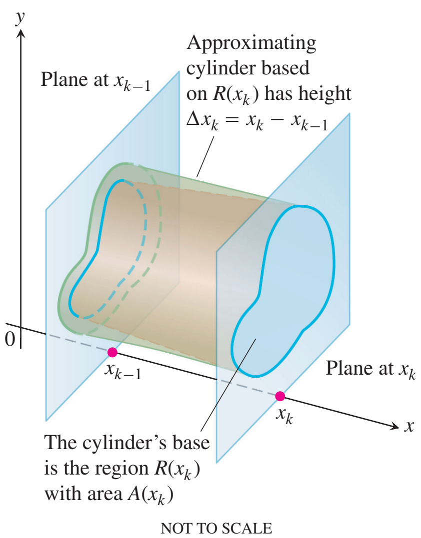


FIGURE 6.4 The solid thin slab in Figure 6.3 is approximated by the cylindrical solid with base $R(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

DEFINITION Volume

The **volume** of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

This definition applies whenever $A(x)$ is continuous, or more generally, when it is integrable. To apply the formula in the definition to calculate the volume of a solid, take the following steps:

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ using the Fundamental Theorem.

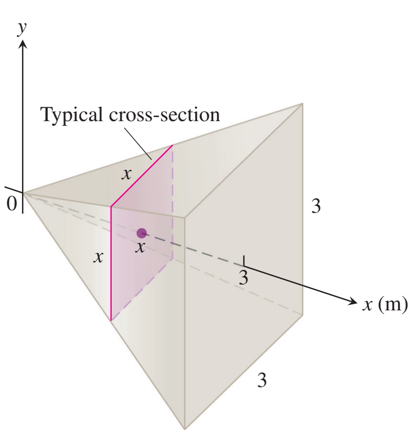


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

EXAMPLE 1 Volume of a Pyramid

You Try It

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

1. *A sketch.* We draw the pyramid with its altitude along the x-axis and its vertex at the origin and include a typical cross-section (Figure 6.5).
2. *A formula for* $A(x)$. The cross-section at x is a square x meters on a side, so its area is
$$A(x) = x^2.$$
3. *The limits of integration.* The squares lie on the planes from $x = 0$ to $x = 3$.
4. *Integrate to find the volume.*

$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right]_0^3 = 9\text{ m}^3$$ ■

HISTORICAL BIOGRAPHY

Bonaventura Cavalieri
(1598–1647)

Biography

EXAMPLE 2 Cavalieri's Principle

Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume (Figure 6.6). This follows immediately from the definition of volume, because the cross-sectional area function $A(x)$ and the interval $[a, b]$ are the same for both solids. ■

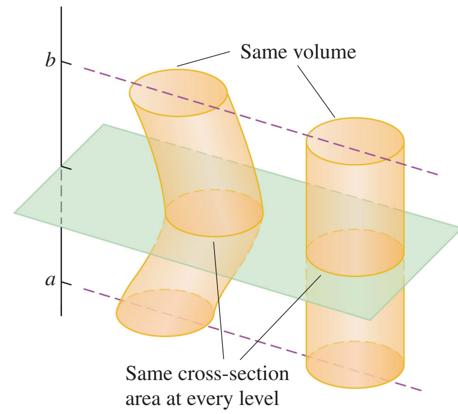


FIGURE 6.6 *Cavalieri's Principle:* These solids have the same volume, which can be illustrated with stacks of coins (Example 2).

EXAMPLE 3 Volume of a Wedge

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

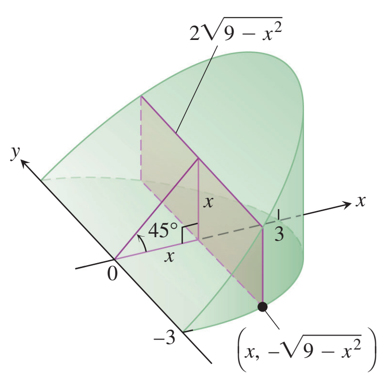


FIGURE 6.7 The wedge of Example 3, sliced perpendicular to the x-axis. The cross-sections are rectangles.

Solution We draw the wedge and sketch a typical cross-section perpendicular to the x-axis (Figure 6.7). The cross-section at x is a rectangle of area

$$A(x) = (\text{height})(\text{width}) = (x)\left(2\sqrt{9-x^2}\right)$$
$$= 2x\sqrt{9-x^2}.$$

The rectangles run from $x = 0$ to $x = 3$, so we have

$$V = \int_a^b A(x)\,dx = \int_0^3 2x\sqrt{9-x^2}\,dx$$
$$= -\frac{2}{3}(9-x^2)^{3/2}\Big]_0^3$$
$$= 0 + \frac{2}{3}(9)^{3/2}$$
$$= 18.$$

Let $u = 9 - x^2$, $du = -2x\,dx$, integrate, and substitute back.

■

Solids of Revolution: The Disk Method

The solid generated by rotating a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume gives

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.

EXAMPLE 4 A Solid of Revolution (Rotation About the x-Axis)

The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

$$V = \int_a^b \pi[R(x)]^2\,dx$$

$$= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx \qquad R(x) = \sqrt{x}$$

$$= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\bigg]_0^4 = \pi\frac{(4)^2}{2} = 8\pi. \quad \blacksquare$$

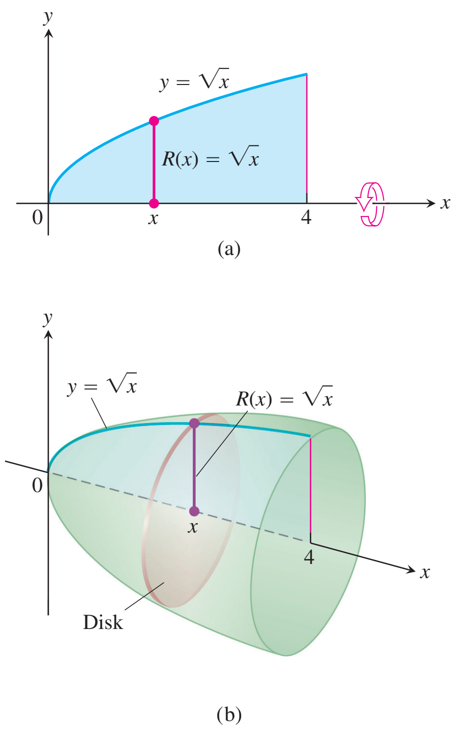


FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

EXAMPLE 5 Volume of a Sphere

The circle

$$x^2 + y^2 = a^2$$

is rotated about the x-axis to generate a sphere. Find its volume.

Solution We imagine the sphere cut into thin slices by planes perpendicular to the x-axis (Figure 6.9). The cross-sectional area at a typical point x between $-a$ and a is

$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

Therefore, the volume is

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

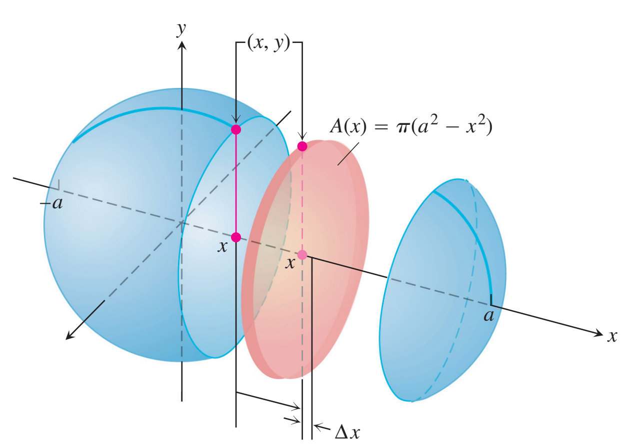


FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x-axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5). $\blacksquare$

The axis of revolution in the next example is not the x-axis, but the rule for calculating the volume is the same: Integrate $\pi(\text{radius})^2$ between appropriate limits.

