Integration and Differentiation of Power Series

Example - 2

Since we have

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad (|z-1| < 1)$$

Differentiating on both sides

$$\frac{-1}{z^2} = \sum_{n=1}^{\infty} (-1)^n \, n \, (z-1)^{n-1}$$

replacing n by n+1

$$\frac{-1}{z^2} = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1)(z-1)^n$$

$$\Rightarrow \frac{1}{z^2} = \sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^n \qquad (|z-1| < 1)$$

Exercise

Q1.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

By differentiating.

$$\frac{-1}{(1-z)^2}(-1) = \sum_{n=1}^{\infty} n \, z^{n-1}$$

replacing n by $n+1$.

$$\boxed{\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \qquad |z|<1}$$

$$(1-z)^{-2} = \sum_{n=0}^{\infty} (n+1) z^n$$

Again differentiating.

$$(-2)(1-z)^{-3}(-1) = \sum_{n=1}^{\infty} (n+1) n\, z^{n-1}$$

$$\frac{2}{(1-z)^3} = \sum_{n=1}^{\infty} n(n+1) z^{n-1}$$

replacing n by $n+1$

$$\boxed{\frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} (n+1)(n+2) z^n \qquad (|z|<1)}$$

Q3 $$\frac{1}{z} = \frac{1}{2+z-2} = \frac{1}{2\left(1+\frac{z-2}{2}\right)} = \frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{2}}$$

we have to find the Taylor series of the given fn. about $z_0 = 2$

Since $$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad |z|<1$$

$$\Rightarrow \frac{1}{1+\frac{z-2}{2}} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{2}\right)^n \qquad |z-2|<2$$

Thus

$$\frac{1}{z} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z-2}{2}\right)^n(-1)^n$$

$$\frac{1}{z} = \sum_{n=0}^{\infty}\frac{(z-2)^n}{2^{n+1}}(-1)^n \qquad |z-2|<2$$

Diff. both sides

$$\frac{-1}{z^2} = \sum_{n\geq 1}\frac{n(z-2)^{n-1}}{2^{n+1}}(-1)^n$$

replace n by n+1

$$\frac{-1}{z^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(n+1)(z-2)^n}{2^{n+2}}$$

$$\frac{-1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(-1)(n+1)(z-2)^n}{2^n}$$

$$\boxed{\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \quad (|z-2|<2)}$$

Q2 Do yourself

Multiplication and Division of Power Series

Example 1

$$f(z) = \frac{\sinh z}{1+z} =$$

sing. point at $z = -1$

Condition of validity for Mac. series will be.

$|z| < 1$

Since $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$

$(|z| < \infty)$

$$\sinh z = z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots$$

Also

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad (|z| < 1)$$

Thus

$$\sinh z \cdot \frac{1}{1+z} = \left(z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right)\left(1 - z + z^2 - z^3 + \cdots\right)$$

Multiply the 1^{st} term of 1^{st} series (i.e "z") by each term of the second series. Then 2^{nd} term of 1^{st} series (i.e "$\frac{z^3}{6}$") by each term of the second series and so on....

we get

$$= z - z^2 + z^3 - z^4 + \cdots$$

$$\frac{z^3}{6} - \frac{z^4}{6} + \frac{z^5}{6} - \cdots$$

$$+ \frac{z^5}{120} - \frac{z^6}{120} + \frac{z^7}{120} - \cdots$$

$$\Rightarrow = z - z^2 + \left(1 + \frac{1}{6}\right) z^3 - \left(1 + \frac{1}{6}\right) z^4 + \left(1 + \frac{1}{6} + \frac{1}{120}\right) z^5 + \cdots$$

$$\frac{\sinh z}{1+z} = z - z^2 + \frac{7}{6} z^3 - \frac{7}{6} z^4 + \left(\frac{120 + 20 + 1}{120}\right) z^5 + \cdots$$

$$\boxed{\frac{\sinh z}{1+z} = z - z^2 + \frac{7}{6} z^3 - \frac{7}{6} z^4 + \frac{141}{120} z^5 + \cdots \qquad |z| < 1}$$

Example-2.

