2

Ancient Egypt

Sesostris . . . made a division of the soil of Egypt among the inhabitants. . . . If the river carried away any portion of a man's lot, . . . the king sent persons to examine, and determine by measurement the exact extent of the loss. . . . From this practice, I think, geometry first came to be known in Egypt, whence it passed into Greece.

Herodotus

The Era and the Sources

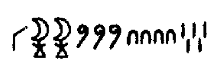
About 450 BCE, Herodotus, the inveterate Greek traveler and narrative historian, visited Egypt. He viewed ancient monuments, interviewed priests, and observed the majesty of the Nile and the achievements of those working along its banks. His resulting account would become a cornerstone for the narrative of Egypt's ancient history. When it came to mathematics, he held that geometry had originated in Egypt, for he believed that the subject had arisen there from the practical need for resurveying after the annual flooding of the river valley. A century later, the philosopher Aristotle speculated on the same subject and attributed the Egyptians' pursuit of geometry to the existence of a priestly leisure class. The debate, extending well beyond the confines of Egypt, about whether to credit progress in mathematics to the practical men (the surveyors, or "rope-stretchers") or to the contemplative elements of society (the priests and the philosophers) has continued to our times. As we shall see, the history of mathematics displays a constant interplay between these two types of contributors.

In attempting to piece together the history of mathematics in ancient Egypt, scholars until the nineteenth century encountered two major obstacles. The first was the inability to read the source materials that existed. The second was the scarcity of such materials. For more than thirty-five centuries, inscriptions used hieroglyphic writing, with variations from purely ideographic to the smoother hieratic and eventually the still more flowing demotic forms. After the third century CE, when they were replaced by Coptic and eventually supplanted by Arabic, knowledge of hieroglyphs faded. The breakthrough that enabled modern scholars to decipher the ancient texts came early in the nineteenth century when the French scholar Jean-François Champollion, working with multilingual tablets, was able to slowly translate a number of hieroglyphs. These studies were supplemented by those of other scholars, including the British physicist Thomas Young, who were intrigued by the Rosetta Stone, a trilingual basalt slab with inscriptions in hieroglyphic, demotic, and Greek writings that had been found by members of Napoleon's Egyptian expedition in 1799. By 1822, Champollion was able to announce a substantive portion of his translations in a famous letter sent to the Academy of Sciences in Paris, and by the time of his death in 1832, he had published a grammar textbook and the beginning of a dictionary.

Although these early studies of hieroglyphic texts shed some light on Egyptian numeration, they still produced no purely mathematical materials. This situation changed in the second half of the nineteenth century. In 1858, the Scottish antiquary Henry Rhind purchased a papyrus roll in Luxor that is about one foot high and some eighteen feet long. Except for a few fragments in the Brooklyn Museum, this papyrus is now in the British Museum. It is known as the Rhind or the Ahmes Papyrus, in honor of the scribe by whose hand it had been copied in about 1650 BCE. The scribe tells us that the material is derived from a prototype from the Middle Kingdom of about 2000 to 1800 BCE. Written in the hieratic script, it became the major source of our knowledge of ancient Egyptian mathematics. Another important papyrus, known as the Golenishchev or Moscow Papyrus, was purchased in 1893 and is now in the Pushkin Museum of Fine Arts in Moscow. It, too, is about eighteen feet long but is only one-fourth as wide as the Ahmes Papyrus. It was written less carefully than the work of Ahmes was, by an unknown scribe of circa. 1890 BCE. It contains twenty-five examples, mostly from practical life and not differing greatly from those of Ahmes, except for two that will be discussed further on. Yet another twelfth-dynasty papyrus, the Kahun, is now in London; a Berlin papyrus is of the same period. Other, somewhat earlier, materials are two wooden tablets from Akhmim of about 2000 BCE and a leather roll containing a list of fractions. Most of this material was deciphered within a hundred years of Champollion's death. There is a striking degree of coincidence between certain aspects of the earliest known inscriptions and the few mathematical texts of the Middle Kingdom that constitute our known source material.

