Chapter 9: Hypothesis Tests of a Single Population

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Introduction

In this chapter we will focus on

- developing **hypothesis testing** procedures that enables us to test the validity of the given claims concerning sample data,
- selecting one option out of two options by the use of hypothesis testing and the estimation procedures that we learned in Chapter 7.

Example

Suppose a ready-to-eat cereal firm claims that each cereal package weights 400 grams. If we collect a random sample of packages and compute the sample mean package weight, then it is possible to check the firm's claim.

Hypothesis Testing Terminology

- **Null Hypothesis** H_0: This is an assumed maintained hypothesis. If we do not find a sufficient evidence for its contrary, this null hypothesis is held to be true.
- **Alternative Hypothesis** H_1: This hypothesis is against to the null hypothesis. After checking the null hypothesis, if the null hypothesis is false then the alternative hypothesis is held to be true.
- **Simple Hypothesis**: A hypothesis that specifies a single value for a population parameter of interest.
- **Composite Hypothesis**: A hypothesis that specifies a range of values for a population parameter.

- **One-sided Alternative**: An alternative hypothesis involving all possible values of a population parameter on either one side or the other of (either greater than or less than) the value specified by a simple null hypothesis.
- **Two-sided Alternative**: An alternative hypothesis involving all possible values of a population parameter other than the value specified by a simple null hypothesis.

Example

- $H_0 : \mu = \mu_0$ is a ___________ ___________ hypothesis.
  - $H_1 : \mu < \mu_0$ is a ___________ ___________ ___________ hypothesis.
  - $H_1 : \mu > \mu_0$ is a ___________ ___________ ___________ hypothesis.
  - $H_1 : \mu \neq \mu_0$ is a ___________ ___________ ___________ hypothesis.
- $H_0 : \mu \leq \mu_0$ is a ___________ ___________ hypothesis.
  - $H_1 : \mu > \mu_0$ is a ___________ ___________ ___________ hypothesis (the obvious alternative hypothesis!).
- $H_0 : \mu \geq \mu_0$ is a ___________ ___________ hypothesis.
  - $H_1 : \mu < \mu_0$ is a ___________ ___________ ___________ hypothesis (the obvious alternative hypothesis!).

- **Hypothesis Test decisions**: A decision rule is formulated, leading the investigator to either reject or fail to reject the null hypothesis on the basis of sample evidence.
- **Type I Error**: The rejection of a true null hypothesis.
- **Type II Error**: The failure to reject a false null hypothesis.
- **Significance Level**: The probability of rejecting a null hypothesis that is true. This probability is sometimes expressed as a percentage, so a test of significance level α is referred to as a $100\alpha\%$-level test.
- **Power of the test**: The probability of rejecting a null hypothesis that is false.

STATES OF NATURE

DECISIONS ON H_0	H_0 is TRUE	H_0 is FALSE
FAIL TO REJECT H_0	Correct Decision with probability $1-\alpha$	Type II Error with probability β
REJECT H_0	Type I Error with probability α	Correct Decision with probability $1-\beta$

Hypothesis Tests of the Mean of a Normal Distribution

We will consider the hypothesis tests of the mean of a normal distribution. Here, we have two cases:

- The population variance σ^2 is known and
- σ^2 is unknown.

Tests of the Mean of a Normal Distribution: KNOWN σ^2

We will present a procedure to develop and implement hypothesis tests which have applications to business and economic problems.

Assumptions:

- We have a random sample of n observations
- from a normally distributed population with mean μ and variance σ^2.
- The observed sample mean is $\bar{x}$.

Then a one-tailed test with significance level α is as follows:

- the **null hypothesis** is:

$$H_0 : \mu = \mu_0 \quad \text{or} \quad H_0 : \mu \leq \mu_0 \quad \text{against} \quad H_1 : \mu > \mu_0,$$

- and the **decision rule** is given as:

$$\text{Reject } H_0 \text{ if } \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_\alpha$$

or equivalently the **decision rule** is given as:

$$\text{Reject } H_0 \text{ if } \bar{x} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}},$$

where z_α is a number such that Z is the *standard normal random variable* and

$$P(Z > z_\alpha) = \alpha.$$

Example: The production manager of Northern windows Inc. has asked you to evaluate a proposed new procedure for producing its Regal line of double-hung windows. The present process has a mean production of 80 units per hour with a population standard deviation of $\sigma = 8$. A random sample of 25 production hours using the proposed new process has a sample mean of 83 windows per hour. The manager indicates that she does not want to change to a new procedure unless there is strong evidence that the mean production level is higher with the new process. Test the null hypothesis that mean production is 80 units against the alternative that it is higher.

