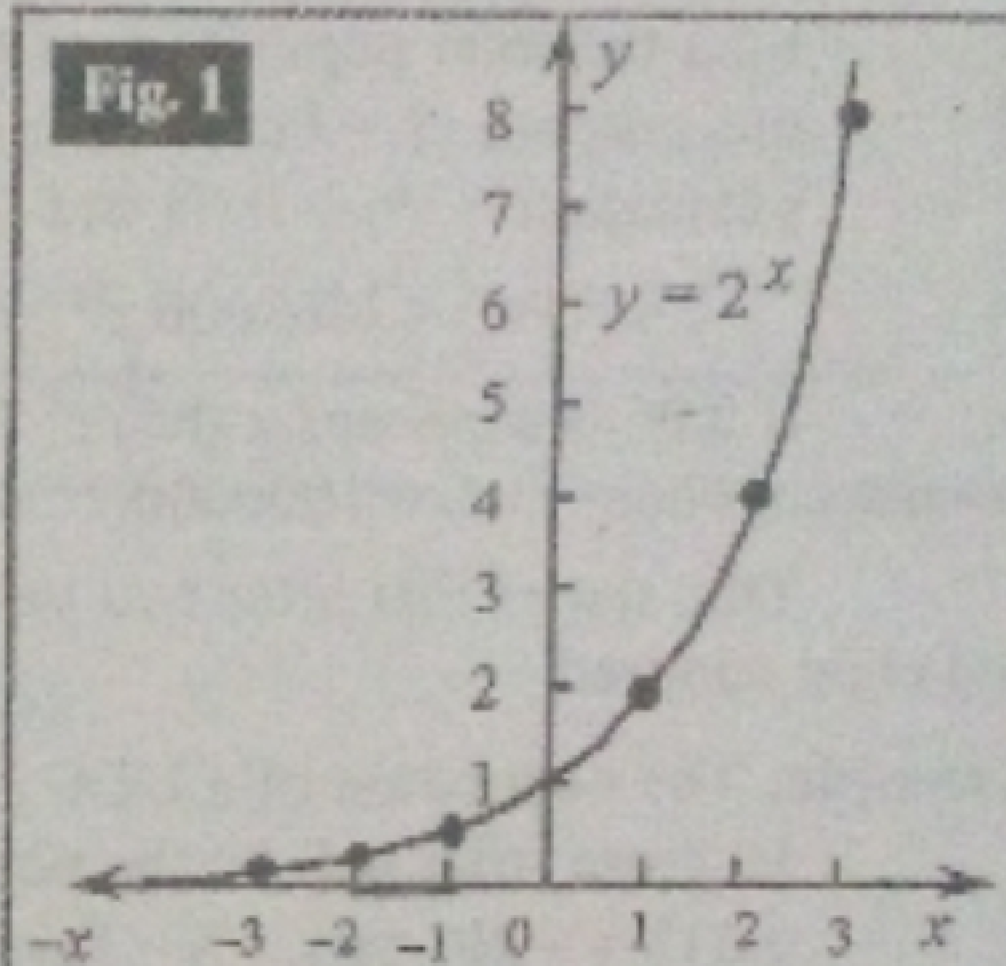
THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

EXPONENTIAL FUNCTION (UOS: 2013, 2014) (UOP: 2010S, 2011A, 2014S,2016) (BZU: 2013) (UAJK: 2013) (UOPR: 2007, 2008, 2009, 2010, 2013)(UOH: 2011)

The function whose base is some constant number and power is a variable like x is called exponential function, as: $y = b^x$ or $y = a^x$. Thus the exponential function whose base is a or b is called common Exponential function. Here $a, b > 0$. The exponential functions represent the constant rate of discontinuous growth. The growth rate which takes place at discontinuous intervals, like at the end of a year or a quarter is shown by exponential function. If we take exponential function $y = 2^x$. Supposing the values of x we can find the values of y and then graph of exponential function can be constructed as the Fig.1 shows.

