65. Negative Powers of $(z-z_0)$

Example-1

$$f(z) = \frac{e^{-z}}{z^2} \qquad \text{sing. pt } z=0$$

We obtain series representation of the given function by using Maclaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty)$$

$$\Rightarrow e^{-z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \qquad (|z| < \infty)$$

$$\Rightarrow \frac{e^{-z}}{z^2} = \frac{1}{z^2}\left[1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \cdots\right]$$

$$= \frac{1}{z^2} - \frac{1}{z^2}\cdot\frac{z}{1!} + \frac{1}{z^2}\cdot\frac{z^2}{2!} + \frac{1}{z^2}\cdot\frac{z^3}{3!} + \cdots$$

$$= \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{1}{3!}z + \frac{1}{4!}z^2 + \cdots$$

This series involves both the positive and negative powers of $(z-z_0)$.

The condition of validity is $(0 < |z| < \infty)$

* The singular point of the given fn $f(z) = \frac{e^{z}}{z^2}$ is $z=0$ and the given series is also centred about $z_0=0$. But this is not the Taylor series.

Example-2

Now $f(z) = z^3 \cosh\left(\frac{1}{z}\right)$ sing pt $z = 0$

Since $\cosh z$ is an entire fn but $z^3\cosh\left(\frac{1}{z}\right)$ has only one singularity i.e $z = 0$ so the condition of validity will be.

$(0 < |z| < \infty)$

Now using Maclaurin series

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty)$$

we have.

$$\cosh\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{2n!} \frac{1}{z^{2n}} \qquad (0 < |z| < \infty)$$

$$\Rightarrow z^3\cosh\left(\frac{1}{z}\right) = z^3 \sum_{n=0}^{\infty} \frac{1}{2n!} \frac{1}{z^{2n}} \qquad (0 < |z| < \infty)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2n!} \frac{1}{z^{2n-3}}$$

$$= \cancel{\frac{1}{z^3} + \frac{1}{2!\,z^5} + \frac{1}{4!\,z^7}}$$

$$= \frac{1}{z^{-3}} + \frac{1}{2!}\cdot\frac{1}{z^{-1}} + \frac{1}{4!}\frac{1}{z} + \frac{1}{6!}\frac{1}{z^3} + \cdots$$

$$= z^3 + \frac{z}{2!} + \frac{1}{4!}\frac{1}{z} + \frac{1}{6!}\frac{1}{z^3} + \cdots$$

we can write the above series as

$$z^3 \cosh\left(\frac{1}{z}\right) = z^3 + \frac{z}{2!} + \sum_{n=2}^{\infty} \frac{1}{2n!} \frac{1}{z^{2n-3}} \qquad (0<|z|<\infty)$$

To make a standard form of the above expansion, we can replace n by $n+1$

$$z^3 \cosh\left(\frac{1}{z}\right) = \frac{z}{2} + z^3 + \sum_{n+1=2}^{\infty} \frac{1}{2(n+1)!} \frac{1}{z^{2(n+1)-3}}$$

$$\boxed{z^3 \cosh\left(\frac{1}{z}\right) = \frac{z}{2} + z^3 + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!} \frac{1}{z^{2n-1}} \qquad (0<|z|<\infty)}$$

Example-3

$$f(z) = \frac{1+2z^2}{z^3+z^5}$$

We have to expand this function to make a convenient form so that we can use the already developed MacLaurin series. Thus

$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1+2z^2}{z^3(1+z^2)} = \frac{2(1+z^2)-1}{z^3(1+z^2)}$$

$$= \frac{1}{z^3} \frac{2(1+z^2)-1}{1+z^2} = \frac{1}{z^3}\left[2 - \frac{1}{1+z^2}\right]$$

We cannot find a Maclaurin series (or Taylor series) of the given function since it is not analytic at $z=0$. But we can use the Maclaurin series.

$$\frac{1}{1-z} = 1+z+z^2+z^3+\cdots \qquad |z|<1$$

replacing z by $-z^2$, we get

$$\frac{1}{1+z^2} = 1+(-z^2)+(-z^2)^2+(-z^2)^3+\cdots \qquad |z|<1$$

$$= 1-z^2+z^4-z^6+z^8-\cdots$$

So when $0<|z|<1$

$$f(z) = \frac{1}{z^3}\left[2-\left(1-z^2+z^4-z^6+z^8-\cdots\right)\right]$$

$$= \frac{1}{z^3}\left[2-1+z^2-z^4+z^6-z^8+\cdots\right]$$

$$= \frac{1}{z^3}\left[1+z^2-z^4+z^6-z^8+\cdots\right]$$

$$= \underbrace{\frac{1}{z^3}+\frac{1}{z}}_{\text{-ve powers of } z} \underbrace{-z+z^3-z^5+\cdots}_{\text{+ve powers of } z}$$

Example-4

We have to expand the function

$$f(z) = \frac{e^z}{(z+1)^2}$$

in powers of $z+1$.

We use Mac. series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty)$$

$$\Rightarrow e^{z+1} = \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!} \qquad (|z+1| < \infty)$$

Since $e^z = e^{z+1} \cdot e^{-1}$

$$\Rightarrow e^z = \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!} \cdot \frac{1}{e}$$

$$\Rightarrow \frac{e^z}{(z+1)^2} = \frac{1}{(z+1)^2} \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!} \cdot \frac{1}{e}. \qquad (0 < |z+1| < \infty)$$

$$= \frac{1}{e} \sum_{n=0}^{\infty} \frac{(z+1)^{n-2}}{n!}$$

$$= \frac{1}{e} \left[(z+1)^{-2} + (z+1)^{-1} + \sum_{n=2}^{\infty} \frac{(z+1)^{n-2}}{n!} \right]$$

$$= \frac{1}{e} \left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=2}^{\infty} \frac{(z+1)^{n-2}}{n!} \right].$$

or replacing n by $n+2$

$$\boxed{\frac{e^z}{(z+1)^2} = \frac{1}{e} \left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} \right]}$$

$(0 < |z+1| < \infty)$

Q10. Derive the expansions ⑥

(a) $$\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \quad (0 < |z| < \infty)$$

Soln

Since we have Maclaurin series

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)$$

$$\frac{1}{z^2}\sinh z = \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad \overset{*}{(0 < |z| < \infty)}$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n+1-2}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n-1}}{(2n+1)!}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n+1)!}$$

Replacing n by $n+1$, we have

$$\frac{1}{z^2}\sinh z = \frac{1}{z} + \sum_{n+1=1}^{\infty} \frac{z^{2(n+1)-1}}{(2(n+1)+1)!}$$

$$\boxed{\frac{1}{z^2}\sinh z = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \quad (0 < |z| < \infty)}$$

(b) Do yourself.

Q11 Show that when $0 < |z| < 4$,

$$\frac{1}{4z-z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

Soln:

$$f(z) = \frac{1}{4z-z^2} = \frac{1}{4z\left(1-\frac{z^2}{4z}\right)} = \frac{1}{4z\left(1-\frac{z}{4}\right)}$$

$4z - z^2 = 0$
$z(4-z) = 0$
$z = 0,\ z = 4$
sing. pts.

By using MacLaurin series.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

we have

$$\frac{1}{1-\frac{z}{4}} = \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n \qquad |z| < 4$$

$\left|\frac{z}{4}\right| < 1$
$\Rightarrow |z| < 4$

Thus

$$\frac{1}{4z-z^2} = \frac{1}{4z} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n \qquad (0 < |z| < 4)$$

$$= \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$$

$$= \frac{1}{z \cdot 4} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{4^{n+1}} \qquad (0 < |z| < 4)$$

$$\boxed{\frac{1}{4z-z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}} \qquad (\because \text{replace } n \text{ by } n+1)$$