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Traces

> Did you bring me a man who cannot number his fingers?
> From the *Egyptian Book of the Dead*

Concepts and Relationships

Contemporary mathematicians formulate statements about abstract concepts that are subject to verification by proof. For centuries, mathematics was considered to be the science of numbers, magnitudes, and forms. For that reason, those who seek early examples of mathematical activity will point to archaeological remnants that reflect human awareness of operations on numbers, counting, or "geometric" patterns and shapes. Even when these vestiges reflect mathematical activity, they rarely evidence much historical significance. They may be interesting when they show that peoples in different parts of the world conducted certain actions dealing with concepts that have been considered mathematical. For such an action to assume historical significance, however, we look for relationships that indicate this action was known to another individual or group that engaged in a related action. Once such a connection has been established, the door is open to more specifically historical studies, such as those dealing with transmission, tradition, and conceptual change.

Mathematical vestiges are often found in the domain of nonliterate cultures, making the evaluation of their significance even more complex. Rules of operation may exist as part of an oral tradition, often in musical or verse form, or they may be clad in the language of magic or ritual. Sometimes they are found in observations of animal behavior, removing them even further from the realm of the historian. While studies of canine arithmetic or avian geometry belong to the zoologist, of the impact of brain lesions on number sense to the neurologist, and of numerical healing incantations to the anthropologist, all of these studies may prove to be useful to the historian of mathematics without being an overt part of that history.

At first, the notions of number, magnitude, and form may have been related to contrasts rather than likenesses—the difference between one wolf and many, the inequality in size of a minnow and a whale, the unlikeness of the roundness of the moon and the straightness of a pine tree. Gradually, there may have arisen, out of the welter of chaotic experiences, the realization that there are samenesses, and from this awareness of similarities in number and form both science and mathematics were born. The differences themselves seem to point to likenesses, for the contrast between one wolf and many, between one sheep and a herd, between one tree and a forest suggests that one wolf, one sheep, and one tree have something in common—their uniqueness. In the same way it would be noticed that certain other groups, such as pairs, can be put into one-to-one correspondence. The hands can be matched against the feet, the eyes, the ears, or the nostrils. This recognition of an abstract property that certain groups hold in common, and that we call "number," represents a long step toward modern mathematics. It is unlikely to have been the discovery of any one individual or any single tribe; it was more probably a gradual awareness that may have developed as early in man's cultural development as the use of fire, possibly some 300,000 years ago.

That the development of the number concept was a long and gradual process is suggested by the fact that some languages, including Greek, have preserved in their grammar a tripartite distinction between 1 and 2 and more than 2, whereas most languages today make only the dual distinction in "number" between singular and plural. Evidently, our very early ancestors at first counted only to 2, and any set beyond this level was designated as "many." Even today, many people still count objects by arranging them into sets of two each.