Fig. 1



x	–3	–2	–1	0	1	2	3
y	1/8	1/4	1/2	1	2	4	8

$y = 2^{-1} = 1/2^1 = 1/2$

$y = 2^{-2} = 1/2^2 = 1/4$

$y = 2^{-3} = 1/2^3 = 1/8$

$y = 2^x$

$y = 2^0 = 1$

$y = 2^1 = 2$

$y = 2^2 = 4$

$y = 2^3 = 8$

> Solve the Question: Find the values if (a): \$ 10 is compounded continually at 5% for 3 years, (b): \$ 60 is compounded continually at 4% for 2 years (UOH: 2011)

The exponential functions in economics are used to show the effects of rate of interest, effects of discount rate and effects of depreciation. For example, the future value (S) of any amount (P) when compound rate of interest (i) is taken and the time period (t) is considered is shown by the following formula: $S = P(1 + i)^t$. If P = 100, t = 2 and $i = 10\%$. Putting them in formula: $S = P(1+i)^t = 100\left(1 + \frac{10}{100}\right)^2 = 100\left(\frac{110}{100} \times \frac{110}{100}\right) = 121$.

Example: In case of depreciation the growth is in negative form. We find the value of a machine after one year which is depreciated 25% annually while its price is Rs. 100000. The formula used will be as: $S = P(1 - i)^t = 100000\,(1 - 0.25)^t = 100000\,(0.75) = 75000$

There is another type of exponential function which is known as Natural Exponential Function. It has a standard form: $y = e^x$ where e is base and x is a power. e is an irrational number whose value is 2.7182. The natural exponential functions describe the constant rate of continuous growth —— the growth rate having no gaps and intervals. In economics the natural exponential functions are used to find the increase in population during certain time period. If we have the specific form of natural exponential function: $y = e^x$. Putting the supposed value of x in the above equation we can get values of y. With these values graph of natural exponential functio.. can be constructed.

x	−2	−1	0	1
y	0.1353	0.3878	1	2.78182

$$y = e^0 = 1 \quad \Rightarrow \quad y = e^1 = 2.7182$$

$$y = e^{-1} = 2.7182^{-1} = \frac{1}{(2.7182)^1} = 0.3678$$

$$y = e^{-2} = 2.7182^{-2} = \frac{1}{(2.7182)^2} = 0.1353$$

Example: The population of a city is 10 lac and it is continuously increasing at the rate of 3%. Find the total population of the city after two years. The following formula is used for this purpose. $S = Pe^{rt}$, where P = present population = 10 lac, e = base, r = growth rate of population = 3% = 0.03 and t = time = 2 years. Putting them in

$$S = Pe^{rt} = 1000000\, e^{(0.03)(2)} = 1000000\, e^{0.06}$$

If 0.06 is supposed x then $e^{0.06} = 1.0618$ (by using table of natural exponential function or calculator) Thus $S = 1000000(1.0618) = 1061800$.

This shows that after two years population of the city will go on to 10 lac, 61 thousand and eight hundred.

Example: The population of a UDC is increasing at the rate of 3.2%, if the present population is 10 crore what will be the population after 20 years. Thus P = 100000000, r = 3.2% or 0.032, t = 20 years. $S = Pert = 100000000\, e^{0.032(20)} = 100000000\, e^{0.64}$

If 0.64 is supposed to be x, then with calculator we find that $e^{0.64} = 1.8965$. Putting it: $S = 100000000(1.8965) = 189650000$

Example: If in a country the cultivable area is prey to erosion due to climatic changes at the rate of 3.5% what will be the total cultivable area after 12 years. Total cultivable area = A, r = −3.5% or r = −0.035, t = 12 years, P = the area of future. Putting them in formula.

$$P = Ae^{-rt} = Ae^{-0.035(12)} = Ae^{-0.42}$$

By supposing −0.42 as x then with calculator we find that $e^{-0.42} = 0.6570$. Putting it $P = A \times 0.6570$ OR $P = 0.66A$.

