66. Laurent Series

Theorem: Suppose that a function 'f' is analytic throughout an annuler domain $R_1 < |z - z_0| < R_2$, centred at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

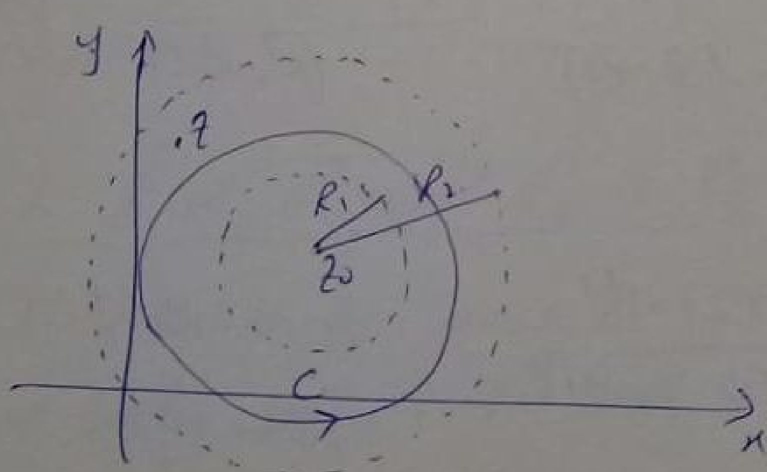
$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (R_1 < |z - z_0| < R_2),$$

where

$$(2) \quad a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots)$$

and

$$(3) \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}} \quad (n = 1, 2, \ldots)$$



By replacing n by $-n$ in the second series of representation (1), we can write.

$$\sum_{n=-\infty}^{-1} \frac{b_{-n}}{(z-z_0)^{-n}}$$

where

$$b_{-n} = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad (n=-1,-2,\cdots) \quad (\because \text{from } (2))$$

Thus (1) becomes

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=-\infty}^{-1} \frac{b_{-n}}{(z-z_0)^{-n}} \qquad (R_1 < |z-z_0| < R_2)$$

$$= \sum_{n=-\infty}^{-1} b_{-n}(z-z_0)^n + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Now if

$$C_n = \begin{cases} b_{-n} & \text{when } n \le -1 \\ a_n & \text{when } n \ge 0 \end{cases}$$

then

$$(4) \quad f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n \qquad (R_1 < |z-z_0| < R_2)$$

where

$$C_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad (n=0,\pm 1,\pm 2,\cdots)$$

$\because$ (from def of a_n and b_{-n})

Thus the expression in (1) and (4) are called Laurent series representation for $f(z)$

Now observe that the integrand in expression (3) can be written as $f(z)(z-z_0)^{n-1}$. This shows that $f(z)$ is analytic throughout the disk $|z-z_0| < R_2$ and the integrand is analytic there too. Thus

$$\int_C f(z)(z-z_0)^{n-1}\,dz = 0 \qquad (\because \text{acc. to Cauchy Goursat formula})$$

Thus all b_n coefficients are zero

i.e $b_n = \frac{1}{2\pi i}\int_C f(z)(z-z_0)^{n-1}\,dz = \frac{1}{2\pi i}(0) = 0$

and moreover

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!} \qquad (\because \text{according to Cauchy Int formula})$$

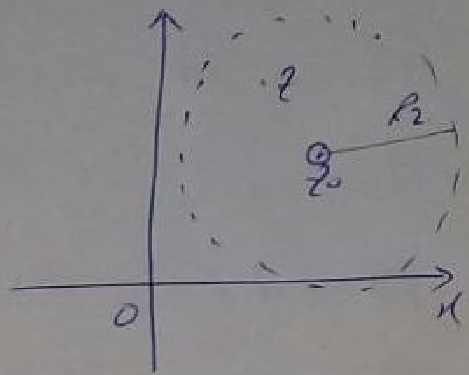
Thus the Laurent series in (1) reduces to Taylor series about z_0

i.e $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \qquad \left(\begin{array}{l}\because b_n = 0 \\ \because a_n = \frac{f^{(n)}(z_0)}{n!}\end{array}\right)$

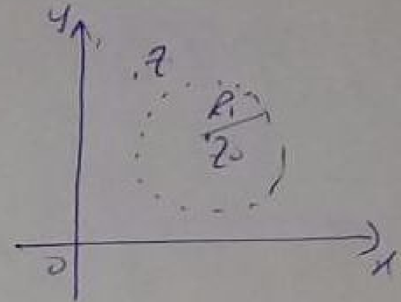
How to fixed R_1 and R_2 in $R_1 < |z-z_0| < R_2$?

* If 'f' fails to be analytic at z_0 but is otherwise analytic in $|z-z_0| < R_2$, then the radius R_1 can be chosen arbitrarily small. Representation

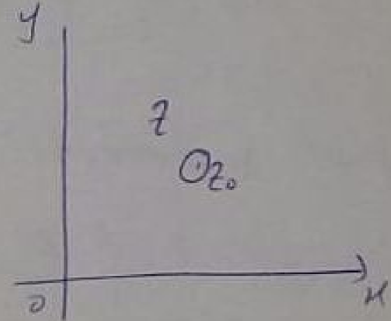
(1) is then valid in the punctured disk.
$0<|z-z_0|<R_2$.



