Ch-6 Residues and Poles 14.04.2020

Definition Singular Point

If 'f' fails to be analytic at z_0 but is analytic at some point in every neighbourhood of z_0, then z_0 is singular point of f.

Definition Isolated Singular Point

A singular point z_0 is said to be isolated if there is a deleted ε neighbourhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout which f is analytic.

Example 1.

$$f(z) = \frac{z-1}{z^5(z^2+9)}$$

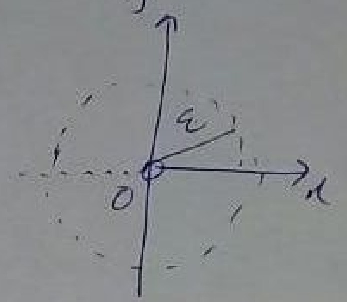
singular points are $z = 0$ and $z = \pm 3i$ which are isolated singular points.

Note that.

The singular points of a rational function, or quotient of ~~a~~ two polynomials are always isolated.

Example - 2

$$f(z) = \text{Log}\, z = \ln r + i\theta \qquad (r > 0, -\pi < \theta < \pi)$$



The origin $z = 0$ is the singular point of principal Logarithmic function but it is not isolated. The reason is that every deleted ϵ nbhd. of $z = 0$ contains points on the -ve real axis (sing pts of Log z) where Log z is not defined.

Example-3

$$f(z) = \frac{1}{\sin(\pi/z)}$$

singular points are where $\sin \pi/z = 0$

or $\pi/z = \sin^{-1}(0) = n\pi$

$$\Rightarrow \frac{\pi}{z} = n\pi \qquad (n = \pm 1, \pm 2, --)$$

$$\Rightarrow \frac{1}{z} = n$$

$$\Rightarrow z = \frac{1}{n} \quad \Rightarrow (n = \pm 1, \pm 2, --)$$

The derivative of f does not exist at $z = 0$ and $z = \frac{1}{n}$ ($n = \pm 1, \pm 2 \ldots$). These are the sing pts. of "f" but $z = 0$ is not the isolated singular point because every ϵ deleted nbhd. of zero contains other singular points

Note that

i) If a function "f" is analytic everywhere inside a simple closed contour C except for a finite number of singular points $z_1, z_2 \cdots z_n$, those points must all be isolated and the deleted neighbourhood about these singular points can be made small enough so that they lie entirely in C.

ii) It is sometime convenient to consider the point at infinity as an isolated singular point.

eg if there is a +ve number R_1 s.t. f is analytic for

$$R_1 < |z| < \infty,$$

Then f is said to have an isolated singular point at $z_0 = \infty$

Residues

When z_0 is an isolated singularity of "f", there is a +ve number R_2 such that f is analytic at each point z for which

$$0 < |z - z_0| < R_2$$

Then $f(z)$ has Laurentz series representation.

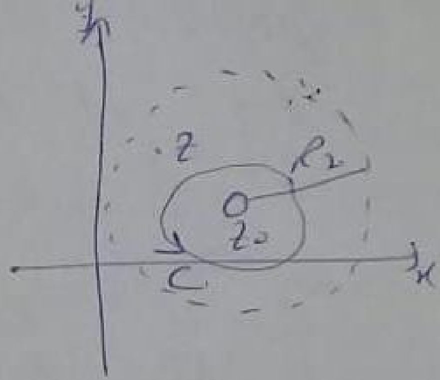
(1) $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

$$(0 < |z-z_0| < R_2)$$

where the coefficient b_n has integral representation

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}} \quad (n = 1, 2, \ldots)$$

where C is any positively oriented simple closed contour around z_0 that lies in the punctured disk. $0 < |z-z_0| < R_2$.



When $n=1$, we have

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz.$$

(2) $$\Rightarrow \int_C f(z)\,dz = 2\pi i\, b_1$$

b_1, the coefficient of $\frac{1}{z-z_0}$ in above expansion is called the residue of f at the isolated singular point z_0 and we write.