Numbers and Fractions

Once Champollion and his contemporaries could decipher inscriptions on tombs and monuments, Egyptian hieroglyphic numeration was easily disclosed. The system, at least as old as the pyramids, dating some 5,000 years ago, was based on the 10 scale. By the use of a simple iterative scheme and of distinctive symbols for each of the first half-dozen powers of 10, numbers greater than a million were carved on stone, wood, and other materials. A single vertical stroke represented a unit, an inverted wicket was used for 10, a snare somewhat resembling a capital C stood for 100, a lotus flower for 1,000, a bent finger for 10,000, a tadpole for 100,000, and a kneeling figure, apparently Heh, the god of the Unending, for 1,000,000. Through repetition of these symbols, the number 12,345, for example, would appear as



Sometimes the smaller digits were placed on the left, and other times the digits were arranged vertically. The symbols themselves were occasionally reversed in orientation, so that the snare might be convex toward either the right or the left.

Egyptian inscriptions indicate familiarity with large numbers at an early date. A museum at Oxford has a royal mace more than 5,000 years old, on which a record of 120,000 prisoners and 1,422,000 captive goats appears. These figures may have been exaggerated, but from other considerations it is clear that the Egyptians were commendably accurate in counting and measuring. The construction of the Egyptian solar calendar is an outstanding early example of observation, measurement, and counting. The pyramids are another famous instance; they exhibit such a high degree of precision in construction and orientation that ill-founded legends have grown up around them.

The more cursive hieratic script used by Ahmes was suitably adapted to the use of pen and ink on prepared papyrus leaves. Numeration remained decimal, but the tedious repetitive principle of hieroglyphic numeration was replaced by the introduction of ciphers or special signs to represent digits and multiples of powers of 10. The number 4, for example, usually was no longer represented by four vertical strokes but by a horizontal bar, and 7 is not written as seven strokes but as a single cipher resembling a sickle. The hieroglyphic form for the number 28 was ; the hieratic form was simply . Note that the cipher for the smaller digit 8 (or two 4s) appears on the left, rather than on the right. The principle of cipherization, introduced by the Egyptians some 4,000 years ago and used in the Ahmes Papyrus, represented an important contribution to numeration, and it is one of the factors that makes our own system in use today the effective instrument that it is.

Egyptian hieroglyphic inscriptions have a special notation for unit fractions—that is, fractions with unit numerators. The reciprocal of any integer was indicated simply by placing over the notation for the integer an elongated oval sign. The fraction $\frac{1}{8}$ thus appeared as and $\frac{1}{20}$ was written as . In the hieratic notation, appearing in papyri, the elongated oval is replaced by a dot, which is placed over the cipher for the corresponding integer (or over the right-hand cipher in the case of the reciprocal of a multidigit number). In the Ahmes Papyrus, for example, the fraction $\frac{1}{8}$ appears as , and $\frac{1}{20}$ is written as . Such unit fractions were freely handled in Ahmes's day, but the general fraction seems to have been an enigma to the Egyptians. They felt comfortable with the fraction $\frac{2}{3}$, for which they had a special hieratic sign ; occasionally, they used special signs for fractions of the form $n/(n+1)$, the complements of the unit fractions. To the fraction $\frac{2}{3}$, the Egyptians assigned a special role in arithmetic processes, so that in finding one-third of a number, they first found two-thirds of it and subsequently took half of the result! They knew and used the fact that two-thirds of the unit fraction $1/p$ is the sum of the two unit fractions $1/2p$ and $1/6p$; they were also aware that double the unit fraction $1/2p$ is the unit fraction $1/p$. Yet it looks as though, apart from the fraction $\frac{2}{3}$, the Egyptians regarded the general proper rational fraction of the form m/n not as an elementary "thing" but as part of an uncompleted process. Where today we think of $\frac{3}{5}$ as a single irreducible fraction, Egyptian scribes thought of it as reducible to the sum of three unit fractions, $\frac{1}{3}$ and $\frac{1}{5}$ and $\frac{1}{15}$.