* If 'f' is analytic at each point in the finite plane exterior to the circle $|z-z_0|=R_1$, the condition of validity is
$R_1<|z-z_0|<\infty$



* If f is analytic everywhere in the finite plane except at z_0, series (1) is valid when
$$0<|z-z_0|<\infty$$



---

8. Examples

We need Maclaurin series (1) to (6) of Sec. 64 to find the Laurent series in this section.

~~Example 1~~

~~[illegible]~~

Example 1

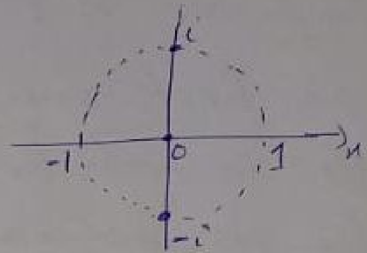
$$f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z} \cdot \frac{1}{1+z^2}$$

Sing. pts : $z = 0$ and $z = \pm i$

We have to find the Laurent series representation of $f(z)$ in the punctured disk $0 < |z| < 1$.

When $|z| < 1$, then $|-z^2| < 1$

So we substitute $-z^2$ for z in the Mac laurin series



$0 < |z| < 1$

(1) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n \qquad |z| < 1$$

Thus

$$f(z) = \frac{1}{z} \cdot \sum_{n=0}^{\infty} (-1)^n z^{2n} \qquad (0 < |z| < 1)$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n-1}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{2n-1}$$

Replacing n by $n+1$

(2) $$\boxed{f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1} \qquad (0 < |z| < 1)}$$

Example-2

$$f(z) = \frac{z+1}{z-1} \qquad \text{sing. pt is } z = 1$$

Thus it is analytic in the domain

$D_1 : |z| < 1$ and $D_2 : 1 < |z| < \infty$

In both these domains, $f(z)$ has series representation in powers of z.

we first consider D_1, where we represent $f(z)$ in Maclaurin series representation (as noticed from its condition of validity)

$$f(z) = \frac{-(z+1)}{1-z} = -(z+1)\frac{1}{1-z}$$

$$f(z) = -(z+1)\sum_{n=0}^{\infty} z^n \qquad (|z|<1)$$

$\because$ using $\frac{1}{1-z} = \sum z^n$ $(|z|<1)$

$$= -z\sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} z^n$$

$$= -\sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} z^n$$

replacing n by $n-1$ in the first series

$$f(z) = -\sum_{n=1}^{\infty} z^n - \sum_{n=0}^{\infty} z^n = -\sum_{n=1}^{\infty} z^n - 1 - \sum_{n=1}^{\infty} z^n$$

$$\boxed{f(z) = -1 - 2\sum_{n=1}^{\infty} z^n \qquad |z| < 1}$$

$D_2: 1 < |z| < \infty$

The representation of $f(z)$ in D_2 (unbounded domain) is a Laurent series, and since $1 < |z|$ in D_2, which implies that $\left|\frac{1}{z}\right| < 1$ when z is a point in D_2, thus we use series (1)

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

to write.

$$f(z) = \frac{z+1}{z-1} = \frac{z+1}{z\left(1-\frac{1}{z}\right)} = \frac{z+1}{z} \cdot \frac{1}{1-\frac{1}{z}}$$

$$= \left(1+\frac{1}{z}\right) \frac{1}{1-\frac{1}{z}}$$

$$= \left(1+\frac{1}{z}\right) \sum_{n=0}^{\infty} \frac{1}{z^n} \qquad (1 < |z| < \infty)$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

$D_2: 1 < |z| < \infty$

$\Rightarrow \frac{1}{|z|} < 1$

$\Rightarrow \left|\frac{1}{z}\right| < 1$

Thus we can use

$$\frac{1}{1-\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{z^n} \qquad \left|\frac{1}{z}\right| < 1$$

replacing n by $n-1$ in second series

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{1}{z^n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=1}^{\infty} \frac{1}{z^n}$$

$$\boxed{f(z) = 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n} \qquad (1 < |z| < \infty)}$$

Example-3.

$$f(z) = e^{1/z} \qquad \text{sing pt } z = 0$$

So the condition of validity is

$$0 < |z| < \infty$$

Since we have Mac. series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad |z| < \infty$$

$$\Rightarrow e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} \qquad 0 < |z| < \infty$$

$$= 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$$

$|z| < \infty$ for e^z

$\left|\frac{1}{z}\right| < \infty$ for $e^{1/z}$

$\Rightarrow \frac{1}{|z|} < \infty$

$\Rightarrow \frac{1}{\infty} < |z|$

$\Rightarrow |z| > 0$

$\Rightarrow 0 < |z| < \infty$

Since there are no positive powers of $z \Rightarrow a_n \neq 0$

note that coefficient of $\frac{1}{z}$ is unity and according to Laurent's theorem, that coefficient is the number b_1

$$b_n = \frac{1}{2\pi i} \int_c \frac{f(z)\,dz}{(z - z_0)^{-n+1}}$$

$$b_1 = \frac{1}{2\pi i} \int_c e^{1/z}\, dz$$

$$1 = \frac{1}{2\pi i} \int_c e^{1/z}\, dz$$

$$\Rightarrow \int_c e^{1/z}\, dz = 2\pi i$$