Special Determinants and Matrices and Their Use in Economics

12.1 THE JACOBIAN

Section 11.1 showed how to test for linear dependence through the use of a simple determinant. In contrast, a *Jacobian determinant* permits testing for functional dependence, both linear *and* nonlinear. A Jacobian determinant $|J|$ is composed of all the first-order partial derivatives of a system of equations, arranged in ordered sequence. Given

$$y_1 = f_1(x_1, x_2, x_3)$$
$$y_2 = f_2(x_1, x_2, x_3)$$
$$y_3 = f_3(x_1, x_2, x_3)$$

$$|J| = \left| \frac{\partial y_1, \partial y_2, \partial y_3}{\partial x_1, \partial x_2, \partial x_3} \right| = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \dfrac{\partial y_2}{\partial x_3} \\ \dfrac{\partial y_3}{\partial x_1} & \dfrac{\partial y_3}{\partial x_2} & \dfrac{\partial y_3}{\partial x_3} \end{vmatrix}$$

Notice that the elements of each row are the partial derivatives of one function y_i with respect to each of the independent variables x_1, x_2, x_3, and the elements of each column are the partial derivatives of each of the functions y_1, y_2, y_3 with respect to one of the independent variables x_j. If $|J| = 0$, the equations are functionally dependent; if $|J| \neq 0$, the equations are functionally independent. See Example 1 and Problems 12.1 to 12.4.

EXAMPLE 1. Use of the Jacobian to test for functional dependence is demonstrated below, given

$$y_1 = 5x_1 + 3x_2$$
$$y_2 = 25x_1^2 + 30x_1x_2 + 9x_2^2$$

First, take the first-order partials,

$$\frac{\partial y_1}{\partial x_1} = 5 \qquad \frac{\partial y_1}{\partial x_2} = 3 \qquad \frac{\partial y_2}{\partial x_1} = 50x_1 + 30x_2 \qquad \frac{\partial y_2}{\partial x_2} = 30x_1 + 18x_2$$

Then set up the Jacobian,

$$|J| = \begin{vmatrix} 5 & 3 \\ 50x_1 + 30x_2 & 30x_1 + 18x_2 \end{vmatrix}$$

and evaluate,

$$|J| = 5(30x_1 + 18x_2) - 3(50x_1 + 30x_2) = 0$$

Since $|J| = 0$, there is functional dependence between the equations. In this, the simplest of cases, $(5x_1 + 3x_2)^2 = 25x_1^2 + 30x_1x_2 + 9x_2^2$.

12.2 THE HESSIAN

Given that the first-order conditions $z_x = z_y = 0$ are met, a sufficient condition for a multivariable function $z = f(x, y)$ to be at an optimum is

(1)
$$z_{xx}, z_{yy} > 0 \quad \text{for a minimum}$$
$$z_{xx}, z_{yy} < 0 \quad \text{for a maximum}$$

(2)
$$z_{xx} z_{yy} > (z_{xy})^2$$

See Section 5.4. A convenient test for this second-order condition is the Hessian. A *Hessian* $|H|$ is a determinant composed of all the second-order partial derivatives, with the second-order direct partials on the principal diagonal and the second-order cross partials off the principal diagonal. Thus,

$$|H| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix}$$

where $z_{xy} = z_{yx}$. If the first element on the principal diagonal, the *first principal minor*, $|H_1| = z_{xx}$ is positive and the *second principal minor*

$$|H_2| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{vmatrix} = z_{xx} z_{yy} - (z_{xy})^2 > 0$$

the second-order conditions for a minimum are met. When $|H_1| > 0$ and $|H_2| > 0$, the Hessian $|H|$ is called *positive definite*. A positive definite Hessian fulfills the second-order conditions for a minimum.

If the first principal minor $|H_1| = z_{xx} < 0$ and the second principal minor

$$|H_2| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{vmatrix} > 0$$

the second-order conditions for a maximum are met. When $|H_1| < 0$, $|H_2| > 0$, the Hessian $|H|$ is *negative definite*. A negative definite Hessian fulfills the second-order conditions for a maximum. See Example 2 and Problems 12.10 to 12.13.