Another one-tailed test with significance level α is as follows:

- the **null hypothesis** is:

$$H_0 : \mu = \mu_0 \quad \text{or} \quad H_0 : \mu \geq \mu_0 \quad \text{against} \quad H_1 : \mu < \mu_0,$$

- and the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} < -z_\alpha$$

or equivalently the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \bar{x} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}},$$

where $-z_\alpha$ is number such that Z is the *standard normal random variable* and

$$P(Z < -z_\alpha) = \alpha.$$

Example: The production manager of Twin Forks Ball Bearing Inc. has asked you to evaluate a modified ball bearing production process. When the process is operating properly, the process produces ball bearings whose weights are normally distributed with a population mean of 5 ounces and a population standard deviation of 0.1 ounce. A new raw material supplier was used for a recent production run, and the manager wants to know if that change has resulted in a lowering of the mean weight of the ball bearings. There is no reason to suspect a problem with the new supplier and the manager will continue to use the new supplier unless there is strong evidence that underweight ball bearings are being produced.

A two-tailed test with significance level α is as follows:

- the **null hypothesis** is:

$$H_0 : \mu = \mu_0 \quad \text{against} \quad H_1 : \mu \neq \mu_0,$$

- and the **decision rule** is given as:

$$\text{Reject } H_0 \text{ if } \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}} < -z_{\frac{\alpha}{2}} \text{ or } \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}} > z_{\frac{\alpha}{2}}$$

  or equivalently the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \bar{x} < \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \quad \text{or} \quad \bar{x} > \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}.$$

Example: The production manager of Circuits Unlimited has asked for your assistance in analyzing a production process. This process involves drilling holes whose diameters are normally distributed with population mean 2 inches and population standard deviation 0.06 inch. A random sample of nine measurements had a sample mean of 1.95 inches. Use a significance level of $\alpha = 0.05$ to determine if the observed sample mean is unusual and suggests that the drilling machine should be adjusted.

Tests of the Mean of a Normal Distribution: UNKNOWN σ^2

Assumptions:

- We have a random sample of n observations
- from a normally distributed population with mean μ.
- The observed sample mean is $\bar{x}$ and the observed sample variance is s^2.

We can use the following tests with significance level α:

Case 1:

- the **null hypothesis** is:

$$H_0 : \mu = \mu_0 \quad \text{or} \quad H_0 : \mu \leqslant \mu_0 \quad \text{against} \quad H_1 : \mu > \mu_0,$$

- and the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1,\alpha}$$

or equivalently the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \bar{x} > \mu_0 + t_{n-1,\alpha} \frac{s}{\sqrt{n}}.$$

Case 2:

- the **null hypothesis** is:

$$H_0 : \mu = \mu_0 \quad \text{or} \quad H_0 : \mu \geq \mu_0 \quad \text{against} \quad H_1 : \mu < \mu_0,$$

- and the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} < -t_{n-1,\alpha}$$

or equivalently the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \bar{x} < \mu_0 - t_{n-1,\alpha} \frac{s}{\sqrt{n}}.$$

Case 3:

- the **null hypothesis** is:

$$H_0 : \mu = \mu_0 \quad \text{against} \quad H_1 : \mu \neq \mu_0,$$

- and the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} < -t_{n-1, \frac{\alpha}{2}} \quad \text{or} \quad \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1, \frac{\alpha}{2}}$$

or equivalently the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \bar{x} < \mu_0 - t_{n-1, \frac{\alpha}{2}} \frac{s}{\sqrt{n}} \quad \text{or} \quad \bar{x} > \mu_0 + t_{n-1, \frac{\alpha}{2}} \frac{s}{\sqrt{n}}.$$

Example: Test the hypotheses

$$H_0 : \mu \leq 100$$
$$H_1 : \mu > 100$$

using a random sample of $n = 25$, with $\alpha = 0.05$, and the following sample statistics:

a) $\bar{x} = 106$, $s = 15$,

b) $\bar{x} = 104$, $s = 10$,

c) $\bar{x} = 95$, $s = 10$,

d) $\bar{x} = 92$, $s = 18$.

Example: Grand Junction Vegetables is a producer of a wide variety of frozen vegetables. The company president has asked you to determine if the weekly sales of 16-ounce packages of frozen broccoli has increased. The weekly sales per store has had a mean of 2400 packages over the past 6 months. You have obtained a random sample of sales data from 134 stores with a sample mean of 3593 and a standard deviation of 4919 for your study. Test the null hypotheses that population mean is 2400 packages against the desired alternative.