To facilitate the reduction of "mixed" proper fractions to the sum of unit fractions, the Ahmes Papyrus opens with a table expressing $2/n$ as a sum of unit fractions for all odd values of n from 5 to 101. The equivalent of $\frac{2}{5}$ is given as $\frac{1}{3}$ and $\frac{1}{15}$, $\frac{2}{11}$ is written as $\frac{1}{6}$ and $\frac{1}{66}$, and $\frac{2}{15}$ is expressed as $\frac{1}{10}$ and $\frac{1}{30}$. The last item in the table decomposes $\frac{2}{101}$ into $\frac{1}{101}$ and $\frac{1}{202}$ and $\frac{1}{303}$ and $\frac{1}{606}$. It is not clear why one form of decomposition was preferred to another of the indefinitely many that are possible. This last entry certainly exemplifies the Egyptian prepossession for halving and taking a third; it is not at all clear to us why the decomposition $2/n = 1/n + 1/2n + 1/3n + 1/2\cdot 3\cdot n$ is better than $1/n + 1/n$. Perhaps one of the objects of the $2/n$ decomposition was to arrive at unit fractions smaller than $1/n$. Certain passages indicate that the Egyptians had some appreciation of general rules and methods above and beyond the specific case at hand, and this represents an important step in the development of mathematics.

Arithmetic Operations

The $2/n$ table in the Ahmes Papyrus is followed by a short $n/10$ table for n from 1 to 9, the fractions again being expressed in terms of the favorites—unit fractions and the fraction $\frac{2}{3}$. The fraction $\frac{9}{10}$, for example, is broken into $\frac{1}{30}$ and $\frac{1}{5}$ and $\frac{2}{3}$. Ahmes had begun his work with the assurance that it would provide a "complete and thorough study of all things . . . and the knowledge of all secrets," and therefore the main portion of the material, following the $2/n$ and $n/10$ tables, consists of eighty-four widely assorted problems. The first six of these require the division of one or two or six or seven or eight or nine loaves of bread among ten men, and the scribe makes use of the $n/10$ table that he has just given. In the first problem, the scribe goes to considerable trouble to show that it is correct to give to each of the ten men one tenth of a loaf! If one man receives $\frac{1}{10}$ of a loaf, two men will receive $\frac{2}{10}$ or $\frac{1}{5}$ and four men will receive $\frac{2}{5}$ of a loaf or $\frac{1}{3} + \frac{1}{15}$ of a loaf. Hence, eight men will receive $\frac{2}{3} + \frac{2}{15}$ of a loaf or $\frac{2}{3} + \frac{1}{10} + \frac{1}{30}$ of a loaf, and eight men plus two men will receive $\frac{2}{3} + \frac{1}{5} + \frac{1}{10} + \frac{1}{30}$, or a whole loaf. Ahmes seems to have had a kind of equivalent to our least common multiple, which enabled him to complete the proof. In the division of seven loaves among ten men, the scribe might have chosen $\frac{1}{2} + \frac{1}{5}$ of a loaf for each, but the predilection for $\frac{2}{3}$ led him instead to $\frac{2}{3}$ and $\frac{1}{30}$ of a loaf for each.

The fundamental arithmetic operation in Egypt was addition, and our operations of multiplication and division were performed in Ahmes's day through successive doubling, or "duplation." Our own word "multiplication," or manifold, is, in fact, suggestive of the Egyptian process. A multiplication of, say, 69 by 19 would be performed by adding 69 to itself to obtain 138, then adding this to itself to reach 276, applying duplation again to get 552, and once more to obtain 1104, which is, of course, 16 times 69. Inasmuch as $19 = 16 + 2 + 1$, the result of multiplying 69 by 19 is $1104 + 138 + 69$, that is, 1311. Occasionally, a multiplication by 10 was also used, for this was a natural concomitant of the decimal hieroglyphic notation. Multiplication of combinations of unit fractions was also a part of Egyptian arithmetic. Problem 13 in the Ahmes Papyrus, for example, asks for the product of $\frac{1}{16} + \frac{1}{112}$ and $1 + \frac{1}{2} + \frac{1}{4}$; the result is correctly found to be $\frac{1}{8}$. For division, the duplation process is reversed, and the *divisor*, instead of the *multiplicand*, is successively doubled. That the Egyptians had developed a high degree of artistry in applying the duplation process and the unit fraction concept is apparent from the calculations in the problems of Ahmes. Problem 70 calls for the quotient when 100 is divided by $7 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$; the result,