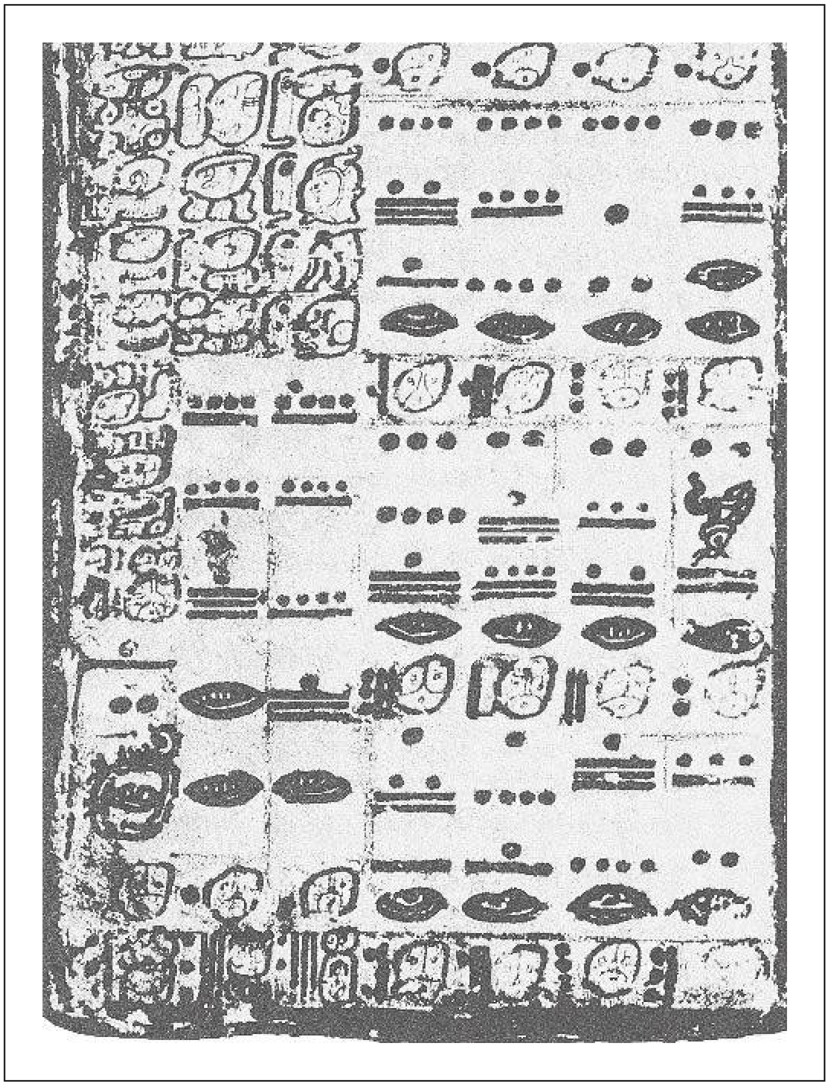
The awareness of number ultimately became sufficiently extended and vivid so that a need was felt to express the property in some way, presumably at first in sign language only. The fingers on a hand can be readily used to indicate a set of two or three or four or five objects, the number 1 generally not being recognized at first as a true "number." By the use of the fingers on both hands, collections containing up to ten elements could be represented; by combining fingers and toes, one could count as high as 20. When the human digits were inadequate, heaps of stones or knotted strings could be used to represent a correspondence with the elements of another set. Where nonliterate peoples used such a scheme of representation, they often piled the stones in groups of five, for they had become familiar with quintuples through observation of the human hand and foot. As Aristotle noted long ago, the widespread use today of the decimal system is but the result of the anatomical accident that most of us are born with ten fingers and ten toes.

Groups of stones are too ephemeral for the preservation of information; hence, prehistoric man sometimes made a number record by cutting notches in a stick or a piece of bone. Few of these records remain today, but in Moravia a bone from a young wolf was found that is deeply incised with fifty-five notches. These are arranged in two series, with twenty-five in the first and thirty in the second: within each series, the notches are arranged in groups of five. It has been dated as being approximately 30,000 years old. Two other prehistoric numerical artifacts were found in Africa: a baboon fibula having twenty-nine notches, dated as being circa 35,000 years old, and the Ishango bone, with its apparent examples of multiplicative entries, initially dated as approximately 8,000 years old but now estimated to be as much as 30,000 years old as well. Such archaeological discoveries provide evidence that the idea of number is far older than previously acknowledged.

Early Number Bases

Historically, finger counting, or the practice of counting by fives and tens, seems to have come later than counter-casting by twos and threes, yet the quinary and decimal systems almost invariably displaced the binary and ternary schemes. A study of several hundred tribes among the American Indians, for example, showed that almost one-third used a decimal base, and about another third had adopted a quinary or a quinary-decimal system; fewer than a third had a binary scheme, and those using a ternary system constituted less than 1 percent of the group. The vigesimal system, with the number 20 as a base, occurred in about 10 percent of the tribes.