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

Eqn (2) then becomes

(3) $$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z)$$

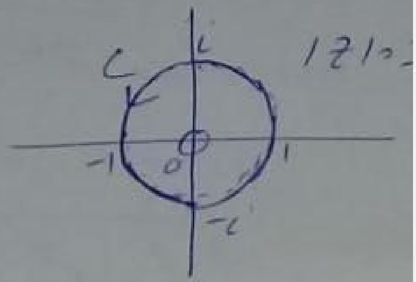
Example-1

Evaluate the integral.

$$\int_C \frac{e^z-1}{z^4}\,dz.$$

C — positively oriented unit circle $|z|=1$.

Since the integrand $\frac{e^z-1}{z^4}$ is analytic everywhere inside C except at the origin $z=0$, it has Laurent series expansion which is valid in $0<|z|<\infty$.



Since
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z|<\infty)$$

$$= 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$$

$$e^z-1 = z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots = \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

$$\frac{e^z-1}{z^4} = \frac{1}{z^4}\sum_{n=1}^{\infty} \frac{z^n}{n!} \qquad (0<|z|<\infty)$$

$$= \sum_{n=1}^{\infty} \frac{z^{n-4}}{n!}$$

$$= z^{-3}+\frac{z^{-2}}{2!}+\frac{z^{-1}}{3!}+\frac{z^0}{4!}+\frac{z}{5!}+\frac{z^2}{6!}+\cdots$$

$$= \frac{1}{z^3}+\frac{1}{z^2 2!}+\frac{1}{3!\,z}+\frac{1}{4!}+\frac{1}{5!}z+\cdots$$

In above series.

b_1 - the coefficient of z_1 is

$$b_1 = \frac{1}{3!}$$

$$\int_c \frac{e^z-1}{z^4}\,dz = 2\pi i\, b_1 = 2\pi i \operatorname*{Res}_{z=z_0} f(z)$$

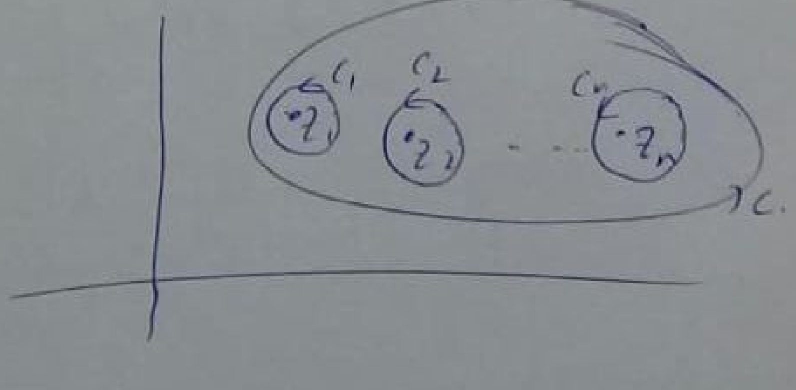
$$= 2\pi i \cdot \frac{1}{3!} = \frac{\pi i}{3}$$

$$\boxed{\int_c \frac{e^z-1}{z^4}\,dz = \frac{\pi i}{3}}$$

Cauchy's Residue Theorem

Theorem: Let C be a simple closed contour, described in the positive sense. If a function 'f' is analytic inside and on C except for a finite number of singular points z_k $(k=1,2,\dots n)$ inside C, then

$$\int_c f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$



Proof:

Let the points z_k $(k=1,2,\dots n)$ be centers of positively oriented circles C_k which are interior to C and are so small that no two of them have points in common. The circles C_k, together with the simple closed contour C, form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain, consisting of the points inside C and exterior to each C_k.

Then according to Cauchy-Goursat theorem for multiply connected domains (Sec. 53)

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{-C_k} f(z)\,dz = 0$$

$$\Rightarrow \int_C f(z)\,dz - \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$$

$$\Rightarrow \int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz.$$

Since from Sec. 75

$$\int_{C_k} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z) \quad (k=1,2,\cdots n)$$

Thus

$$\int_C f(z)\,dz = \sum_{k=1}^{n} 2\pi i \operatorname*{Res}_{z=z_k} f(z)$$

$$= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

The theorem is proved