Tests of the Population Proportion (Large Samples)

Another important set of business and economics problems involves population proportion (p). Inference about the population proportion p, based on the sample proportion $\hat{p}$, is an important application of hypothesis testing.

From our work in chapters 6 and 7, we know that $\hat{p}$ can be approximated by the standard normal statistic as follows:

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

If $np(1-p) > 5$, then the following tests have significance level α:

Case 1:

- the **null hypothesis** is:

$$H_0 : p = p_0 \quad \text{or} \quad H_0 : p \leq p_0 \quad \text{against} \quad H_1 : p > p_0,$$

- and the **decision rule** is given as:

$$\text{Reject } H_0 \text{ if } \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} > z_\alpha.$$

Case 2:

- the **null hypothesis** is:

$$H_0 : p = p_0 \quad \text{or} \quad H_0 : p \geq p_0 \quad \text{against} \quad H_1 : p < p_0,$$

- and the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} < -z_\alpha.$$

Case 3:

- the **null hypothesis** is:

$$H_0 : p = p_0 \quad \text{against} \quad H_1 : p \neq p_0,$$

- and the **decision rule** is given as:

$$\text{Reject } H_0 \quad \text{if} \quad \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} < -z_{\frac{\alpha}{2}} \quad \text{or} \quad \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} > z_{\frac{\alpha}{2}}.$$

Example: Market Research, Inc., wants to know if shoppers are sensitive to the prices of items sold in a supermarket. A random sample of 802 shoppers was obtained, and 378 of those supermarket shoppers were able to state the correct price of an item immediately after putting it into their cart. Test at the 7% level the null hypothesis that at least one-half of all shoppers are able to state the correct price.

Exercises

Example: Test the hypotheses

$$H_0 : \mu = 100$$
$$H_1 : \mu < 100$$

using a random sample of $n = 36$, a probability of Type I error equal to 0.05, and the following sample statistics:

a) $\bar{x} = 106$, $s = 15$,

b) $\bar{x} = 95$, $s = 10$.

Example: A statistics instructor is interested in the ability of students to assess the difficulty of a test they have taken. This test was taken by a large group of students, and the average score was 78.5. A random sample of eight students was asked to predict their average score. Their predictions were as follows:

$$72 \quad 83 \quad 78 \quad 65 \quad 69 \quad 77 \quad 81 \quad 71$$

Assuming a normal distribution, test the null hypothesis that the population mean prediction would be 78.5. Use a two-sided alternative and a 10% significance level.

Example: Of a random sample of 199 auditors, 104 indicated some measure of agreement with this statement: "Cash flow is an important indication of profitability." Test at the 10% significance level against a two-sided alternative the null hypothesis that one-half of the members of this population would agree with this statement.

Example: A random sample of 50 university admissions officers was asked about expectations in application interviews. Of these sample members, 28 agreed that the interviewer usually expects the interviewee to have volunteer experience doing community projects. Test the null hypothesis that one-half of all interviewers have this expectation against the alternative that the population proportion is larger than one-half. Use $\alpha = 0.05$.

Example: A manufacturer of detergent claims that the contents of boxes sold weigh on average at least 16 ounces. The distribution of weight is known to be normal, with a standard deviation of 0.4 ounce. A random sample of 16 boxes yielded a sample mean weight of 15.84 ounces. Test at the 10% significance level the null hypothesis that the population mean is at least 16 ounces.

Example: You have been asked to evaluate single-employer plans after the establishment of the Health Benefit Guarantee Corporation. A random sample of 76 percentage changes in promised health benefits was observed. The sample mean percentage change was 0.078, and the sample standard deviation was 0.201. Test the null hypothesis that the population mean percentage change is 0 against a two-sided alternative.

Example: An engineering research center claims that through the use of a new computer control system, automobiles should achieve, on average, an additional 3 miles per gallon of gas. A random sample of 100 automobiles was used to evaluate this product. The sample mean increase in miles per gallon achieved was 2.4, and the sample standard deviation was 1.8 miles per gallon. Test the hypothesis that the population mean is at least 3 miles per gallon.

Example: A pharmaceutical manufacturer is concerned that the impurity concentration in pills should not exceed 3%. It is known that from a particular production run impurity concentrations follow a normal distribution with a standard deviation of 0.4%. A random sample of 64 pills from a production run was checked, and the sample mean impurity concentration was found to be 3.07%.

a) Test at the 5% level the null hypothesis that the population mean impurity is 3% against the alternative that it is more than 3%.

b) Suppose that the alternative hypothesis had been two-sided, rather than one-sided, with the null hypothesis $H_0 : \mu = 3$.