77. Residue At Infinity

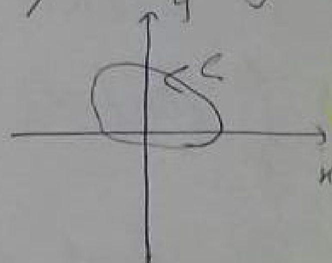
Theorem:

If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

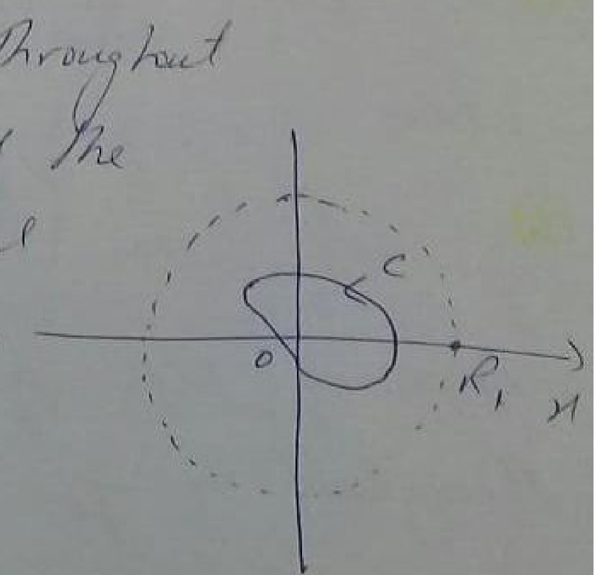
$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Proof:

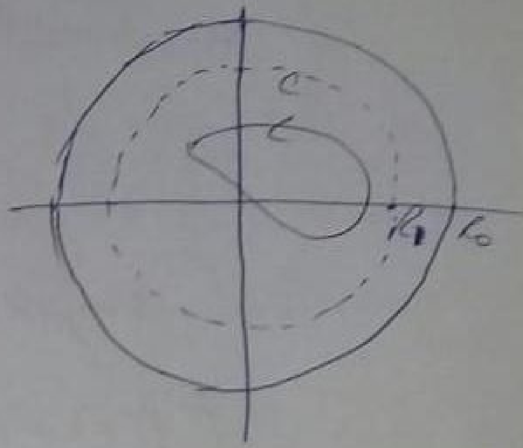
Suppose that the function f is analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C.



Next let R_1 denote the positive number which is large enough that C lies inside the circle $|z| = R_1$. The fn 'f' is evidently analytic throughout the domain $R_1 < |z| < \infty$, and the point at infinity is considered as an isolated singular point of f. (according to sec. 74)



Now let C_0 denote a circle $|z| = R_0$, negatively oriented, where $R_0 > R_1$. The residue of f at infinity is defined by means of equation:



(1) $$\int_{C_0} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\infty} f(z)$$

Since f is analytic throughout the closed region bounded by C and C_0, the principle of deformation of paths (sec 53) tells us that

$$\int_C f(z)\,dz = \int_{-C_0} f(z)\,dz = -\int_{C_0} f(z)\,dz$$

So according to definition (1)

(2) $$\int_C f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z)$$

To find the residue, we write the Laurent series

(3) $$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \qquad (R_1 < |z| < \infty)$$

where

(4) $$c_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)\,dz}{z^{n+1}} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

Replacing z by $\frac{1}{z}$ in (3) and then multiplying throughout by $\frac{1}{z^2}$, we get

$$f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} C_n \frac{1}{z^n} \qquad \left(R_1 < \left|\frac{1}{z}\right| < \infty\right)$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \sum C_n \frac{1}{z^n} \qquad \left(\frac{1}{R_1} > |z| > \frac{1}{\infty}\right)$$

$$= \sum_{n=-\infty}^{\infty} C_n \frac{1}{z^{n+2}} \qquad \left(0 < |z| < \frac{1}{R_1}\right)$$

replacing n by $n-2$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} C_{n-2} \frac{1}{z^n}$$

from above series, we get the Residue of $f(z)$ at $z=0$ for $n=1$,

i.e $C_{-1} = \underset{z=0}{\text{Res}}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$

Putting $n=-1$ in (4), we get

$$C_{-1} = \frac{1}{2\pi i} \int_{-C_0} f(z)\,dz$$

$$\Rightarrow \int_{-C_0} f(z)\,dz = 2\pi i\, C_{-1} = 2\pi i \underset{z=0}{\text{Res}}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

$$\Rightarrow -\int_{C_0} f(z)\,dz = 2\pi i \underset{z=0}{\text{Res}}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