$$f(z) = \frac{1}{\sinh z}$$

The zero of $\sinh z$ are the singular points of $f(z)$, that are $z = n\pi i$ $(n = 0, \pm 1, \pm 2 \cdots)$

Since $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

$= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$

Thus

$$\frac{1}{\sinh z} = \frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}$$

$$= \frac{1}{z\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)}$$

$$= \frac{1}{z} \cdot \frac{1}{\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)}$$

The given fn. has a Laurent series represent. in the punctured disk $0 < |z| < \pi$

$$1 - \frac{1}{3!} z^2 + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^4$$

$$1 + \frac{1}{3!} z^2 + \frac{1}{5!} z^4 + \cdots \;\Big)\; 1$$

$$-1 - \frac{1}{3!} z^2 - \frac{1}{5!} z^4 + \cdots$$

$$-\frac{1}{3!} z^2 - \frac{1}{5!} z^4 - \cdots$$

$$+\frac{1}{3!} z^2 + \frac{1}{3!3!} z^4 + \frac{1}{3!5!} z^6 + \cdots$$

$$\left(-\frac{1}{5!} + \frac{1}{3!}\frac{1}{3!}\right) z^4 + \frac{1}{3!5!} z^6$$

Thus

$$\frac{1}{\sinh z} = \frac{1}{z}\left(1 - \frac{1}{3!}z^2 + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)z^4 + \cdots\right)$$

$$\boxed{\frac{1}{\sinh z} = \frac{1}{z} - \frac{1}{3!}z + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)z^3 - \cdots \quad (0 < |z| < \pi)}$$

Pg # 224

Exercise.

Q1. Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z^2 - \frac{5}{6}z^3 + \cdots$$

$(0 < |z| <$

Soln Since the given fn. has sing. point $z = 0$, $z = \pm i$

$\Rightarrow$ the condition of validity for the Laurent series will be.

$0 < |z| < 1$

Now since

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (|z|$$

and

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots \quad (|z| < 1)$$

Thus

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(1 - z^2 + z^4 - z^6 + \cdots\right)$$

$$= \frac{1}{z}\Big(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (0 < |z| < 1)$$

$$- z^2 - z^3 - \frac{z^4}{2!} - \cdots$$

$$+ z^4 + z^5 + \frac{z^6}{2!} + \cdots\Big)$$

$$= \frac{1}{z}\left(1 + z + \left(\frac{1}{2!} - 1\right)z^2 + \left(\frac{1}{3!} - 1\right)z^3 + \cdots\right)$$

$$= \frac{1}{z}\left(1 + z + \left(\frac{1}{2} - 1\right)z^2 + \left(\frac{1}{6} - 1\right)z^3 + \cdots\right)$$

$$= \frac{1}{z}\left(1 + z - \frac{1}{2}z^2 - \frac{5}{6}z^3 + \cdots\right)$$

$$\boxed{\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots \quad (0 < |z| < 1)}$$

Q2. Do yourself.

(7)

Q4

$$f(z) = \frac{1}{e^z - 1}$$

$|z| < 8$

Since $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$\frac{1}{e^z - 1} = \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$$

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2!} + \frac{1}{12} z^2 + \cdots$$

$(0 < |z| < 2\pi)$

For sing points

$e^z - 1 = 0$

$e^z = 1$

$\Rightarrow z = 0,$

$$\frac{1}{z} - \frac{1}{2!} + \frac{1}{12} z$$

$$z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \;\Big)\; 1$$

$$1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

$$-\frac{z}{2!} - \frac{z^2}{3!} - \cdots$$

$$-\frac{z}{2!} - \frac{z^2}{2!2!} - \frac{z^3}{2!3!}$$

$$\left(\frac{1}{(2!)^2} - \frac{1}{3!}\right) z^2 + \frac{1}{2!3!} z^3 + \cdots$$

$$\left(\frac{1}{4} - \frac{1}{6}\right) z^2 + \frac{1}{(2)(6)} z^3$$

$$\left(\frac{3-2}{12}\right) z^2 + \frac{1}{12} z^3$$

$$\frac{1}{12} z^2 + \frac{1}{12} z^3$$