You Try It

EXAMPLE 6 A Solid of Revolution (Rotation About the Line $y = 1$)

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.10). The volume is

$$\begin{aligned} V &= \int_1^4 \pi[R(x)]^2\,dx \\ &= \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx \\ &= \pi\int_1^4 \left[x - 2\sqrt{x} + 1\right] dx \\ &= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}. \end{aligned}$$

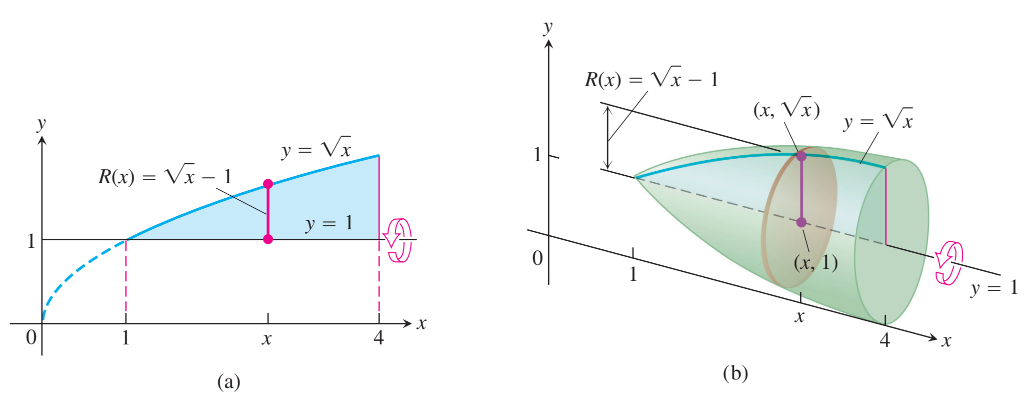


FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6. ■

To find the volume of a solid generated by revolving a region between the y-axis and a curve $x = R(y)$, $c \le y \le d$, about the y-axis, we use the same method with x replaced by y. In this case, the circular cross-section is

$$A(y) = \pi[\text{radius}]^2 = \pi[R(y)]^2.$$

Video

EXAMPLE 7 Rotation About the y-Axis

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.11). The volume is

$$\begin{aligned} V &= \int_1^4 \pi[R(y)]^2\,dy \\ &= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy \\ &= \pi\int_1^4 \frac{4}{y^2}\,dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right] \\ &= 3\pi. \end{aligned}$$

■

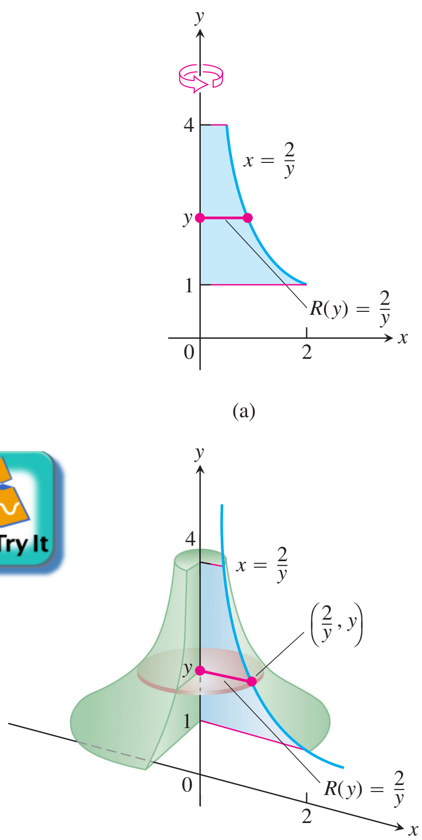


FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

EXAMPLE 8 Rotation About a Vertical Axis

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.12). Note that the cross-sections are perpendicular to the line $x = 3$. The volume is

$$\begin{aligned} V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2\,dy \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2\,dy \qquad R(y) = 3 - (y^2+1) = 2 - y^2 \\ &= \pi\int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\,dy \\ &= \pi\left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} \\ &= \frac{64\pi\sqrt{2}}{15}. \end{aligned}$$

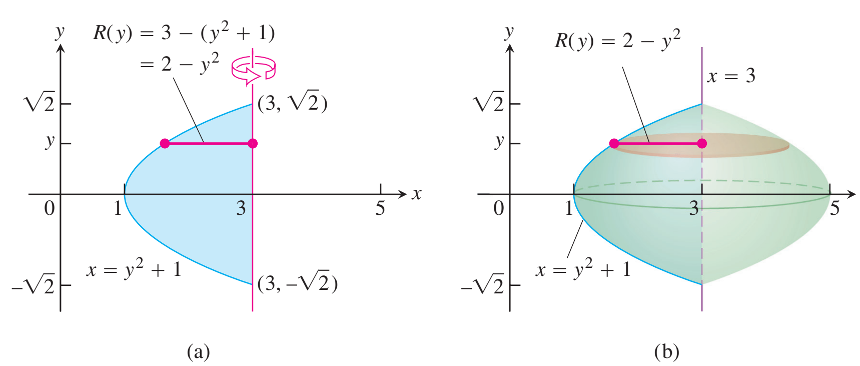


FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8. ■

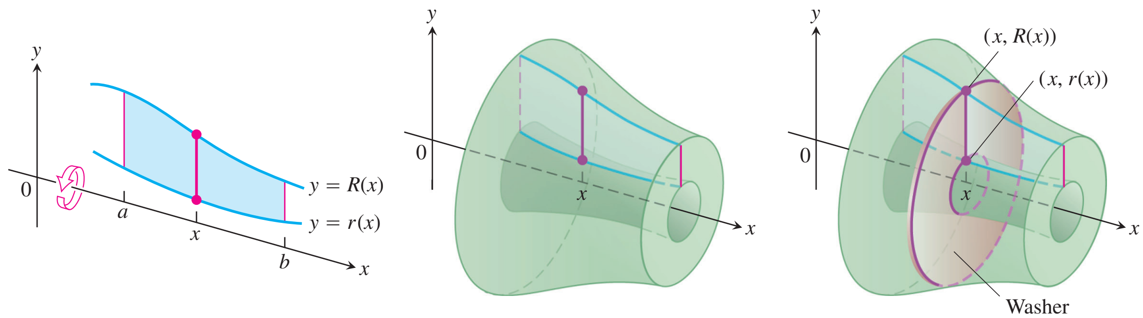


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are washers (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

Outer radius:	$R(x)$
Inner radius:	$r(x)$

The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume gives

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

This method for calculating the volume of a solid of revolution is called the **washer method** because a slab is a circular washer of outer radius $R(x)$ and inner radius $r(x)$.

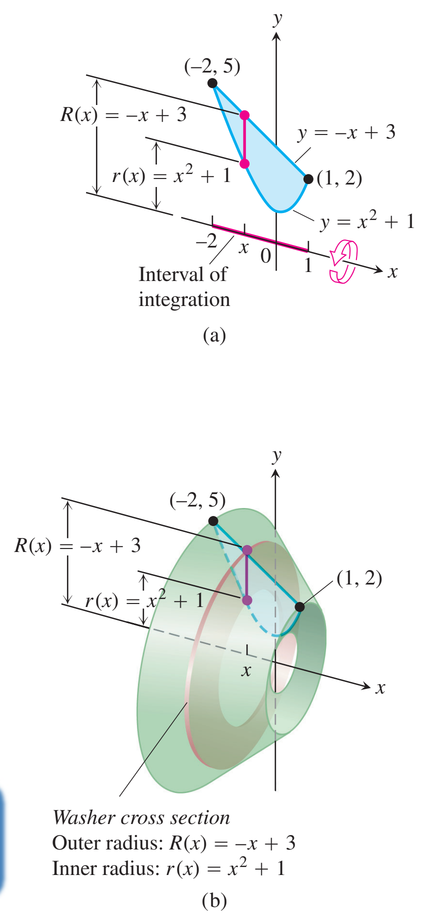


FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

EXAMPLE 9 A Washer Cross-Section (Rotation About the x-Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Figure 6.14).
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.

These radii are the distances of the ends of the line segment from the axis of revolution (Figure 6.14).

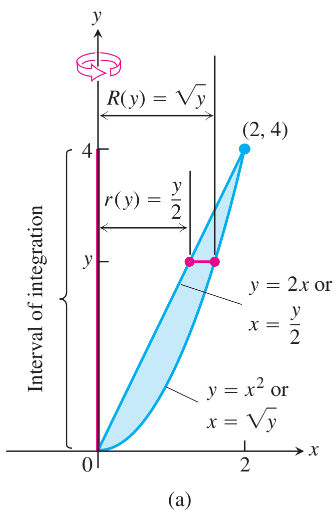
$$\text{Outer radius:} \quad R(x) = -x + 3$$
$$\text{Inner radius:} \quad r(x) = x^2 + 1$$

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line in Figure 6.14a.

$$\begin{aligned} x^2 + 1 &= -x + 3 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x = -2, \quad x &= 1 \end{aligned}$$

4. Evaluate the volume integral.

$$\begin{aligned} V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx \\ &= \int_{-2}^{1} \pi((-x + 3)^2 - (x^2 + 1)^2)\,dx \qquad \text{Values from Steps 2 and 3} \\ &= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\,dx \\ &= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5} \end{aligned}$$



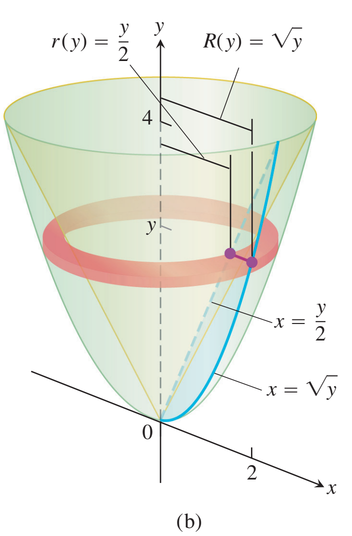


FIGURE 6.15 (a) The region being rotated about the y-axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

To find the volume of a solid formed by revolving a region about the y-axis, we use the same procedure as in Example 9, but integrate with respect to y instead of x. In this situation the line segment sweeping out a typical washer is perpendicular to the y-axis (the axis of revolution), and the outer and inner radii of the washer are functions of y.