$12 + \frac{2}{3} + \frac{1}{42} + \frac{1}{126}$, is obtained as follows. Doubling the divisor successively, we first obtain $15 + \frac{1}{2} + \frac{1}{4}$, then $31 + \frac{1}{2}$, and finally 63, which is 8 times the divisor. Moreover, $\frac{2}{3}$ of the divisor is known to be $5 + \frac{1}{4}$. Hence, the divisor when multiplied by $8 + 4 + \frac{2}{3}$ will total $99\frac{3}{4}$, which is $\frac{1}{4}$ short of the product 100 that is desired. Here a clever adjustment was made. Inasmuch as 8 times the divisor is 63, it follows that the divisor when multiplied by $\frac{2}{63}$ will produce $\frac{1}{4}$. From the $2/n$ table, one knows that $\frac{2}{63}$ is $\frac{1}{42} + \frac{1}{126}$; hence, the desired quotient is $12 + \frac{2}{3} + \frac{1}{42} + \frac{1}{126}$. Incidentally, this procedure makes use of a commutative principle in multiplication, with which the Egyptians evidently were familiar.

Many of Ahmes's problems show knowledge of manipulations of proportions equivalent to the "rule of three." Problem 72 calls for the number of loaves of bread of "strength" 45, which are equivalent to 100 loaves of "strength" 10, and the solution is given as $100 / 10 \times 45$, or 450 loaves. In bread and beer problems, the "strength," or *pesu*, is the reciprocal of the grain density, being the quotient of the number of loaves or units of volume divided by the amount of grain. Bread and beer problems are numerous in the Ahmes Papyrus. Problem 63, for example, requires the division of 700 loaves of bread among four recipients if the amounts they are to receive are in the continued proportion $\frac{2}{3} : \frac{1}{2} : \frac{1}{3} : \frac{1}{4}$. The solution is found by taking the ratio of 700 to the sum of the fractions in the proportion. In this case, the quotient of 700 divided by $1\frac{3}{4}$ is found by multiplying 700 by the reciprocal of the divisor, which is $\frac{1}{2} + \frac{1}{14}$. The result is 400; by taking $\frac{2}{3}$ and $\frac{1}{2}$ and $\frac{1}{3}$ and $\frac{1}{4}$ of this, the required shares of bread are found.

"Heap" Problems

The Egyptian problems so far described are best classified as arithmetic, but there are others that fall into a class to which the term "algebraic" is appropriately applied. These do not concern specific concrete objects, such as bread and beer, nor do they call for operations on known numbers. Instead, they require the equivalent of solutions of linear equations of the form $x + ax = b$ or $x + ax + bx = c$, where a and b and c are known and x is unknown. The unknown is referred to as "aha," or heap. Problem 24, for instance, calls for the value of heap if heap and $\frac{1}{7}$ of heap is 19. The solution given by Ahmes is not that of modern textbooks but is characteristic of a procedure now known as the "method of false position," or the "rule of false." A specific value, most likely a false one, is assumed for heap, and the operations indicated on the left-hand side of the equality sign are performed on this assumed number. The result of these operations is then compared with the result desired, and by the use of proportions the correct answer is found. In problem 24, the tentative value of the unknown is taken as 7, so that $x + \frac{1}{7}x$ is 8, instead of the desired answer, which was 19. Inasmuch as $8(2 + \frac{1}{4} + \frac{1}{8}) = 19$, one must multiply 7 by $2 + \frac{1}{4} + \frac{1}{8}$ to obtain the correct heap; Ahmes found the answer to be $16 + \frac{1}{2} + \frac{1}{8}$. Ahmes then "checked" his result by showing that if to $16 + \frac{1}{2} + \frac{1}{8}$ one adds $\frac{1}{7}$ of this (which is $2 + \frac{1}{4} + \frac{1}{8}$), one does indeed obtain 19. Here we see another significant step in the development of mathematics, for the check is a simple instance of a proof. Although the method of false position was generally used by Ahmes, there is one problem (Problem 30) in which $x + \frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x = 37$ is solved by factoring the left-hand side of the equation and dividing 37 by $1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{7}$, the result being $16 + \frac{1}{56} + \frac{1}{679} + \frac{1}{776}$.

Many of the "aha" calculations in the Rhind (Ahmes) Papyrus appear to be practice exercises for young students. Although a large proportion of them are of a practical nature, in some places the scribe seemed to have had puzzles or mathematical recreations in mind. Thus, Problem 79 cites only "seven houses, 49 cats, 343 mice, 2401 ears of spelt, 16807 hekats." It is presumed that the scribe was dealing with a problem, perhaps quite well known, where in each of seven houses there are seven cats, each of which eats seven mice, each of which would have eaten seven ears of grain, each of which would have produced seven measures of grain. The problem evidently called not for the practical answer, which would be the number of measures of grain that were saved, but for the impractical sum of the numbers of houses, cats, mice, ears of spelt, and measures of grain. This bit of fun in the Ahmes Papyrus seems to be a forerunner of our familiar nursery rhyme:

> As I was going to St. Ives,
> I met a man with seven wives;
> Every wife had seven sacks,
> Every sack had seven cats,
> Every cat had seven kits,
> Kits, cats, sacks, and wives,
> How many were going to St. Ives?

Geometric Problems

It is often said that the ancient Egyptians were familiar with the Pythagorean theorem, but there is no hint of this in the papyri that have come down to us. There are nevertheless some geometric problems in the Ahmes Papyrus. Problem 51 of Ahmes shows that the area of an isosceles triangle was found by taking half of what we would call the base and multiplying this by the altitude. Ahmes justified his method of finding the area by suggesting that the isosceles triangle can be thought of as two right triangles, one of which can be shifted in position, so that together the two triangles form a rectangle. The isosceles trapezoid is similarly handled in Problem 52, in which the larger base of a trapezoid is 6, the smaller base is 4, and the distance between them is 20. Taking $\frac{1}{2}$ of the sum of the bases, "so as to make a rectangle," Ahmes multiplied this by 20 to find the area. In transformations such as these, in which isosceles triangles and trapezoids are converted into rectangles, we may see the beginnings of a theory of congruence and the idea of proof in geometry, but there is no evidence that the Egyptians carried such work further. Instead, their geometry lacks a clear-cut distinction between relationships that are exact and those that are only approximations.

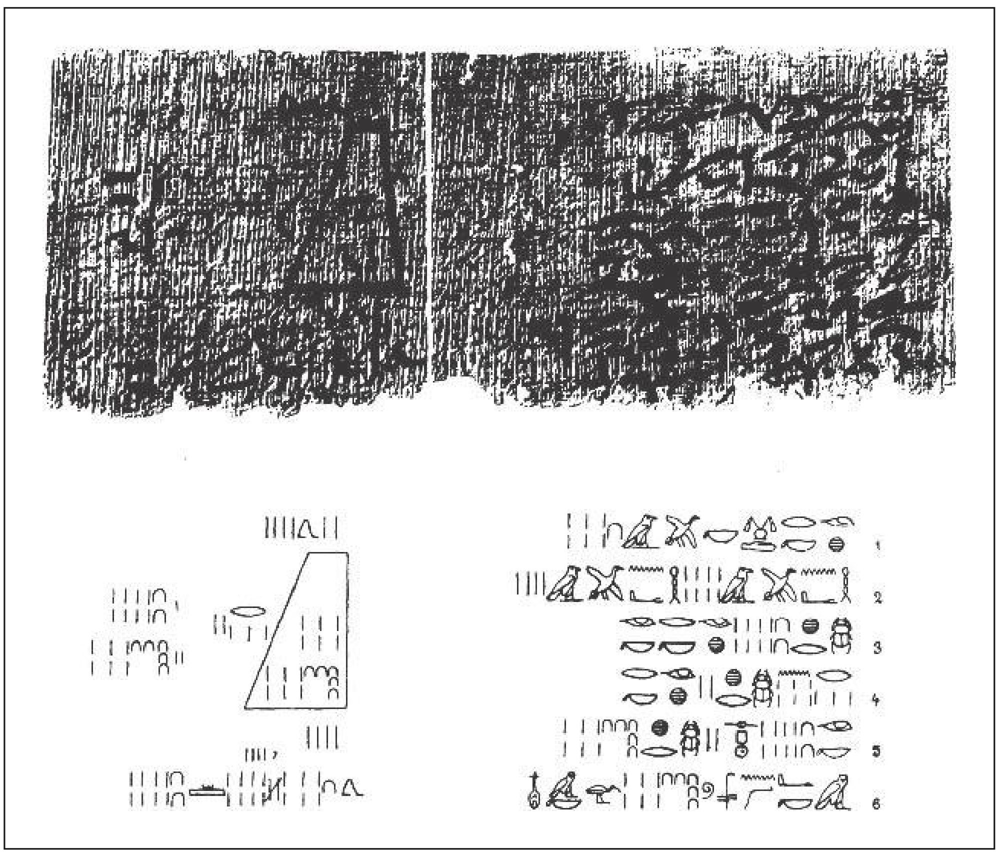
A surviving deed from Edfu, dating from a period some 1,500 years after Ahmes, gives examples of triangles, trapezoids, rectangles, and more general quadrilaterals. The rule for finding the area of the general quadrilateral is to take the product of the arithmetic means of the opposite sides. Inaccurate though the rule is, the author of the deed deduced from it a corollary—that the area of a triangle is half of the sum of two sides multiplied by half of the third side. This is a striking instance of the search for relationships among geometric figures, as well as an early use of the zero concept as a replacement for a magnitude in geometry.

The Egyptian rule for finding the area of a circle has long been regarded as one of the outstanding achievements of the time. In Problem 50, the scribe Ahmes assumed that the area of a circular field with a diameter of 9 units is the same as the area of a square with a side of 8 units. If we compare this assumption with the modern formula $A = \pi r^2$, we find the Egyptian rule to be equivalent to giving π a value of about $3\frac{1}{6}$, a commendably close approximation, but here again we miss any hint that Ahmes was aware that the areas of his circle and square were not exactly equal. It is possible that Problem 48 gives a hint to the way in which the Egyptians were led to their area of the circle. In this problem, the scribe formed an octagon from a square having sides of 9 units by trisecting the sides and cutting off the four corner isosceles triangles, each having an area of $4\frac{1}{2}$ units. The area of the octagon, which does not differ greatly from that of a circle inscribed within the square, is 63 units, which is not far removed from the area of a square with 8 units on a side. That the number $4(8/9)^2$ did indeed play a role comparable to our constant π seems to be confirmed by the Egyptian rule for the circumference of a circle, according to which the ratio of the area of a circle to the circumference is the same as the ratio of the area of the circumscribed square to its perimeter. This observation represents a geometric relationship of far greater precision and mathematical significance than the relatively good approximation for π.

Degree of accuracy in approximation is not a good measure of either mathematical or architectural achievement, and we should not overemphasize this aspect of Egyptian work. Recognition by the Egyptians of interrelationships among geometric figures, on the other hand, has too often been overlooked, and yet it is here that they came closest in attitude to their successors, the Greeks. No theorem or formal proof is known in Egyptian mathematics, but some of the geometric comparisons made in the Nile Valley, such as those on the perimeters and the areas of circles and squares, are among the first exact statements in history concerning curvilinear figures.

The value of $\frac{22}{7}$ is often used today for π; but we must recall that Ahmes's value for π is about $3\frac{1}{6}$, not $3\frac{1}{7}$. That Ahmes's value was also used by other Egyptians is confirmed in a papyrus roll from the twelfth dynasty (the Kahun Papyrus), in which the volume of a cylinder is found by multiplying the height by the area of the base, the base being determined according to Ahmes's rule.

Associated with Problem 14 in the Moscow Papyrus is a figure that looks like an isosceles trapezoid (see Fig. 2.1), but the calculations associated with it indicate that a frustum of a square pyramid is intended. Above and below the figure are signs for 2 and 4, respectively, and within the figure are the hieratic symbols for 6 and 56. The directions



Reproduction (top) of a portion of the Moscow Papyrus, showing the problem of the volume of a frustum of a square pyramid, together with hieroglyphic transcription (below)

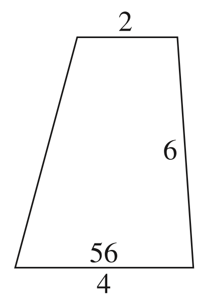


FIG. 2.1

alongside make it clear that the problem calls for the volume of a frustum of a square pyramid 6 units high if the edges of the upper and lower bases are 2 and 4 units, respectively. The scribe directs one to square the numbers 2 and 4 and to add to the sum of these squares the product of 2 and 4, the result being 28. This result is then multiplied by a third of 6, and the scribe concludes with the words "See, it is 56; you have found it correctly." That is, the volume of the frustum has been calculated in accordance with the modern formula $V = h(a^2 + ab + b^2)/3$, where h is the altitude and a and b are the sides of the square bases. Nowhere is this formula written out, but in substance it evidently was known to the Egyptians. If, as in the deed from Edfu, one takes $b = 0$, the formula reduces to the familiar formula, one-third the base times the altitude, for the volume of a pyramid.

How these results were arrived at by the Egyptians is not known. An empirical origin for the rule on the volume of a pyramid seems to be a possibility, but not for the volume of the frustum. For the latter, a theoretical basis seems more likely, and it has been suggested that the Egyptians may have proceeded here as they did in the cases of the isosceles triangle and the isosceles trapezoid—they may mentally have broken the frustum into parallelepipeds, prisms, and pyramids. On replacing the pyramids and the prisms by equal rectangular blocks, a plausible grouping of the blocks leads to the Egyptian formula. One could, for example, have begun with a pyramid having a square base and with the vertex directly over one of the base vertices. An obvious decomposition of the frustum would be to break it into four parts as in Fig. 2.2—a rectangular parallelepiped having a volume b^2h, two triangular prisms, each with a volume of $b(a - b)h/2$, and a pyramid of volume $(a - b)^2h/3$. The prisms can be combined into a rectangular parallelepiped with dimensions b and $a - b$ and h; and the pyramid can be thought of as a rectangular parallelepiped with dimensions $a - b$ and $a - b$ and $h/3$. On cutting up the tallest parallelepipeds so that all altitudes are $h/3$, one can easily arrange the slabs so as to form three layers, each of altitude $h/3$, and having cross-sectional areas of a^2 and ab and b^2, respectively.

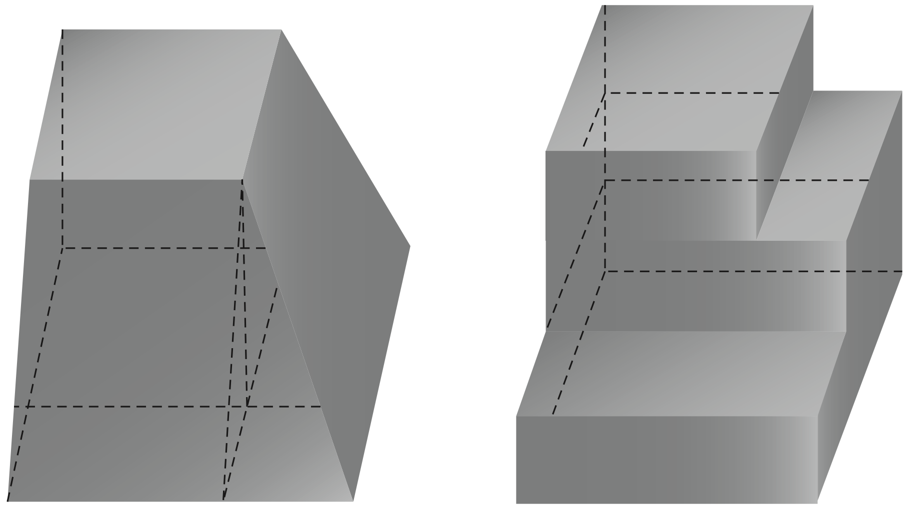


FIG. 2.2

Problem 10 in the Moscow Papyrus presents a more difficult question of interpretation than does Problem 14. Here the scribe asks for the surface area of what looks like a basket with a diameter of $4\frac{1}{2}$. He proceeds as though he were using the equivalent of a formula $S = (1 - \frac{1}{9})^2(2x) \cdot x$, where x is $4\frac{1}{2}$, obtaining an answer of 32 units. Inasmuch as $(1 - \frac{1}{9})^2$ is the Egyptian approximation of $\pi/4$, the answer 32 would correspond to the surface of a hemisphere of diameter $4\frac{1}{2}$, and this was the interpretation given to the problem in 1930. Such a result, antedating the oldest known calculation of a hemispherical surface by some 1,500 years, would have been amazing, and it seems, in fact, to have been too good to be true. Later analysis indicates that the "basket" may have been a roof—somewhat like that of a Quonset hut in the shape of a half-cylinder of diameter $4\frac{1}{2}$ and length $4\frac{1}{2}$. The calculation in this case calls for nothing beyond knowledge of the length of a semicircle, and the obscurity of the text makes it admissible to offer still more primitive interpretations, including the possibility that the calculation is only a rough estimate of the area of a domelike barn roof. In any case, we seem to have here an early estimation of a curvilinear surface area.