An interesting example of a vigesimal system is that used by the Maya of Yucatan and Central America. This was deciphered some time before the rest of the Maya languages could be translated. In their representation of time intervals between dates in their calendar, the Maya used a place value numeration, generally with 20 as the primary base and with 5 as an auxiliary. (See the following illustration.) Units were represented by dots and fives by horizontal bars, so that the number



From the Dresden Codex of the Maya, displaying numbers. The second column on the left, reading down from above, displays the numbers 9, 9, 16, 0, 0, which stand for $9 \times 144{,}000 + 9 \times 7{,}200 + 16 \times 360 + 0 + 0 = 1{,}366{,}560$. In the third column are the numerals 9, 9, 9, 16, 0, representing 1,364,360. The original appears in black and red. (Taken from Morley 1915, p. 266.)

17, for example, would appear as  (that is, as $3(5) + 2$). A vertical positional arrangement was used, with the larger units of time above; hence, the notation  denoted 352 (that is, $17(20) + 12$). Because the system was primarily for counting days within a calendar that had 360 days in a year, the third position usually did not represent multiples of (20)(20), as in a pure vigesimal system, but (18)(20). Beyond this point, however, the base 20 again prevailed. Within this positional notation, the Maya indicated missing positions through the use of a symbol, which appeared in variant forms, somewhat resembling a half-open eye.

In their scheme, then, the notation  denoted $17(20 \cdot 18 \cdot 20) + 0(18 \cdot 20) + 13(20) + 0$.

Number Language and Counting

It is generally believed that the development of language was essential to the rise of abstract mathematical thinking. Yet words expressing numerical ideas were slow in arising. Number *signs* probably preceded number *words*, for it is easier to cut notches in a stick than it is to establish a well-modulated phrase to identify a number. Had the problem of language not been so difficult, rivals to the decimal system might have made greater headway. The base 5, for example, was one of the earliest to leave behind some tangible written evidence, but by the time that language became formalized, 10 had gained the upper hand. The modern languages of today are built almost without exception around the base 10, so that the number 13, for example, is not described as 3 and 5 and 5, but as 3 and 10. The tardiness in the development of language to cover abstractions such as number is also seen in the fact that primitive numerical verbal expressions invariably refer to specific concrete collections—such as "two fishes" or "two clubs"—and later some such phrase would be adopted conventionally to indicate all sets of two objects. The tendency for language to develop from the concrete to the abstract is seen in many of our present-day measures of length. The height of a horse is measured in "hands," and the words "foot" and "ell" (or elbow) have similarly been derived from parts of the body.

The thousands of years required for man to separate out the abstract concepts from repeated concrete situations testify to the difficulties that must have been experienced in laying even a very primitive basis for mathematics. Moreover, there are a great many unanswered questions relating to the origins of mathematics. It is usually assumed that the subject arose in answer to practical needs, but anthropological studies suggest the possibility of an alternative origin. It has been suggested that the art of counting arose in connection with primitive religious ritual and that the ordinal aspect preceded the quantitative concept. In ceremonial rites depicting creation myths, it was necessary to call the participants onto the scene in a specific order, and perhaps counting was invented to take care of this problem. If theories of the ritual origin of counting are correct, the concept of the ordinal number may have preceded that of the cardinal number. Moreover, such an origin would tend to point to the possibility that counting stemmed from a unique origin, spreading subsequently to other areas of the world. This view, although far from established, would be in harmony with the ritual division of the integers into odd and even, the former being regarded as male, the latter as female. Such distinctions were known to civilizations in all corners of the earth, and myths regarding the male and female numbers have been remarkably persistent.

The concept of the whole number is one of the oldest in mathematics, and its origin is shrouded in the mists of prehistoric antiquity. The notion of a rational fraction, however, developed relatively late and was not in general closely related to systems for the integers. Among nonliterate tribes, there seems to have been virtually no need for fractions. For quantitative needs, the practical person can choose units that are sufficiently small to obviate the necessity of using fractions. Hence, there was no orderly advance from binary to quinary to decimal fractions, and the dominance of decimal fractions is essentially the product of the modern age.