EXAMPLE 10 A Washer Cross-Section (Rotation About the y-Axis)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y-axis). See Figure 6.15a.

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figure 6.15).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$\begin{aligned} V &= \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy \\ &= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right)dy \\ &= \pi\int_0^4 \left(y - \frac{y^2}{4}\right)dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi. \end{aligned}$$

Summary

In all of our volume examples, no matter how the cross-sectional area $A(x)$ of a typical slab is determined, the definition of volume as the definite integral $V = \int_a^b A(x)\,dx$ is the heart of the calculations we made.

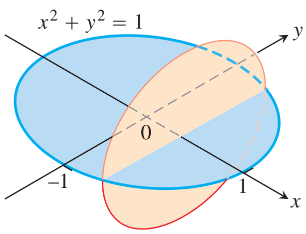
EXERCISES 6.1

Cross-Sectional Areas

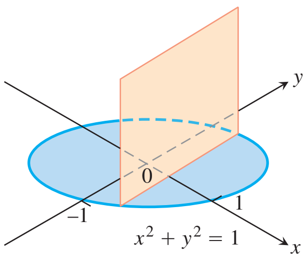
In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross-sections of the solid perpendicular to the x-axis.

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. In each case, the cross-sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

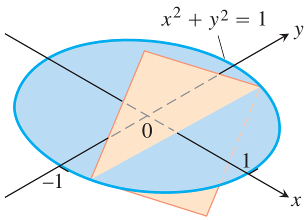
a. The cross-sections are circular disks with diameters in the xy-plane.



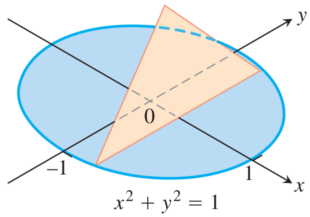
b. The cross-sections are squares with bases in the xy-plane.



c. The cross-sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)

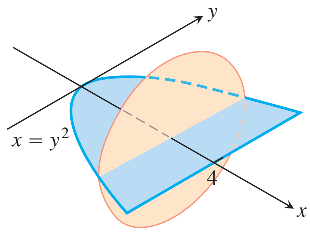


d. The cross-sections are equilateral triangles with bases in the xy-plane.

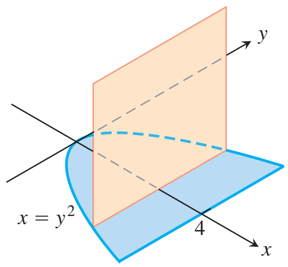


2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the x-axis between these planes run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

a. The cross-sections are circular disks with diameters in the xy-plane.



b. The cross-sections are squares with bases in the xy-plane.



c. The cross-sections are squares with diagonals in the xy-plane.

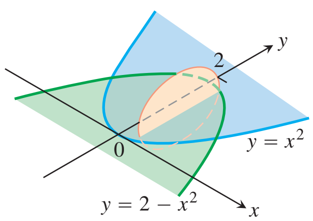
d. The cross-sections are equilateral triangles with bases in the xy-plane.

Volumes by Slicing

Find the volumes of the solids in Exercises 3–10.

3. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \leq x \leq 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

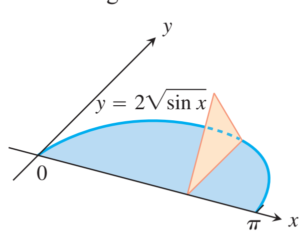


5. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

6. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

7. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are

a. equilateral triangles with bases running from the x-axis to the curve as shown in the figure.



b. squares with bases running from the x-axis to the curve.

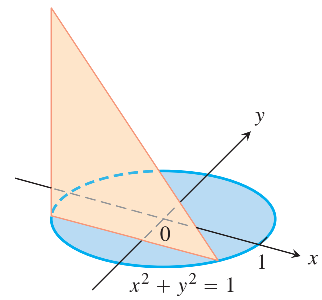
8. The solid lies between planes perpendicular to the x-axis at $x = -\pi/3$ and $x = \pi/3$. The cross-sections perpendicular to the x-axis are

a. circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

b. squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$.

9. The solid lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$.

10. The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.

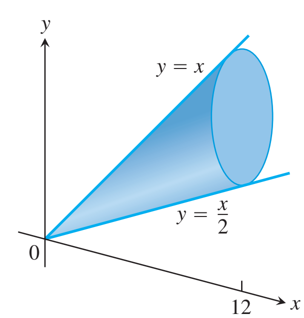


11. A twisted solid A square of side length s lies in a plane perpendicular to a line L. One vertex of the square lies on L. As this square moves a distance h along L, the square turns one revolution about L to generate a corkscrew-like column with square cross-sections.

a. Find the volume of the column.

b. What will the volume be if the square turns twice instead of once? Give reasons for your answer.

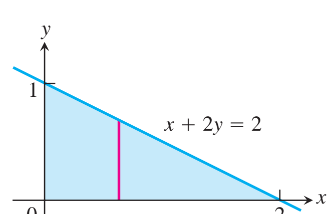
12. Cavalieri's Principle A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 12$. The cross-sections by planes perpendicular to the x-axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the accompanying figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



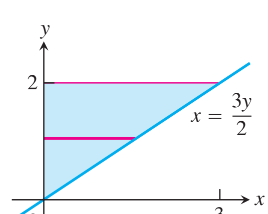
Volumes by the Disk Method

In Exercises 13–16, find the volume of the solid generated by revolving the shaded region about the given axis.

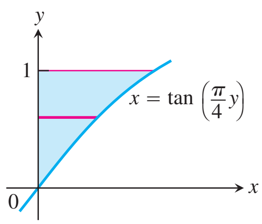
13. About the x-axis



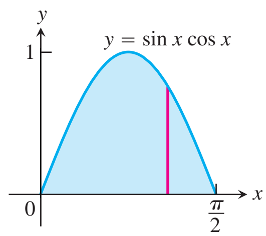
14. About the y-axis



15. About the y-axis



16. About the x-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 17–22 about the x-axis.

17. $y = x^2, \quad y = 0, \quad x = 2$

18. $y = x^3, \quad y = 0, \quad x = 2$

19. $y = \sqrt{9 - x^2}, \quad y = 0$

20. $y = x - x^2, \quad y = 0$

21. $y = \sqrt{\cos x}, \quad 0 \le x \le \pi/2, \quad y = 0, \quad x = 0$

22. $y = \sec x, \quad y = 0, \quad x = -\pi/4, \quad x = \pi/4$

In Exercises 23 and 24, find the volume of the solid generated by revolving the region about the given line.

23. The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$

24. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 25–30 about the y-axis.

25. The region enclosed by $x = \sqrt{5}y^2, \quad x = 0, \quad y = -1, \quad y = 1$

26. The region enclosed by $x = y^{3/2}, \quad x = 0, \quad y = 2$

27. The region enclosed by $x = \sqrt{2 \sin 2y}, \quad 0 \le y \le \pi/2, \quad x = 0$

28. The region enclosed by $x = \sqrt{\cos(\pi y/4)}, \quad -2 \le y \le 0, \quad x = 0$

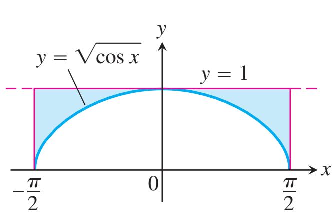
29. $x = 2/(y + 1), \quad x = 0, \quad y = 0, \quad y = 3$

30. $x = \sqrt{2y}/(y^2 + 1), \quad x = 0, \quad y = 1$

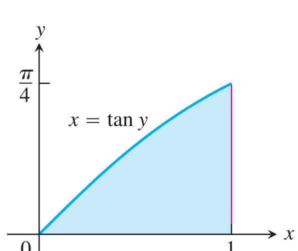
Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 31 and 32 about the indicated axes.

31. The x-axis



32. The y-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 33–38 about the x-axis.

33. $y = x, \quad y = 1, \quad x = 0$

34. $y = 2\sqrt{x}, \quad y = 2, \quad x = 0$

35. $y = x^2 + 1, \quad y = x + 3$

36. $y = 4 - x^2, \quad y = 2 - x$

37. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$

38. $y = \sec x, \quad y = \tan x, \quad x = 0, \quad x = 1$

In Exercises 39–42, find the volume of the solid generated by revolving each region about the y-axis.

39. The region enclosed by the triangle with vertices (1, 0), (2, 1), and (1, 1)

40. The region enclosed by the triangle with vertices (0, 1), (1, 0), and (1, 1)

41. The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line $x = 2$

42. The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

In Exercises 43 and 44, find the volume of the solid generated by revolving each region about the given axis.

43. The region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis, and on the right by the line $x = 1$, about the line $x = -1$

44. The region in the second quadrant bounded above by the curve $y = -x^3$, below by the x-axis, and on the left by the line $x = -1$, about the line $x = -2$

Volumes of Solids of Revolution

45. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

a. the x-axis. **b.** the y-axis.

c. the line $y = 2$. **d.** the line $x = 4$.

46. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

a. the line $x = 1$. **b.** the line $x = 2$.

47. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

a. the line $y = 1$. **b.** the line $y = 2$.

c. the line $y = -1$.

48. By integration, find the volume of the solid generated by revolving the triangular region with vertices (0, 0), (b, 0), (0, h) about

a. the x-axis. **b.** the y-axis.

Theory and Applications

49. The volume of a torus The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b$ ($b > a$) to generate a solid shaped like a doughnut and called a *torus*. Find its volume. (*Hint:* $\int_{-a}^{a} \sqrt{a^2 - y^2}\, dy = \pi a^2/2$, since it is the area of a semicircle of radius a.)

50. Volume of a bowl A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y-axis.

a. Find the volume of the bowl.

b. Related rates If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep?

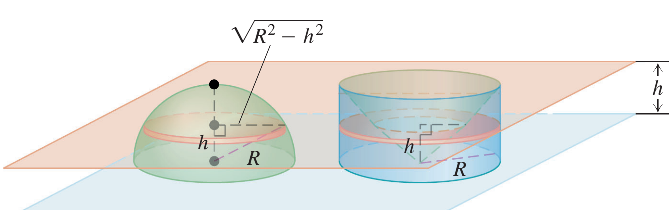
51. Volume of a bowl

a. A hemispherical bowl of radius a contains water to a depth h. Find the volume of water in the bowl.

b. Related rates Water runs into a sunken concrete hemispherical bowl of radius 5 m at the rate of 0.2 m^3/sec. How fast is the water level in the bowl rising when the water is 4 m deep?

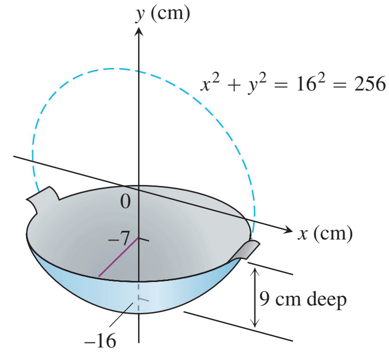
52. Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it.

53. Volume of a hemisphere Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross-sections with the cross-sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed as suggested by the accompanying figure.

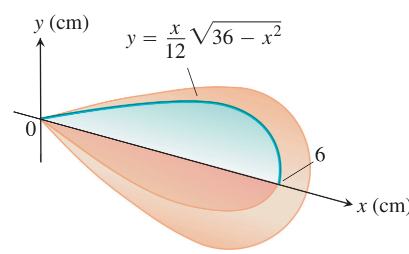


54. Volume of a cone Use calculus to find the volume of a right circular cone of height h and base radius r.

55. Designing a wok You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? (1 L = 1000 cm^3.)



56. Designing a plumb bob Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here. Find the plumb bob's volume. If you specify a brass that weighs 8.5 g/cm^3, how much will the plumb bob weigh (to the nearest gram)?



57. Max-min The arch $y = \sin x$, $0 \le x \le \pi$, is revolved about the line $y = c$, $0 \le c \le 1$, to generate the solid in Figure 6.16.

a. Find the value of c that minimizes the volume of the solid. What is the minimum volume?

b. What value of c in [0, 1] maximizes the volume of the solid?

T **c.** Graph the solid's volume as a function of c, first for $0 \le c \le 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from [0, 1]? Does this make sense physically? Give reasons for your answers.

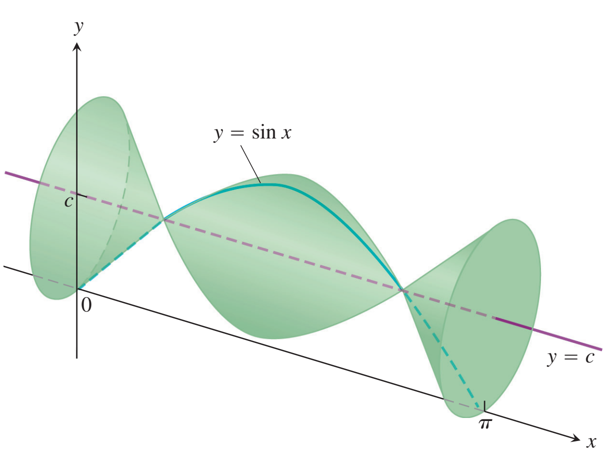


FIGURE 6.16

58. An auxiliary fuel tank You are designing an auxiliary fuel tank that will fit under a helicopter's fuselage to extend its range. After some experimentation at your drawing board, you decide to shape the tank like the surface generated by revolving the curve $y = 1 - (x^2/16)$, $-4 \le x \le 4$, about the x-axis (dimensions in feet).

a. How many cubic feet of fuel will the tank hold (to the nearest cubic foot)?

b. A cubic foot holds 7.481 gal. If the helicopter gets 2 mi to the gallon, how many additional miles will the helicopter be able to fly once the tank is installed (to the nearest mile)?

6.2 Volumes by Cylindrical Shells

In Section 6.1 we defined the volume of a solid S as the definite integral

$$V = \int_a^b A(x)\,dx,$$

where $A(x)$ is an integrable cross-sectional area of S from $x = a$ to $x = b$. The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x-axis. In this section we use the same integral definition for volume, but obtain the area by slicing through the solid in a different way. Now we slice through the solid using circular cylinders of increasing radii, like cookie cutters. We slice straight down through the solid perpendicular to the x-axis, with the axis of the cylinder parallel to the y-axis. The vertical axis of each cylinder is the same line, but the radii of the cylinders increase with each slice. In this way the solid S is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings. Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx. This allows us to apply the same integral definition for volume as before. Before describing the method in general, let's look at an example to gain some insight.

Video You Try It

EXAMPLE 1 Finding a Volume Using Shells

The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate the shape of a solid (Figure 6.17). Find the volume of the solid.

Solution Using the washer method from Section 6.1 would be awkward here because we would need to express the x-values of the left and right branches of the parabola in terms

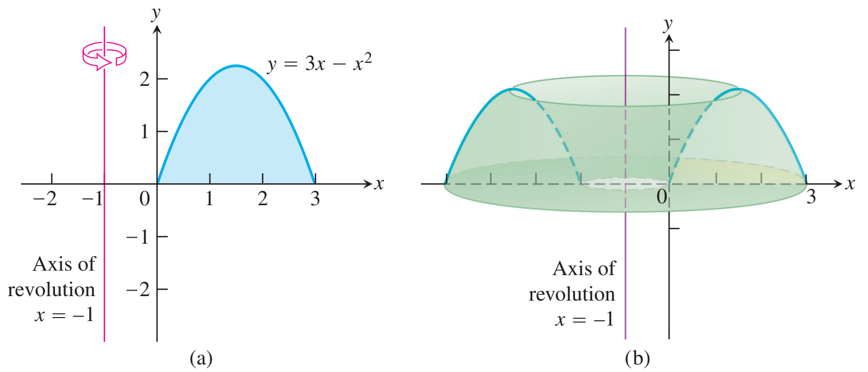


FIGURE 6.17 (a) The graph of the region in Example 1, before revolution. (b) The solid formed when the region in part (a) is revolved about the axis of revolution $x = -1$.

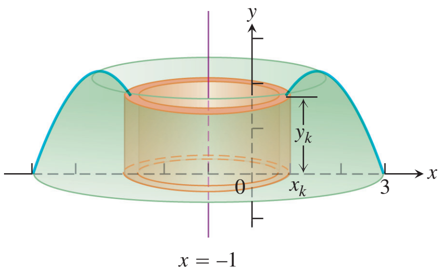


FIGURE 6.18 A cylindrical shell of height y_k obtained by rotating a vertical strip of thickness Δx about the line $x = -1$. The outer radius of the cylinder occurs at x_k, where the height of the parabola is $y_k = 3x_k - x_k^2$ (Example 1).

of y. (These x-values are the inner and outer radii for a typical washer, leading to complicated formulas.) Instead of rotating a horizontal strip of thickness Δy, we rotate a *vertical strip* of thickness Δx. This rotation produces a *cylindrical shell* of height y_k above a point x_k within the base of the vertical strip, and of thickness Δx. An example of a cylindrical shell is shown as the orange-shaded region in Figure 6.18. We can think of the cylindrical shell shown in the figure as approximating a slice of the solid obtained by cutting straight down through it, parallel to the axis of revolution, all the way around close to the inside hole. We then cut another cylindrical slice around the enlarged hole, then another, and so on, obtaining n cylinders. The radii of the cylinders gradually increase, and the heights of the cylinders follow the contour of the parabola: shorter to taller, then back to shorter (Figure 6.17a).

Each slice is sitting over a subinterval of the x-axis of length (width) Δx. Its radius is approximately $(1 + x_k)$, and its height is approximately $3x_k - x_k^2$. If we unroll the cylinder at x_k and flatten it out, it becomes (approximately) a rectangular slab with thickness Δx (Figure 6.19). The outer circumference of the kth cylinder is $2\pi \cdot \text{radius} = 2\pi(1 + x_k)$, and this is the length of the rolled-out rectangular slab. Its volume is approximated by that of a rectangular solid,

$$\begin{aligned}\Delta V_k &= \text{circumference} \times \text{height} \times \text{thickness} \\ &= 2\pi(1 + x_k) \cdot \left(3x_k - x_k^2\right) \cdot \Delta x.\end{aligned}$$

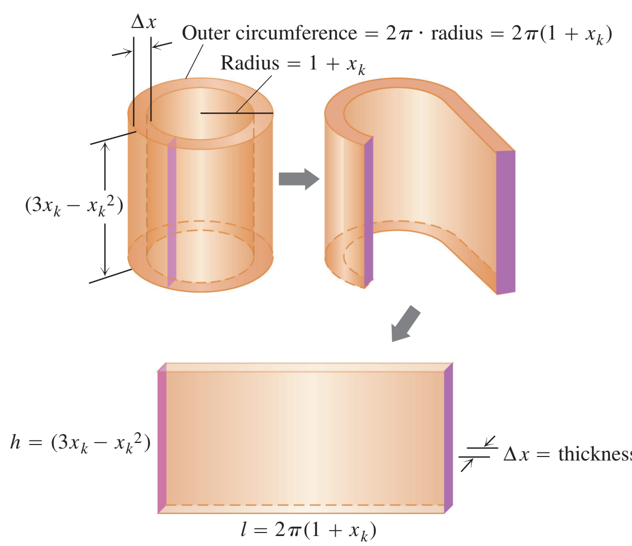


FIGURE 6.19 Imagine cutting and unrolling a cylindrical shell to get a flat (nearly) rectangular solid (Example 1).

Summing together the volumes ΔV_k of the individual cylindrical shells over the interval [0, 3] gives the Riemann sum

$$\sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi(x_k + 1)\left(3x_k - x_k^2\right) \Delta x.$$

Taking the limit as the thickness $\Delta x \to 0$ gives the volume integral

$$
\begin{aligned}
V &= \int_0^3 2\pi(x+1)(3x - x^2)\,dx \\
&= \int_0^3 2\pi(3x^2 + 3x - x^3 - x^2)\,dx \\
&= 2\pi \int_0^3 (2x^2 + 3x - x^3)\,dx \\
&= 2\pi\left[\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{4}x^4\right]_0^3 \\
&= \frac{45\pi}{2}.
\end{aligned}
$$

■

We now generalize the procedure used in Example 1.

The Shell Method

Suppose the region bounded by the graph of a nonnegative continuous function $y = f(x)$ and the x-axis over the finite closed interval $[a, b]$ lies to the right of the vertical line $x = L$ (Figure 6.20a). We assume $a \geq L$, so the vertical line may touch the region, but not pass through it. We generate a solid S by rotating this region about the vertical line L.

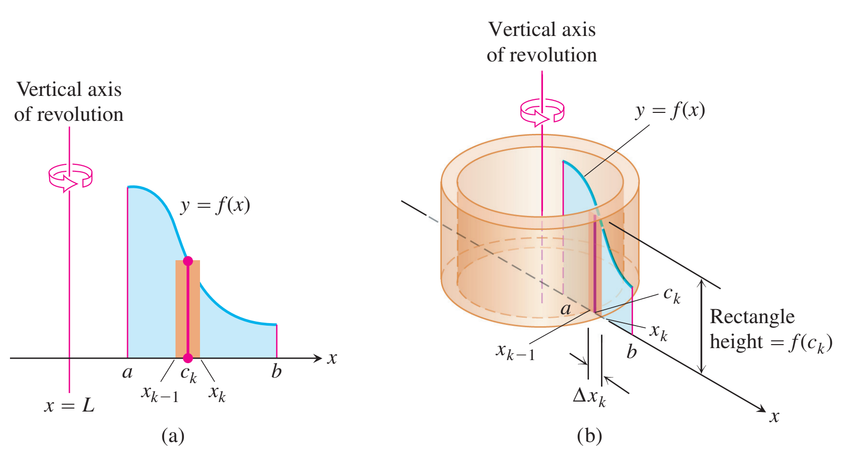


FIGURE 6.20 When the region shown in (a) is revolved about the vertical line $x = L$, a solid is produced which can be sliced into cylindrical shells. A typical shell is shown in (b).

Let P be a partition of the interval $[a, b]$ by the points $a = x_0 < x_1 < \cdots < x_n = b$, and let c_k be the midpoint of the kth subinterval $[x_{k-1}, x_k]$. We approximate the region in Figure 6.20a with rectangles based on this partition of $[a, b]$. A typical approximating rectangle has height $f(c_k)$ and width $\Delta x_k = x_k - x_{k-1}$. If this rectangle is rotated about the vertical line $x = L$, then a shell is swept out, as in Figure 6.20b. A formula from geometry tells us that the volume of the shell swept out by the rectangle is

$$
\begin{aligned}
\Delta V_k &= 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness} \\
&= 2\pi \cdot (c_k - L) \cdot f(c_k) \cdot \Delta x_k .
\end{aligned}
$$

We approximate the volume of the solid S by summing the volumes of the shells swept out by the n rectangles based on P:

$$V \approx \sum_{k=1}^{n} \Delta V_k.$$

The limit of this Riemann sum as $\|P\| \to 0$ gives the volume of the solid as a definite integral:

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\,dx.$$

$$= \int_a^b 2\pi(x - L)f(x)\,dx.$$

We refer to the variable of integration, here x, as the **thickness variable**. We use the first integral, rather than the second containing a formula for the integrand, to emphasize the *process* of the shell method. This will allow for rotations about a horizontal line L as well.

Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx.$$

EXAMPLE 2 Cylindrical Shells Revolving About the y-Axis

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.21a). Label the segment's height (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.21b, but you need not do that.)

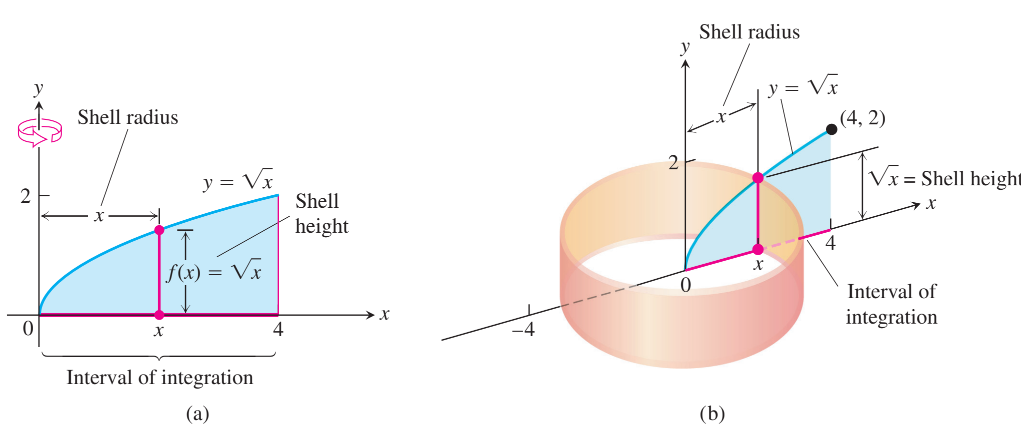


FIGURE 6.21 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx.

The shell thickness variable is x, so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is then

$$\begin{aligned} V &= \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx \\ &= \int_0^4 2\pi(x)\left(\sqrt{x}\right) dx \\ &= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}. \end{aligned}$$ ■

So far, we have used vertical axes of revolution. For horizontal axes, we replace the x's with y's.

EXAMPLE 3 Cylindrical Shells Revolving About the x-Axis

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.22a). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.22b, but you need not do that.)