EXAMPLE 2. In Problem 5.10(*a*) it was found that

$$z = 3x^2 - xy + 2y^2 - 4x - 7y + 12$$

is optimized at $x_0 = 1$ and $y_0 = 2$. The second partials were $z_{xx} = 6$, $z_{yy} = 4$, and $z_{xy} = -1$. Using the Hessian to test the second-order conditions,

$$|H| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -1 \\ -1 & 4 \end{vmatrix}$$

Taking the principal minors, $|H_1| = 6 > 0$ and

$$|H_2| = \begin{vmatrix} 6 & -1 \\ -1 & 4 \end{vmatrix} = 6(4) - (-1)(-1) = 23 > 0$$

With $|H_1| > 0$ and $|H_2| > 0$, the hessian $|H|$ is positive definite, and z is minimized at the critical values.

12.3 THE DISCRIMINANT

Determinants may be used to test for positive or negative definiteness of any quadratic form. The determinant of a quadratic form is called a *discriminant* $|D|$. Given the quadratic form

$$z = ax^2 + bxy + cy^2$$

the discriminant is formed by placing the coefficients of the squared terms on the principal diagonal and dividing the coefficients of the nonsquared term equally between the off-diagonal positions. Thus,

$$|D| = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$$

Then evaluate the principal minors as in the Hessian test, where

$$|D_1| = a \qquad \text{and} \qquad |D_2| = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix} = ac - \frac{b^2}{4}$$

If $|D_1|, |D_2| > 0$, $|D|$ is positive definite and z is positive for all nonzero values of x and y. If $|D_1| < 0$ and $|D_2| > 0$, z is negative definite and z is negative for all nonzero values of x and y. See Example 3 and Problems 12.5 to 12.7.

EXAMPLE 3. To test for sign definiteness, given the quadratic form

$$z = 2x^2 + 5xy + 8y^2$$

form the discriminant as explained in Section 12.3.

$$|D| = \begin{vmatrix} 2 & 2.5 \\ 2.5 & 8 \end{vmatrix}$$

Then evaluate the principal minors as in the Hessian test.

$$|D_1| = 2 > 0 \qquad |D_2| = \begin{vmatrix} 2 & 2.5 \\ 2.5 & 8 \end{vmatrix} = 16 - 6.25 = 9.75 > 0$$

Thus, z is positive definite, meaning that it will be greater than zero for all nonzero values of x and y.

12.4 HIGHER-ORDER HESSIANS

Given $y = f(x_1, x_2, x_3)$, the third-order Hessian is

$$|H| = \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix}$$

where the elements are the various second-order partial derivatives of y:

$$y_{11} = \frac{\partial^2 y}{\partial x_1^2} \qquad y_{12} = \frac{\partial^2 y}{\partial x_2\,\partial x_1} \qquad y_{23} = \frac{\partial^2 y}{\partial x_3\,\partial x_2} \qquad \text{etc.}$$

Conditions for a relative minimum or maximum depend on the signs of the first, second, and third principal minors, respectively. If $|H_1| = y_{11} > 0$,

$$|H_2| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} > 0 \qquad \text{and} \qquad |H_3| = |H| > 0$$

where $|H_3|$ is the *third principal minor*, $|H|$ is positive definite and fulfills the second-order conditions for a minimum. If $|H_1| = y_{11} < 0$,

$$|H_2| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} > 0 \qquad \text{and} \qquad |H_3| = |H| < 0$$

$|H|$ is negative definite and will fulfill the second-order conditions for a maximum. Higher-order Hessians follow in analogous fashion. If all the principal minors of $|H|$ are positive, $|H|$ is positive definite and the second-order conditions for a relative minimum are met. If all the principal minors of $|H|$ alternate in sign between negative and positive, $|H|$ is negative definite and the second-order conditions for a relative maximum are met. See Example 4 and Problems 12.8, 12.9, and 12.14 to 12.22.

EXAMPLE 4. The function

$$y = -5x_1^2 + 10x_1 + x_1x_3 - 2x_2^2 + 4x_2 + 2x_2x_3 - 4x_3^2$$

is optimized as follows, using the Hessian to test the second-order conditions.