Slope Problems

In the construction of the pyramids, it had been essential to maintain a uniform slope for the faces, and it may have been this concern that led the Egyptians to introduce a concept equivalent to the cotangent of an angle. In modern technology, it is customary to measure the steepness of a straight line through the ratio of the "rise" to the "run." In Egypt, it was customary to use the reciprocal of this ratio. There, the word "seqt" meant the horizontal departure of an oblique line from the vertical axis for every unit change in the height. The seqt thus corresponded, except for the units of measurement, to the *batter* used today by architects to describe the inward slope of a masonry wall or pier. The vertical unit of length was the cubit, but in measuring the horizontal distance, the unit used was the "hand," of which there were seven in a cubit. Hence, the seqt of the face of a pyramid was the ratio of run to rise, the former measured in hands, the latter in cubits.

In Problem 56 of the Ahmes Papyrus, one is asked to find the seqt of a pyramid that is 250 ells or cubits high and has a square base 360 ells on a side. The scribe first divided 360 by 2 and then divided the result by 250, obtaining $\frac{1}{2} + \frac{1}{5} + \frac{1}{50}$ in ells. Multiplying the result by 7, he gave the seqt as $5\frac{1}{25}$ in hands per ell. In other pyramid problems in the Ahmes Papyrus, the seqt turns out to be $5\frac{1}{4}$, agreeing somewhat better with that of the great Cheops Pyramid, 440 ells wide and 280 high, the seqt being $5\frac{1}{2}$ hands per ell.

Arithmetic Pragmatism

The knowledge indicated in extant Egyptian papyri is mostly of a practical nature, and calculation was the chief element in the questions. Where some theoretical elements appear to enter, the purpose may have been to facilitate technique. Even the once-vaunted Egyptian geometry turns out to have been mainly a branch of applied arithmetic. Where elementary congruence relations enter, the motive seems to be to provide mensurational devices. The rules of calculation concern specific concrete cases only. The Ahmes and Moscow papyri, our two chief sources of information, may have been only manuals intended for students, but they nevertheless indicate the direction and tendencies in Egyptian mathematical instruction. Further evidence provided by inscriptions on monuments, fragments of other mathematical papyri, and documents from related scientific fields serves to confirm the general impression. It is true that our two chief mathematical papyri are from a relatively early period, a thousand years before the rise of Greek mathematics, but Egyptian mathematics seems to have remained remarkably uniform throughout its long history. It was at all stages built around the operation of addition, a disadvantage that gave to Egyptian computation a peculiar primitivity, combined with occasionally astonishing complexity.

The fertile Nile Valley has been described as the world's largest oasis in the world's largest desert. Watered by one of the most gentlemanly of rivers and geographically shielded to a great extent from foreign invasion, it was a haven for peace-loving people who pursued, to a large extent, a calm and unchallenged way of life. Love of the beneficent gods, respect for tradition, and preoccupation with death and the needs of the dead all encouraged a high degree of stagnation. Geometry may have been a gift of the Nile, as Herodotus believed, but the available evidence suggests that Egyptians used the gift but did little to expand it. The mathematics of Ahmes was that of his ancestors and of his descendants. For more progressive mathematical achievements, one must look to the more turbulent river valley known as Mesopotamia.

3

Mesopotamia

How much is one god beyond the other god?
An Old Babylonian astronomical text

The Era and the Sources

The fourth millennium before our era was a period of remarkable cultural development, bringing with it the use of writing, the wheel, and metals. As in Egypt during the first dynasty, which began toward the end of this extraordinary millennium, so also in the Mesopotamian Valley there was at the time a high order of civilization. There the Sumerians had built homes and temples decorated with artistic pottery and mosaics in geometric patterns. Powerful rulers united the local principalities into an empire that completed vast public works, such as a system of canals to irrigate the land and control flooding between the Tigris and Euphrates rivers, where the overflow of the rivers was not predictable, as was the inundation of the Nile Valley. The cuneiform pattern of writing that the Sumerians had developed during the fourth millennium probably antedates the Egyptian hieroglyphic system.

The Mesopotamian civilizations of antiquity are often referred to as Babylonian, although such a designation is not strictly correct. The city of