This means that in future the cultivable area (P) will decrease to 0.66 of total area (A).

LOGARITHMIC FUNCTION (UAJK: 2013) (UOP: 2010-S) (UOPR: 2008) (UOS: 2012-A)

Before we define log function the concept of log is given. 'The log of any number is a power which is given to a base to get that number'. The base of common log is 10. It is written, as $\log_{10}$ or log. Thus

Exponential form		*Log form*
$10^1 = 10$	⟶	$\text{Log}_{10} 10 = 1$
$10^2 = 100$	⟶	$\text{Log}_{10} 100 = 2$

No we define log. function, "by log function we mean a function where dependent variable (y) has been expressed in the log of independent variable (x)". y = log x. It is reminded that the log function having the base 'a' is called log function. In Economics the Log function are used in production function, total sales after a particular period, calculation of interest, finding of present value and depreciation of machinery after a specific period. **(BZU: 2013)**

If y = log x, by supposing different values of x like x = 1, 2, 3, 4, 5, we can find values of y which will be as: log 1, log 2, log 3. log 4, log 5. These log values can be found with the help of calculator. They are as:

x	1	2	3	4	5
log x	0	0.30	0.477	0.6	0.69

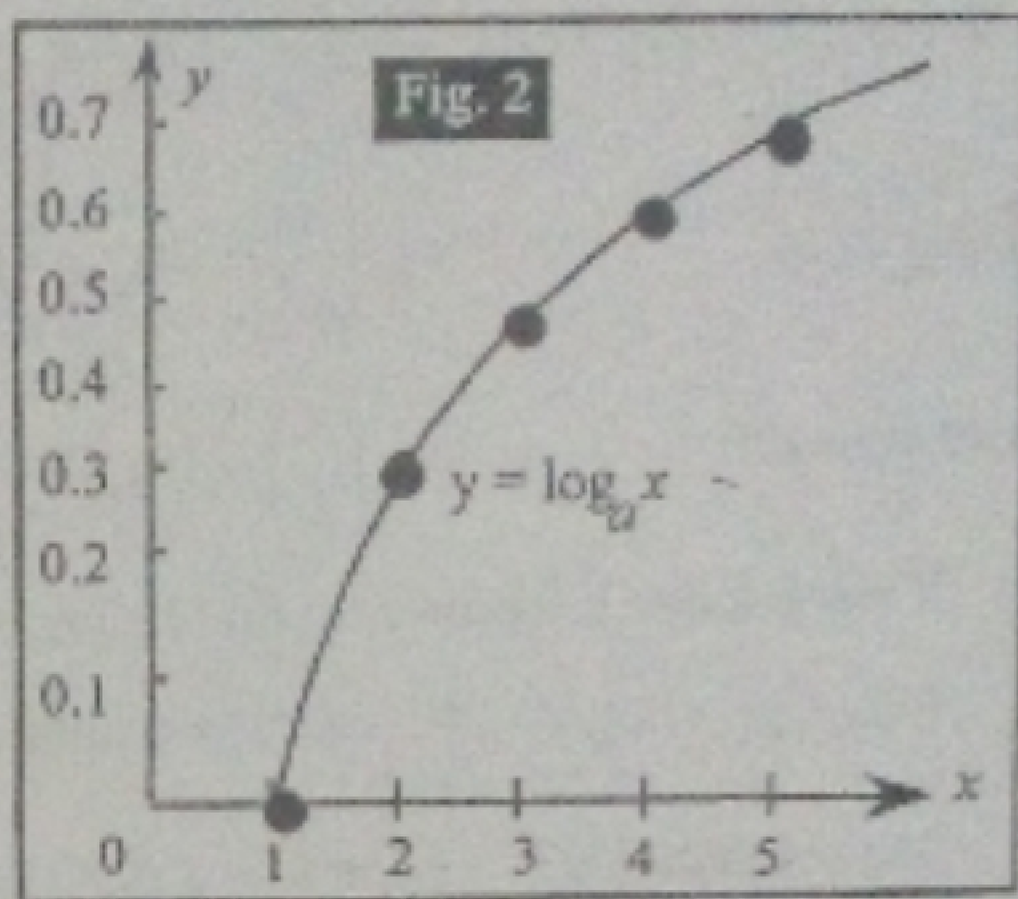


Fig. 2

With these values Fig.2 has been constructed.

RULES OF LOGARITHMS

1. If two numbers are multiplying to get their *log* their *logs* are added, as

$$\log_a (x)(y) = \log_a x + \log_a y$$

Algebraic solution of (3)(4) = 12

The log solution of product of 3 and 4.

Log (3) (4) = log 3 + log 4 ⇒ = 0.4771 + 0.6021 = 1.06792

Taking its antilog. Antilog 1.06792 = 12

Proof of Law:

$\log_a (x)(y) = \log_a x + \log_a y$

Supposing $\log_a (x) = m \Rightarrow \log_a (y) = n$

Giving them exponential form $\log_a (x) = m \longrightarrow a^m = x \Rightarrow \log_a (y) = n \longrightarrow a^n = y$