In this case, the shell thickness variable is y, so the limits of integration for the shell formula method are $a = 0$ and $b = 2$ (along the y-axis in Figure 6.22). The volume of the solid is

$$\begin{aligned} V &= \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy \\ &= \int_0^2 2\pi(y)(4 - y^2)\, dy \\ &= \int_0^2 2\pi(4y - y^3)\, dy \\ &= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi. \end{aligned}$$ ■

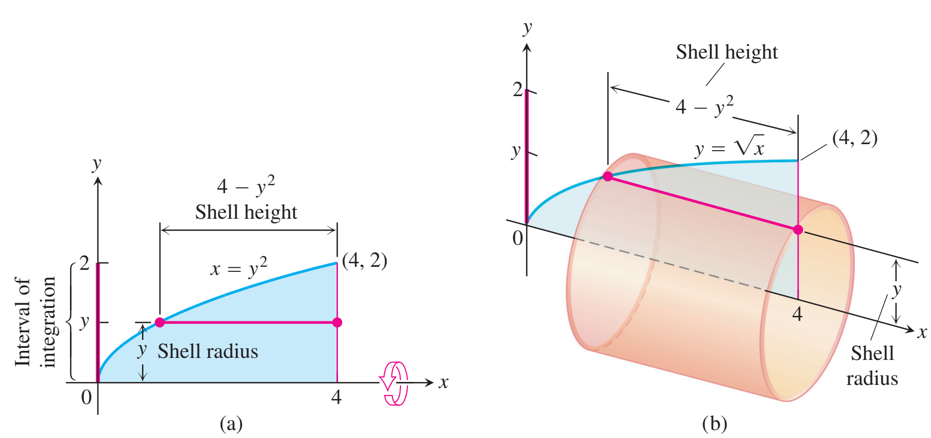


FIGURE 6.22 (a) The region, shell dimensions, and interval of integration in Example 3. (b) The shell swept out by the horizontal segment in part (a) with a width Δy.

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

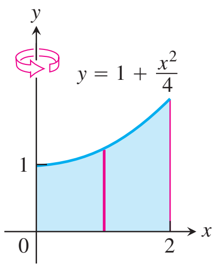
1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find* the limits of integration for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

The shell method gives the same answer as the washer method when both are used to calculate the volume of a region. We do not prove that result here, but it is illustrated in Exercises 33 and 34. Both volume formulas are actually special cases of a general volume formula we look at in studying double and triple integrals in Chapter 15. That general formula also allows for computing volumes of solids other than those swept out by regions of revolution.

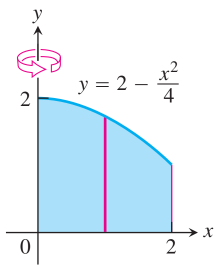
EXERCISES 6.2

In Exercises 1–6, use the shell method to find the volumes of the solids generated by revolving the shaded region about the indicated axis.

1.



2.



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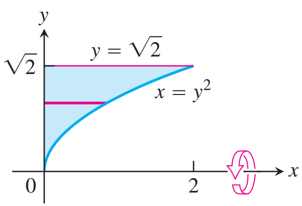

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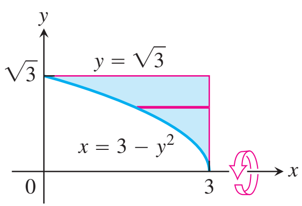
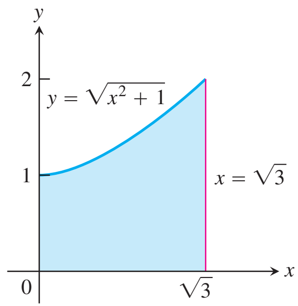
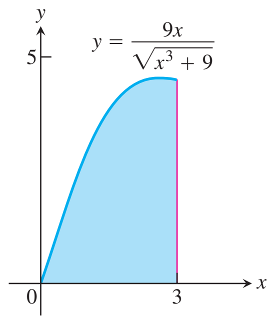


5. The y-axis



6. The y-axis



Revolution About the y-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–14 about the y-axis.

7. $y = x, \quad y = -x/2, \quad x = 2$

8. $y = 2x, \quad y = x/2, \quad x = 1$

9. $y = x^2, \quad y = 2 - x, \quad x = 0$, for $x \geq 0$

10. $y = 2 - x^2, \quad y = x^2, \quad x = 0$

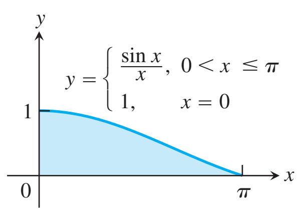
11. $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$

12. $y = 3/(2\sqrt{x}),\quad y = 0,\quad x = 1,\quad x = 4$

13. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \leq \pi \\ 1, & x = 0 \end{cases}$

a. Show that $x f(x) = \sin x$, $0 \leq x \leq \pi$.

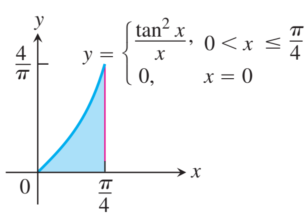
b. Find the volume of the solid generated by revolving the shaded region about the y-axis.



14. Let $g(x) = \begin{cases} (\tan x)^2/x, & 0 < x \leq \pi/4 \\ 0, & x = 0 \end{cases}$

a. Show that $x g(x) = (\tan x)^2$, $0 \leq x \leq \pi/4$.

b. Find the volume of the solid generated by revolving the shaded region about the y-axis.



Revolution About the x-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x-axis.

15. $x = \sqrt{y},\quad x = -y,\quad y = 2$

16. $x = y^2,\quad x = -y,\quad y = 2,\quad y \geq 0$

17. $x = 2y - y^2,\quad x = 0$

18. $x = 2y - y^2,\quad x = y$

19. $y = |x|,\quad y = 1$

20. $y = x,\quad y = 2x,\quad y = 2$

21. $y = \sqrt{x},\quad y = 0,\quad y = x - 2$

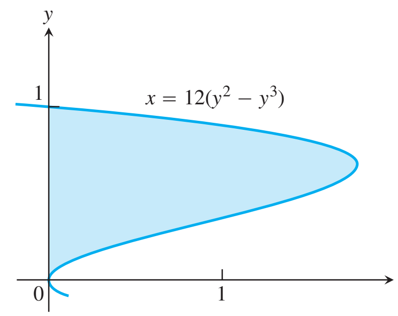
22. $y = \sqrt{x},\quad y = 0,\quad y = 2 - x$

Revolution About Horizontal Lines

In Exercises 23 and 24, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

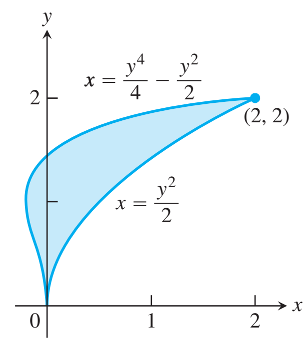
23. **a.** The x-axis   **b.** The line $y = 1$

c. The line $y = 8/5$ **d.** The line $y = -2/5$



24. **a.** The x-axis   **b.** The line $y = 2$

c. The line $y = 5$ **d.** The line $y = -5/8$



Comparing the Washer and Shell Models

For some regions, both the washer and shell methods work well for the solid generated by revolving the region about the coordinate axes, but this is not always the case. When a region is revolved about the y-axis, for example, and washers are used, we must integrate with respect to y. It may not be possible, however, to express the integrand in terms of y. In such a case, the shell method allows us to integrate with respect to x instead. Exercises 25 and 26 provide some insight.

25. Compute the volume of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about each coordinate axis using

a. the shell method.   **b.** the washer method.

26. Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4$, $y = x$, and $x = 0$ about

a. the x-axis using the washer method.

b. the y-axis using the shell method.

c. the line $x = 4$ using the shell method.

d. the line $y = 8$ using the washer method.

Choosing Shells or Washers

In Exercises 27–32, find the volumes of the solids generated by revolving the regions about the given axes. If you think it would be better to use washers in any given instance, feel free to do so.

27. The triangle with vertices (1, 1), (1, 2), and (2, 2) about

a. the x-axis **b.** the y-axis

c. the line $x = 10/3$ **d.** the line $y = 1$

28. The region bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$ about

a. the x-axis **b.** the y-axis

c. the line $x = 4$ **d.** the line $y = 2$

29. The region in the first quadrant bounded by the curve $x = y - y^3$ and the y-axis about

a. the x-axis **b.** the line $y = 1$

30. The region in the first quadrant bounded by $x = y - y^3$, $x = 1$, and $y = 1$ about

a. the x-axis **b.** the y-axis

c. the line $x = 1$ **d.** the line $y = 1$

31. The region bounded by $y = \sqrt{x}$ and $y = x^2/8$ about

a. the x-axis **b.** the y-axis

32. The region bounded by $y = 2x - x^2$ and $y = x$ about

a. the y-axis **b.** the line $x = 1$

33. The region in the first quadrant that is bounded above by the curve $y = 1/x^{1/4}$, on the left by the line $x = 1/16$, and below by the line $y = 1$ is revolved about the x-axis to generate a solid. Find the volume of the solid by

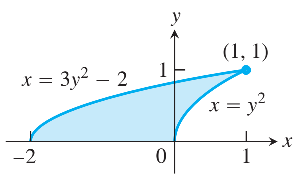
a. the washer method. **b.** the shell method.

34. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by

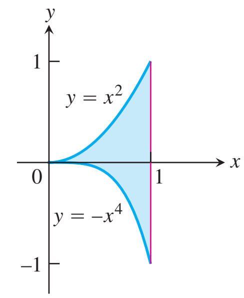
a. the washer method. **b.** the shell method.

Choosing Disks, Washers, or Shells

35. The region shown here is to be revolved about the x-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Explain.



36. The region shown here is to be revolved about the y-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Give reasons for your answers.



6.3 Lengths of Plane Curves

HISTORICAL BIOGRAPHY

Archimedes
(287–212 B.C.)

We know what is meant by the length of a straight line segment, but without calculus, we have no precise notion of the length of a general winding curve. The idea of approximating the length of a curve running from point A to point B by subdividing the curve into many pieces and joining successive points of division by straight line segments dates back to the ancient Greeks. Archimedes used this method to approximate the circumference of a circle by inscribing a polygon of n sides and then using geometry to compute its perimeter

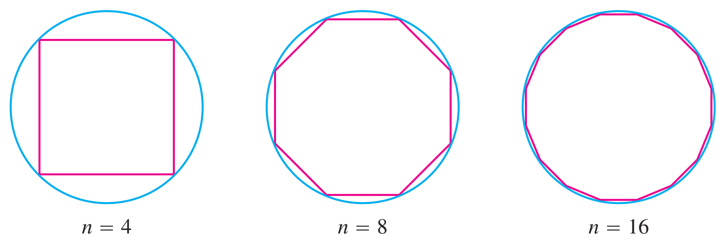


FIGURE 6.23 Archimedes used the perimeters of inscribed polygons to approximate the circumference of a circle. For $n = 96$ the approximation method gives $\pi \approx 3.14103$ as the circumference of the unit circle.

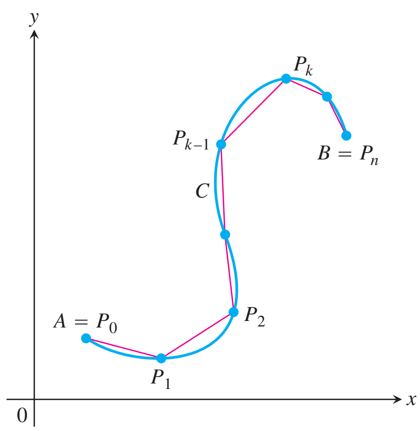


FIGURE 6.24 The curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1, and so on, ending at $B = P_n$.

(Figure 6.23). The extension of this idea to a more general curve is displayed in Figure 6.24, and we now describe how that method works.

Length of a Parametrically Defined Curve

Let C be a curve given parametrically by the equations

$$x = f(t) \quad \text{and} \quad y = g(t), \quad a \le t \le b.$$

We assume the functions f and g have continuous derivatives on the interval $[a, b]$ that are not simultaneously zero. Such functions are said to be **continuously differentiable**, and the curve C defined by them is called a **smooth curve**. It may be helpful to imagine the curve as the path of a particle moving from point $A = (f(a), g(a))$ at time $t = a$ to point $B = (f(b), g(b))$ in Figure 6.24. We subdivide the path (or arc) AB into n pieces at points $A = P_0, P_1, P_2, \ldots, P_n = B$. These points correspond to a partition of the interval $[a, b]$ by $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, where $P_k = (f(t_k), g(t_k))$. Join successive points of this subdivision by straight line segments (Figure 6.24). A representative line segment has length

$$\begin{aligned} L_k &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2} \end{aligned}$$

(see Figure 6.25). If Δt_k is small, the length L_k is approximately the length of arc $P_{k-1}P_k$. By the Mean Value Theorem there are numbers t_k^* and t_k^{**} in $[t_{k-1}, t_k]$ such that

$$\begin{aligned} \Delta x_k &= f(t_k) - f(t_{k-1}) = f'(t_k^*)\,\Delta t_k, \\ \Delta y_k &= g(t_k) - g(t_{k-1}) = g'(t_k^{**})\,\Delta t_k. \end{aligned}$$

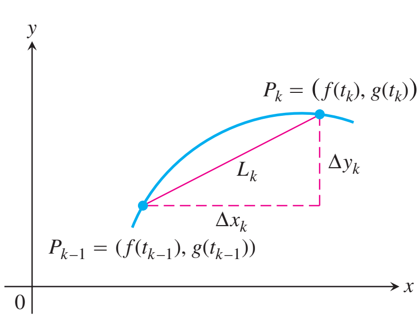


FIGURE 6.25 The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

Assuming the path from A to B is traversed exactly once as t increases from $t = a$ to $t = b$, with no doubling back or retracing, an intuitive approximation to the "length" of the curve AB is the sum of all the lengths L_k:

$$\begin{aligned} \sum_{k=1}^{n} L_k &= \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\,\Delta t_k. \end{aligned}$$

Although this last sum on the right is not exactly a Riemann sum (because f' and g' are evaluated at different points), a theorem in advanced calculus guarantees its limit, as the norm of the partition tends to zero, to be the definite integral

$$\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Therefore, it is reasonable to define the length of the curve from A to B as this integral.

DEFINITION Length of a Parametric Curve

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

A smooth curve C does not double back or reverse the direction of motion over the time interval $[a, b]$ since $(f')^2 + (g')^2 > 0$ throughout the interval.

If $x = f(t)$ and $y = g(t)$, then using the Leibniz notation we have the following result for arc length:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{1}$$

What if there are two different parametrizations for a curve C whose length we want to find; does it matter which one we use? The answer, from advanced calculus, is no, as long as the parametrization we choose meets the conditions stated in the definition of the length of C (see Exercise 29).

EXAMPLE 1 The Circumference of a Circle

You Try It

Find the length of the circle of radius r defined parametrically by

$$x = r\cos t \quad \text{and} \quad y = r\sin t, \quad 0 \le t \le 2\pi.$$

Solution As t varies from 0 to 2π, the circle is traversed exactly once, so the circumference is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We find

$$\frac{dx}{dt} = -r\sin t, \quad \frac{dy}{dt} = r\cos t$$

and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2.$$

So

$$L = \int_0^{2\pi} \sqrt{r^2}\, dt = r\Big[t\Big]_0^{2\pi} = 2\pi r. \qquad \blacksquare$$

EXAMPLE 2 Applying the Parametric Formula for Length of a Curve

Find the length of the astroid (Figure 6.26)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

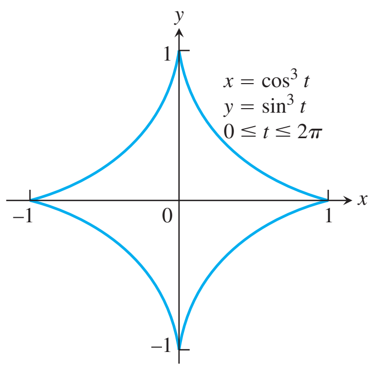


FIGURE 6.26 The astroid in Example 2.

Solution Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first-quadrant portion. We have

$$x = \cos^3 t, \qquad y = \sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t(-\sin t)]^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t(\cos t)]^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^2 t \sin^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}}$$

$$= \sqrt{9\cos^2 t \sin^2 t}$$

$$= 3|\cos t \sin t|$$

$$= 3\cos t \sin t. \qquad \cos t \sin t \ge 0 \text{ for } 0 \le t \le \pi/2$$

Therefore,

$$\text{Length of first-quadrant portion} = \int_0^{\pi/2} 3\cos t \sin t\, dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t\, dt \qquad \cos t \sin t = (1/2)\sin 2t$$

$$= -\frac{3}{4}\cos 2t\Big]_0^{\pi/2} = \frac{3}{2}.$$

The length of the astroid is four times this: $4(3/2) = 6$. ■

Historical Biography

Gregory St. Vincent
(1584–1667)

Length of a Curve $y = f(x)$

Given a continuously differentiable function $y = f(x)$, $a \le x \le b$, we can assign $x = t$ as a parameter. The graph of the function f is then the curve C defined parametrically by

$$x = t \quad \text{and} \quad y = f(t), \qquad a \le t \le b,$$

a special case of what we considered before. Then,

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = f'(t).$$

From our calculations in Section 3.5, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = f'(t)$$

giving

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 1 + [f'(t)]^2 \\ &= 1 + \left(\frac{dy}{dx}\right)^2 \\ &= 1 + [f'(x)]^2.\end{aligned}$$

Substitution into Equation (1) gives the arc length formula for the graph of $y = f(x)$.