The first-order conditions are

$$\frac{\partial y}{\partial x_1} = y_1 = -10x_1 + 10 + x_3 = 0$$

$$\frac{\partial y}{\partial x_2} = y_2 = -4x_2 + 2x_3 + 4 = 0$$

$$\frac{\partial y}{\partial x_3} = y_3 = x_1 + 2x_2 - 8x_3 = 0$$

which can be expressed in matrix form as

$$\begin{bmatrix} -10 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \\ 0 \end{bmatrix} \qquad (12.1)$$

Using Cramer's rule (see Section 11.9) and taking the different determinants, $|A| = -10(28) + 1(4) = -276 \neq 0$. Since $|A|$ in this case is the Jacobian and does not equal zero, the three equations are functionally independent.

$$|A_1| = -10(28) + 1(-8) = -288 \qquad |A_2| = -10(32) - (-10)(-2) + 1(4) = -336$$

$$|A_3| = -10(8) - 10(4) = -120$$

Thus,
$$\bar{x}_1 = \frac{|A_1|}{|A|} = \frac{-288}{-276} \cong 1.04 \qquad \bar{x}_2 = \frac{|A_2|}{|A|} = \frac{-336}{-276} \cong 1.22 \qquad \bar{x}_3 = \frac{|A_3|}{|A|} = \frac{-120}{-276} \cong 0.43$$

Taking the second partial derivatives from the first-order conditions to prepare the Hessian,

$$\begin{matrix} y_{11} = -10 & y_{12} = 0 & y_{13} = 1 \\ y_{21} = 0 & y_{22} = -4 & y_{23} = 2 \\ y_{31} = 1 & y_{32} = 2 & y_{33} = -8 \end{matrix}$$

Thus,
$$|H| = \begin{vmatrix} -10 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -8 \end{vmatrix}$$

which has the same elements as the coefficient matrix in (12.1) since the first-order partials are all linear. Finally, applying the Hessian test, by checking the signs of the first, second, and third principal minors, respectively,

$$|H_1| = -10 < 0 \qquad |H_2| = \begin{vmatrix} -10 & 0 \\ 0 & -4 \end{vmatrix} = 40 > 0 \qquad |H_3| = |H| = |A| = -276 < 0$$

Since the principal minors alternate correctly in sign, the Hessian is negative definite and the function is maximized at $\bar{x}_1 = 1.04$, $\bar{x}_2 = 1.22$, and $\bar{x}_3 = 0.43$.

12.5 THE BORDERED HESSIAN FOR CONSTRAINED OPTIMIZATION

To optimize a function $f(x, y)$ subject to a constraint $g(x, y)$, Section 5.5 showed that a new function could be formed $F(x, y, \lambda) = f(x, y) + \lambda[k - g(x, y)]$, where the first-order conditions are $F_x = F_y = F_\lambda = 0$.

The second-order conditions can now be expressed in terms of a *bordered Hessian* $|\bar{H}|$ in either of two ways:

$$|\bar{H}| = \begin{vmatrix} F_{xx} & F_{xy} & g_x \\ F_{yx} & F_{yy} & g_y \\ g_x & g_y & 0 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 0 & g_x & g_y \\ g_x & F_{xx} & F_{xy} \\ g_y & F_{yx} & F_{yy} \end{vmatrix}$$

which is simply the plain Hessian

$$\begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix}$$

bordered by the first derivatives of the constraint with zero on the principal diagonal. The order of a bordered principal minor is determined by the order of the principal minor being bordered. Hence $|\bar{H}|$ above represents a second bordered principal minor $|\bar{H}_2|$, because the principal minor being bordered is 2×2.

For a function in n variables $f(x_1, x_2, \ldots, x_n)$, subject to $g(x_1, x_2, \ldots, x_n)$,

$$|\bar{H}| = \begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} & g_1 \\ F_{21} & F_{22} & \cdots & F_{2n} & g_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} & g_n \\ g_1 & g_2 & \cdots & g_n & 0 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & F_{11} & F_{12} & \cdots & F_{1n} \\ g_2 & F_{21} & F_{22} & \cdots & F_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & F_{n1} & F_{n2} & \cdots & F_{nn} \end{vmatrix}$$

where $|\bar{H}| = |\bar{H}_n|$, because of the $n \times n$ principal minor being bordered.

If $|\bar{H}_2|, |\bar{H}_3|, \ldots, |\bar{H}_n| < 0$, the bordered Hessian is positive definite, which is a sufficient condition for a minimum. Note that the test starts with $|\bar{H}_2|$, and *not* $|\bar{H}_1|$.

If $|\bar{H}_2| > 0$, $|\bar{H}_3| < 0$, $|\bar{H}_4| > 0$, etc., the bordered Hessian is negative definite, which is a sufficient condition for a maximum. If a given $|\bar{H}|$ meets the criteria, one is assured of a minimum or a maximum. Further tests beyond the scope of the present book are needed if the criteria are not met, since the given criteria represent sufficient conditions, and not necessary conditions. See Examples 5 and 6 and Problems 12.23 to 12.31. For a 4×4 bordered Hessian, see Problem 12.32.