As x and y are multiplying their values will also multiply. $(x)(y) = (a^m)(a^n)$

As the bases are same their powers will be added.

$(x)(y) = a^{m+n}$

Converting it in *log* form. $\log_a (x)(y) = m + n$ As $m = \log_a x$, $m = \log_a y$ Putting them

$\log_a (x)(y) = \log_a x + \log_a y$ Prove yourself $\log_b (K)(L) = \log_b x + \log_b L$

2. If two numbers are dividing, to find their *log*, their *logs* are subtracted, as:

$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ Algebraic solution of: $\frac{12}{4} = 3$.....(1)

log solution: $\log\left(\frac{12}{4}\right) = \log 12 - \log 4 = 1.067 - 0.6020 = 0.4771$

Taking anti-log 0.4771 = 3. ..... (2) Thus, results (1) & (2) are similar.

RELATIONSHIP BETWEEN
LOGARITHMIC AND EXPONENTIAL FUNCTIONS

Log functions are inverse of so many exponential functions. Therefore, the log functions can be converted into exponential function. Again, the exponential function can be converted into log function. As $\log_a x = y$, its conversion in exponential function is $y = a^x$. This means that x is a power which is given to some base (a) to get the number y.

Exponential form		Log form.
$y = a^x$	$\Longleftrightarrow$	$x = \log_a y$

Thus we have following relationships.

$10^2 = 100$	$\Longleftrightarrow$	$\log_{10} 100 = 2,$
$10^3 = 1000$	$\Longleftrightarrow$	$\log_{10} 1000 = 3$

Log Form		Exponential Form
$\log_{10} y = 2x$	$\Longleftrightarrow$	$y = 10^{2x}$
$\log_a y = xz$	$\Longleftrightarrow$	$y = a^{xz}$
$\ln y = 5t$	$\Longleftrightarrow$	$y = e^{5t}$

Exponential Form		Log Form
$y = a^{3x}$	$\Longleftrightarrow$	$\log_a y = 3x$
$y = 10^{6x}$	$\Longleftrightarrow$	$\log_{10} y = 6x$
$y = e^{t+1}$	$\Longleftrightarrow$	$\ln y = t + 1$

Examples of Natural Exponential Functions

Example 2: If $Q = 1000K^{0.4} L^{0.5}$

If $L = 220$, $K = 80$, find total output (Q) of the firm with *log*.