Formula for the Length of $y = f(x)$, $a \le x \le b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (2)$$

EXAMPLE 3 Applying the Arc Length Formula for a Graph

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution We use Equation (2) with $a = 0$, $b = 1$, and

$$\begin{aligned} y &= \frac{4\sqrt{2}}{3} x^{3/2} - 1 \\ \frac{dy}{dx} &= \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2} = 2\sqrt{2}x^{1/2} \\ \left(\frac{dy}{dx}\right)^2 &= \left(2\sqrt{2}x^{1/2}\right)^2 = 8x. \end{aligned}$$

The length of the curve from $x = 0$ to $x = 1$ is

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx \qquad \text{Eq. (2) with } a = 0,\ b = 1 \\ &= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6}. \qquad \text{Let } u = 1 + 8x\text{, integrate, and replace } u \text{ by } 1 + 8x. \end{aligned}$$

■

Dealing with Discontinuities in dy/dx

At a point on a curve where dy/dx fails to exist, dx/dy may exist and we may be able to find the curve's length by expressing x as a function of y and applying the following analogue of Equation (2):

You Try It

Formula for the Length of $x = g(y)$, $c \leq y \leq d$
If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \qquad (3)$$

EXAMPLE 4 Length of a Graph Which Has a Discontinuity in dy/dx

Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (2).

We therefore rewrite the equation to express x in terms of y:

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2} \qquad \text{Raise both sides to the power 3/2.}$$

$$x = 2y^{3/2}. \qquad \text{Solve for } x.$$

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Figure 6.27).

The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$. We may therefore use Equation (3) to find the curve's length:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy \qquad \text{Eq. (3) with } c = 0, d = 1.$$

$$= \frac{1}{9} \cdot \frac{2}{3}(1 + 9y)^{3/2}\Big]_0^1 \qquad \text{Let } u = 1 + 9y,\ du/9 = dy, \text{ integrate, and substitute back.}$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.27.$$

■

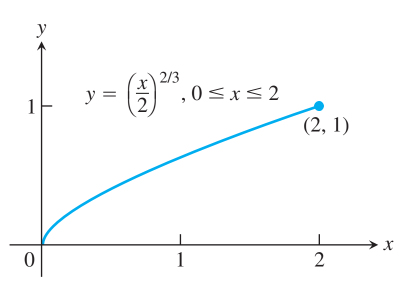


FIGURE 6.27 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 4).

Historical Biography

James Gregory
(1638–1675)

Biography

The Short Differential Formula

Equation (1) is frequently written in terms of differentials in place of derivatives. This is done formally by writing $(dt)^2$ under the radical in place of the dt outside the radical, and then writing

$$\left(\frac{dx}{dt}\right)^2 (dt)^2 = \left(\frac{dx}{dt}\,dt\right)^2 = (dx)^2$$

and

$$\left(\frac{dy}{dt}\right)^2 (dt)^2 = \left(\frac{dy}{dt}\,dt\right)^2 = (dy)^2.$$

It is also customary to eliminate the parentheses in $(dx)^2$ and write dx^2 instead, so that Equation (1) is written

$$L = \int \sqrt{dx^2 + dy^2}. \tag{4}$$

We can think of these differentials as a way to summarize and simplify the properties of integrals. Differentials are given a precise mathematical definition in a more advanced text.

To do an integral computation, dx and dy must both be expressed in terms of one and the same variable, and appropriate limits must be supplied in Equation (4).

A useful way to remember Equation (4) is to write

$$ds = \sqrt{dx^2 + dy^2} \tag{5}$$

and treat ds as the differential of arc length, which can be integrated between appropriate limits to give the total length of a curve. Figure 6.28a gives the exact interpretation of ds corresponding to Equation (5). Figure 6.28b is not strictly accurate but is to be thought of as a simplified approximation of Figure 6.28a.

With Equation (5) in mind, the quickest way to recall the formulas for arc length is to remember the equation

$$\text{Arc length} = \int ds.$$

If we write $L = \int ds$ and have the graph of $y = f(x)$, we can rewrite Equation (5) to get

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + \frac{dy^2}{dx^2}\,dx^2} = \sqrt{1 + \frac{dy^2}{dx^2}}\,dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx,$$

resulting in Equation (2). If we have instead $x = g(y)$, we rewrite Equation (5)

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dy^2 + \frac{dx^2}{dy^2}\,dy^2} = \sqrt{1 + \frac{dx^2}{dy^2}}\,dy = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy,$$

and obtain Equation (3).

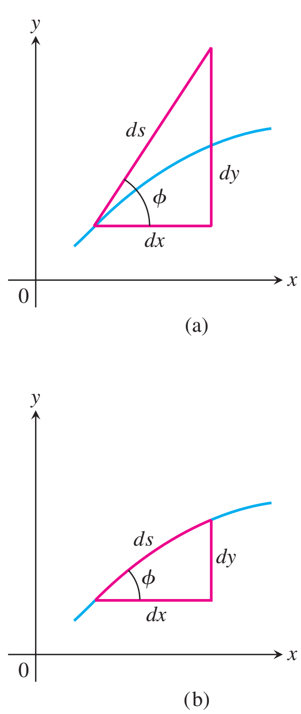


FIGURE 6.28 Diagrams for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.

EXERCISES 6.3

Lengths of Parametrized Curves

Find the lengths of the curves in Exercises 1–6.

1. $x = 1 - t, \quad y = 2 + 3t, \quad -2/3 \le t \le 1$

2. $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$

3. $x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}$

4. $x = t^2/2, \quad y = (2t + 1)^{3/2}/3, \quad 0 \le t \le 4$

5. $x = (2t + 3)^{3/2}/3, \quad y = t + t^2/2, \quad 0 \le t \le 3$

6. $x = 8\cos t + 8t\sin t, \quad y = 8\sin t - 8t\cos t, \quad 0 \le t \le \pi/2$

Finding Lengths of Curves

Find the lengths of the curves in Exercises 7–16. If you have a grapher, you may want to graph these curves to see what they look like.

7. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$

8. $y = x^{3/2}$ from $x = 0$ to $x = 4$

9. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

10. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

11. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

12. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

13. $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5, \quad 1 \le x \le 8$

14. $y = (x^3/3) + x^2 + x + 1/(4x + 4), \quad 0 \le x \le 2$

15. $x = \displaystyle\int_0^y \sqrt{\sec^4 t - 1}\,dt, \quad -\pi/4 \le y \le \pi/4$

16. $y = \displaystyle\int_{-2}^x \sqrt{3t^4 - 1}\,dt, \quad -2 \le x \le -1$

T Finding Integrals for Lengths of Curves

In Exercises 17–24, do the following.

a. Set up an integral for the length of the curve.

b. Graph the curve to see what it looks like.

c. Use your grapher's or computer's integral evaluator to find the curve's length numerically.

17. $y = x^2, \quad -1 \le x \le 2$

18. $y = \tan x, \quad -\pi/3 \le x \le 0$

19. $x = \sin y, \quad 0 \le y \le \pi$

20. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$

21. $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$

22. $y = \sin x - x\cos x, \quad 0 \le x \le \pi$

23. $y = \displaystyle\int_0^x \tan t\,dt, \quad 0 \le x \le \pi/6$

24. $x = \displaystyle\int_0^y \sqrt{\sec^2 t - 1}\,dt, \quad -\pi/3 \le y \le \pi/4$

Theory and Applications

25. Is there a smooth (continuously differentiable) curve $y = f(x)$ whose length over the interval $0 \le x \le a$ is always $\sqrt{2}a$? Give reasons for your answer.

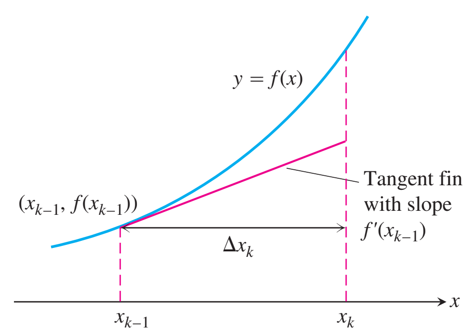
26. Using tangent fins to derive the length formula for curves Assume that f is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$, construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$, as shown in the accompanying figure.

a. Show that the length of the kth tangent fin over the interval $[x_{k-1}, x_k]$ equals $\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\,\Delta x_k)^2}$.

b. Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2}\,dx,$$

which is the length L of the curve $y = f(x)$ from a to b.



27. a. Find a curve through the point $(1, 1)$ whose length integral is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\,dx.$$

b. How many such curves are there? Give reasons for your answer.

28. a. Find a curve through the point $(0, 1)$ whose length integral is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\,dy.$$

b. How many such curves are there? Give reasons for your answer.

29. Length is independent of parametrization To illustrate the fact that the numbers we get for length do not depend on the way