(5) $$\int_{C_0} f(z)\,dz = -2\pi i \underset{z=0}{\text{Res}}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Now it follows from (1) and (5) that

$$\operatorname*{Res}_{z=\infty} f(z) = \frac{1}{2\pi i}\int_{C_0} f(z)\,dz$$

$$= \frac{1}{2\pi i}\left[-2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]\right]$$

(6) $$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Example

$$f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}$$

The singular points of the given function are

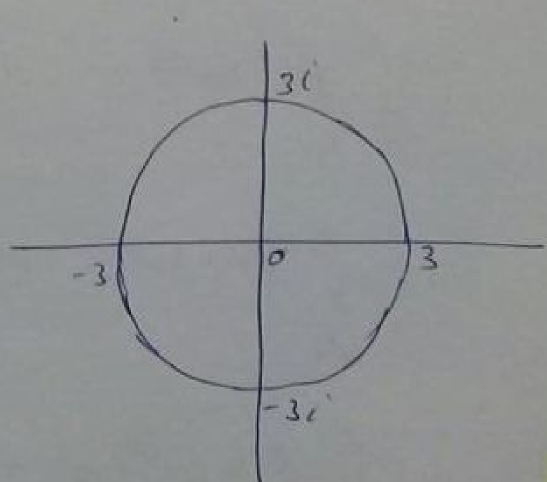
$1+z=0$, $1+2z^4=0$

$z=-1$ $\qquad 2z^4=-1$

$z^4 = \frac{-1}{2} \Rightarrow z^2 = \pm\frac{1}{\sqrt{2}}i$

$C - |z| = 3$

Since all sing. points lie inside the circle $|z|=3$. So to use the above theorem we write.



$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{\frac{1}{z^3}\left(1 - \frac{3}{z}\right)}{\left(1 + \frac{1}{z}\right)\left(1 + \frac{2}{z^4}\right)}$$

$$= \frac{1}{z^2} \frac{\frac{1}{z^3}\left(\frac{z-3}{z}\right)}{\left(\frac{z+1}{z}\right)\left(\frac{z^4+2}{z^4}\right)}$$

$$= \frac{1}{z^2} \frac{\frac{z-3}{z^4}}{\frac{(z+1)(z^4+2)}{z^5}}$$

$$= \frac{1}{z^2} \cdot \frac{z^5}{z^4} \frac{z-3}{(z+1)(z^4+2)}$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z} \frac{z-3}{(z+1)(z^4+2)}$$

Since the quotient $\frac{z-3}{(z+1)(z^4+2}$ is analytic at the origin so it has Maclaurin series representation. Thus, we have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{z-3}{z} \frac{1}{1+z} \cdot \frac{1}{2\left(1 + \frac{z^4}{2}\right)}$$

$$= \left(1 - \frac{3}{z}\right)\frac{1}{2}\left(1 - z + z^2 - \cdots\right)\left(1 - \frac{z^4}{2} + \frac{z^8}{2^2} - \cdots\right)$$

$$= \left(\frac{1}{2} - \frac{3}{2}\frac{1}{z}\right)\left(1 - z + z^2 \cdots - \frac{z^4}{2} + \frac{z^5}{2} - \cdots\right)$$

$=$

Therefore

$$\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -\frac{3}{2}$$

and thus

$$\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)} = -2\pi i \operatorname*{Res}_{z=\infty} f(z)$$

$$= -2\pi i\left[-\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]\right]$$

$$= -2\pi i\left(+\frac{3}{2}\right)$$

$$\boxed{\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)} = -3\pi i}$$

Q3 $C - |z| = 2$

$$f(z) = \frac{4z-5}{z(z-1)} \, dz.$$

Since both the singular points lie inside the circle $|z| = 2$, so we can write.

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{4/z - 5}{\frac{1}{z}\left(\frac{1}{z} - 1\right)}$$

$$= \frac{1}{z^2} \frac{\frac{4-5z}{z}}{\frac{1-z}{z^2}}$$

$$= \frac{1}{z^2} \cdot \frac{z^2}{z} \quad \frac{4-5z}{1-z}$$

$$= \frac{1}{z}(4-5z)\frac{1}{1-z}$$

$$= \left(\frac{4}{z}-5\right)\sum_{n=0}^{\infty} z^n \qquad 0<|z|<1$$

$$= \left(\frac{4}{z}-5\right)(1+z+z^2+z^3+\cdots)$$

$$= \frac{4}{z}+4+4z+\cdots-5-5z-5z^2\cdots$$

$$\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 4$$

$$\Rightarrow \operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

$$= -4$$

$$\Rightarrow \int_C \frac{4z-5}{z(z-1)}\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z)$$

$$= -2\pi i(-4)$$

$$= 8\pi i$$