Spatial Relationships

Statements about the origins of mathematics, whether of arithmetic or geometry, are of necessity hazardous, for the beginnings of the subject are older than the art of writing. It is only during the last half-dozen millennia, in a passage that may have spanned thousands of millennia, that human beings have been able to put their records and thoughts into written form. For data about the prehistoric age, we must depend on interpretations based on the few surviving artifacts, on evidence provided by current anthropology, and on a conjectural backward extrapolation from surviving documents. Neolithic peoples may have had little leisure and little need for surveying, yet their drawings and designs suggest a concern for spatial relationships that paved the way for geometry. Pottery, weaving, and basketry show instances of congruence and symmetry, which are in essence parts of elementary geometry, and they appear on every continent. Moreover, simple sequences in design, such as that in Fig. 1.1, suggest a sort of applied group theory, as well as

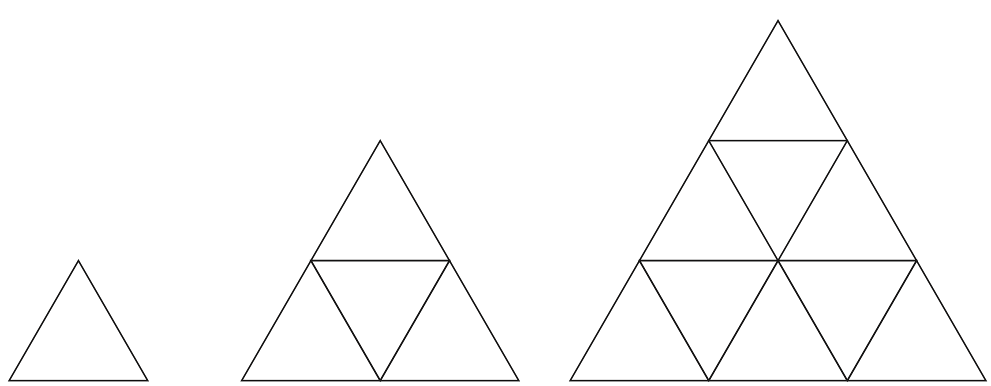


FIG. 1.1

propositions in geometry and arithmetic. The design makes it immediately obvious that the areas of triangles are to one another as squares on a side, or, through counting, that the sums of consecutive odd numbers, beginning from unity, are perfect squares. For the prehistoric period there are no documents; hence, it is impossible to trace the evolution of mathematics from a specific design to a familiar theorem. But ideas are like hardy spores, and sometimes the presumed origin of a concept may be only the reappearance of a much more ancient idea that had lain dormant.

The concern of prehistoric humans for spatial designs and relationships may have stemmed from their aesthetic feeling and the enjoyment of beauty of form, motives that often actuate the mathematician of today. We would like to think that at least some of the early geometers pursued their work for the sheer joy of doing mathematics, rather than as a practical aid in mensuration, but there are alternative theories. One of these is that geometry, like counting, had an origin in primitive ritualistic practice. Yet the theory of the origin of geometry in a secularization of ritualistic practice is by no means established. The development of geometry may just as well have been stimulated by the practical needs of construction and surveying or by an aesthetic feeling for design and order.

We can make conjectures about what led people of the Stone Age to count, to measure, and to draw. That the beginnings of mathematics are older than the oldest civilizations is clear. To go further and categorically identify a specific origin in space or time, however, is to mistake conjecture for history. It is best to suspend judgment on this matter and to move on to the safer ground of the history of mathematics as found in the written documents that have come down to us.

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Ancient Egypt

Sesostris . . . made a division of the soil of Egypt among the inhabitants. . . . If the river carried away any portion of a man's lot, . . . the king sent persons to examine, and determine by measurement the exact extent of the loss. . . . From this practice, I think, geometry first came to be known in Egypt, whence it passed into Greece.
Herodotus

The Era and the Sources

About 450 BCE, Herodotus, the inveterate Greek traveler and narrative historian, visited Egypt. He viewed ancient monuments, interviewed priests, and observed the majesty of the Nile and the achievements of those working along its banks. His resulting account would become a cornerstone for the narrative of Egypt's ancient history. When it came to mathematics, he held that geometry had originated in Egypt, for he believed that the subject had arisen there from the practical need for resurveying after the annual flooding of the river valley. A century later, the philosopher Aristotle speculated on the same subject and attributed the Egyptians' pursuit of geometry to the existence of a priestly leisure class. The debate, extending