$$\log Q = \log 1000 + 0.4 \log (80) + 0.5 \log (220)$$

$$= 3 + 0.4 (1.9031) + 0.5 (2.3424) = 4.9324$$

$$Q = \text{Anti} \log 4.9324 = 85580$$

Example 3: If $Z = x^{0.3} y^{0.4}$ is a homogenous production function, multiplying each input by k we get new production function (Z^*) :

$$Z^* = (kx)^{0.3} (ky)^{0.4} = k^{0.3+0.4} (x^{0.3} y^{0.4}) = k^{0.7} (x^{0.3} y^{0.4})$$

Example 4: A firm whose present sales are Rs. 1 lac it plans to increase its sales at the rate 12% what will be its total sales after 5 years. It is solved with the formula.

$$S = P(1+i)^t = 100000 \left(1+\frac{12}{100}\right)^5 = 100000 (1.12)^5$$

$$\log S = \log 100000 + 5 (\log 1.12) \qquad \text{Taking } \textit{log} \text{ of both sides.}$$

$$= 5 + 5 = (0.0493) = 5.246$$

$$S = \text{Antilog } (5.246) = 176200$$

Example 5: If principal amount (P) is 1000, find total sum (S) at 6% when (1) annual interest is computed (2) bi–annual interest is calculated and (3) quarterly interest is computed. The total time period (t) is 3 year.

$$S = P(1+i)^t \Rightarrow S = 1000 (1+0.06)^3 \qquad (1)$$

$$\log S = \log 1000 + 3 \log 1.06 \qquad \text{Taking } \textit{logs.}$$

$$\log S = 3 + 3 (0.0253) = 3.0759$$

$$S = \text{Anti} \log 3.0759 = 1191$$

$$S = P\left(1+\frac{i}{m}\right)^{mt} = 1000 \left(1+\frac{0.06}{2}\right)^{2(3)} \qquad (2)$$

$$S = 1000 (1+0.06)^3$$

$$\log S = \log 1000 + 6 \log 1.03$$

$$\log S = 3 + 6 (0.0128) = 3.0768$$

$$S = \text{Anti} \log 3.0768 = 1194$$

$$S = 1000 \left(1+\frac{0.06}{4}\right)^{4(3)} = 1000 (1.015)^{12} \qquad (3)$$

$$\log S = \log 1000 + 12 \log (1.015) = 3 + 12 (0.0065) = 3.0780$$

$$S = \text{Anti} \log 3.0780 = 1197$$

Example 6: What will be the present value of Rs. 750 whose payment is made after 4 years, when rate of interest is 10% (1) if annual interest is taken and (2) bi–annual interest is taken.

$$P = S(1+i)^{t} \Rightarrow P = 750(1+0.10)^{-4} \quad (1)$$

$$\log P = \log 750 - 4\log 1.10 = 2.7095 \Rightarrow P = \text{Anti} \log 2.7095 = 512.3$$

$$P = S\left(1+\frac{i}{m}\right)^{-mt} = 750\left(1+\frac{0.10}{2}\right)^{-2(4)} \quad (2)$$

$$P = 750(1+0.05)^{-8} \log \Rightarrow P = \log 750 - 8\log 1.05$$

$$\log P = 2.8751 - 8(0.0212) = 2.7055$$

$$P = \text{Anti} \log 2.7055 = 507.6$$

Example 7: A firm hopes to get Rs. 1000/– from a machine each year. This machine lasts for 4 years. If the discount rate is 4% what will be present value of Rs. 1000/–.

$$P = S\left(\frac{1}{i}\right)\left(1-\frac{1}{(1+i)^{t}}\right) = 1000\left(\frac{1}{0.4}\right)\left(1-\frac{1}{(1.04)^{4}}\right)$$

$$\log\left(\frac{1}{(1.4)^{4}}\right) = \log 1 - 4\log(1.04) = 0 - 4(0.0170) = -0.0680$$

$$P = \text{Anti}\log -0.0680 = 0.855$$

$$P = 1000\left(\frac{1}{0.04}\right)[1-0.855] = 2500(0.45) = 3625$$

Example 8: