CHAPTER-5 SERIES

60. Convergence of Sequences

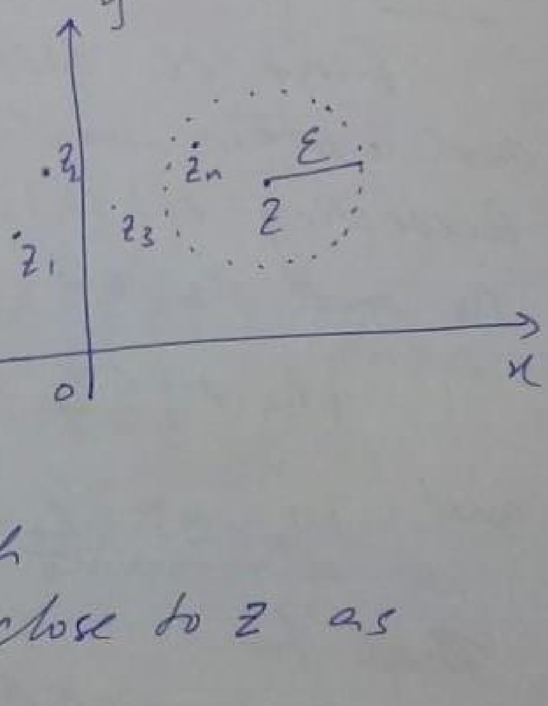
Definition

An infinite sequence $z_1, z_2 \cdots z_n \cdots$ of a complex numbers has a limit z if $\forall \varepsilon > 0$. There exist a positive integer n_0 such that

$$|z_n - z| < \varepsilon \quad \text{whenever } n > n_0 \quad \text{--- (1)}$$

When the limit z exists, the sequence is said to converge to z and written as

$$\lim_{n \to \infty} z_n = z$$



Geometrically, this means that for sufficiently large values of n. The points z_n lie in ε-nbhd of z (as in fig). Since ε can be chosen very small, for which the points z_n become arbitrary close to z as their subscript increases.

Note that:

1) The value of n_0 in (1) generally depends upon the value of ε
2) The sequence can have at most one limit.

i.e limit z is unique if it exists.

3) If the sequence has no limit, it diverges.

Theorem.

Suppose that $z_n = x_n + iy_n \quad (n = 1, 2, \ldots)$

and $z = x + iy$.

Then
$$\lim_{n \to \infty} z_n = z \qquad (3)$$
if and only if
$$\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y \qquad (4)$$

Proof:

First we assume that condition (4) holds and obtain condition (3) from it.

According to (4), for each $\varepsilon > 0$, $\exists$ positive integers n_1 and n_2 s.t.

$$|x_n - x| < \frac{\varepsilon}{2} \text{ whenever } n > n_1$$

$\because \lim_{n \to \infty} z_n = z$
$\Rightarrow |z_n - z| < \varepsilon$
whenever $n > n_0$

and $|y_n - y| < \frac{\varepsilon}{2}$ whenever $n > n_2$

Then for

$n_0 = \max(n_1, n_2)$, we have

$$|x_n - x| < \frac{\varepsilon}{2} \text{ and } |y_n - y| < \frac{\varepsilon}{2} \text{ whenever } n > n_0$$

Since

$z_n = x_n + iy_n$ & $z = x + iy$

$|z_n - z| = |(x_n + iy_n) - (x + iy)|$

$$|z_n - z| = |(x_n - x) + i(y_n - y)|$$
$$\leq |x_n - x| + |i(y_n - y)| \quad \left(\because \text{Using triangular inequality}\right)$$
$$= |x_n - x| + |y_n - y| \quad (\because |iy| = |i||y| = |y|)$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow |z_n - z| < \varepsilon$ whenever $n > n_0$

Thus equation (3) holds

Now conversely, we start with condition (3).

According to it, for each $\varepsilon > 0$, there exist a $n_0 > 0$ s.t.

$$|z_n - z| < \varepsilon \quad \text{whenever } n > n_0$$

$$\Rightarrow |(x_n + iy_n) - (x + iy)| < \varepsilon$$
$$\Rightarrow |(x_n - x) + i(y_n - y)| < \varepsilon$$

Since

$$\left.\begin{aligned} |x_n - x| &= |Re(z_n - z)| \leq |z_n - z| < \varepsilon \\ \text{and } |y_n - y| &= |Im(z_n - z)| \leq |z_n - z| < \varepsilon \end{aligned}\right\} \forall n > n_0$$

$\Rightarrow |x_n - x| < \varepsilon$ & $|y_n - y| < \varepsilon$ whenever $n > n_0$

$\Rightarrow \lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$

Thus condition (4) holds.

By the theorem, we can write.

$$\lim_{n \to \infty}(x_n + iy_n) = \lim_{n \to \infty} x_n + i \lim_{n \to \infty} y_n$$

Example - 1

$$\lim_{n\to\infty}\left(-1 + i\frac{(-1)^n}{n^2}\right) = \lim_{n\to\infty}(-1) + i\lim_{n\to\infty}\frac{(-1)^n}{n^2}$$

$$= -1 + i(0)$$

$$= -1$$

Definition (1) can also be used

$|z_n - z| < \varepsilon$ whenever $n > n_0$

$$\Rightarrow \left|-1 + i\frac{(-1)^n}{n^2} - (-1)\right| = \left|i\frac{(-1)^n}{n^2}\right| = |i|\left|\frac{(-1)^n}{n^2}\right|$$

$$= \frac{1}{n^2} < \varepsilon$$

$\Rightarrow |z_n - (-1)| < \varepsilon$ whenever $n > \frac{1}{\sqrt{\varepsilon}}$

Example - 2

$$z_n = -1 + i\frac{(-1)^n}{n^2}.$$

If we use polar coordinates.

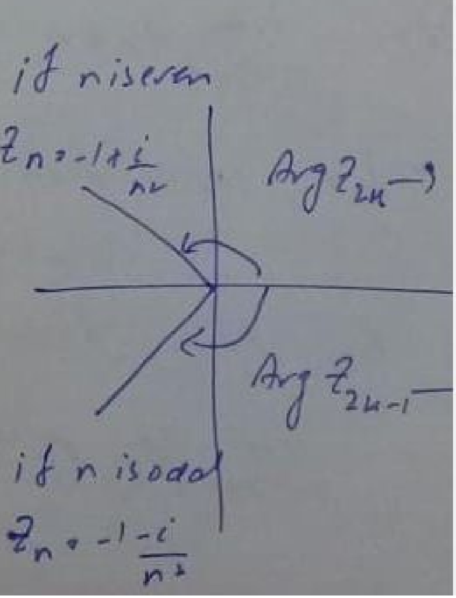
$$r_n = |z_n| = \left|-1 + i\frac{(-1)^n}{n^2}\right| = \sqrt{(-1)^2 + \left(\frac{(-1)^n}{n^2}\right)^2} = \sqrt{1 + \frac{1}{n^4}}$$

& $\lim_{n\to\infty} r_n = \lim_{n\to\infty}\sqrt{1 + \frac{1}{n^4}} = 1$

and $\Theta_n = \operatorname{Arg} z_n \quad (-\pi < \Theta_n \le \pi)$.

$\lim_{n\to\infty}\Theta_{2n} = \pi$ & $\lim_{n\to\infty}\Theta_{2n-1} = -\pi$

$\Rightarrow \lim_{n\to\infty}\Theta_n$ does not exist



6.1 Convergence of Series

Definition:

An infinite series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots \quad \text{——— (1)}$$

of complex numbers converges to the sum S if the sequence of partial sums

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N \quad N = 1, 2, 3 \text{ ——— (2)}$$

$S_1 = z_1$
$S_2 = z_1 + z_2$
$S_3 = z_1 + z_2 + z_3$
$\vdots$

converges to S

i.e $\lim_{N \to \infty} S_N = S$.

We then write

$$\sum_{n=1}^{\infty} z_n = S$$

\# Note that

i) A series can have atmost one sum.

ii) When a series does not converge, we say that it diverges.

Theorem: Suppose that

$z_n = x_n + i y_n \quad (n = 1, 2, \ldots)$ and $S = X + iY$.

Then

$$\sum_{n=1}^{\infty} z_n = S \quad \text{——— (3)}$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y \quad \text{——— (4)}$$

To prove the theorem, we write the partial sums (2) as

$$S_N = X_N + i Y_N \quad \text{—— (5)}$$

where

$$X_N = \sum_{n=1}^{N} x_n \quad \text{and} \quad Y_N = \sum_{n=1}^{N} y_n.$$

Now statement (3) is true if and only if

$$\lim_{N\to\infty} S_N = S \quad \text{—— (6)}$$

$\left(\because \sum_{n=1}^{\infty} z_n = S \Leftrightarrow \lim_{N\to\infty} S_N = S \right.$

and in view of relation (5) and the theorem of seq. in Sec. 60, limit in (6) holds

$$\Leftrightarrow \lim_{N\to\infty} X_N = X \quad \text{and} \quad \lim_{N\to\infty} Y_N = Y \quad \text{—— (7)}$$

$$\Rightarrow \sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y$$

$$\Rightarrow \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n = X + iY$$

$$\Rightarrow \sum_{n=1}^{\infty} (x_n + i y_n) = X + iY$$

$$\Rightarrow \sum_{n=1}^{\infty} z_n = S$$

Thus (3) is established.

Corollary 1.

If a series of complex numbers converges. The nth term converges to zero as n tends to infinity.

Proof:

We assume that series of complex numbers $\sum_{n\geq 1}^{\infty} z_n$ converges.

Now since $z_n = x_n + iy_n$.

Then from the above theorem, each of the series

$$\sum_{n\geq 1}^{\infty} x_n \quad \text{and} \quad \sum_{n\geq 1}^{\infty} y_n$$

also converges.

Since from calculus, we know that the nth term of a convergent series of real number approaches zero as n tends to infinity

i.e $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$

Thus

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(x_n + iy_n)$$

$$= \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

$$= 0 + i.0 = 0$$

Hence the corollary is proved.

Definition Absolutely Convergent Series

A series $\sum_{n=1}^{\infty} z_n$ is said to be absolutely convergent series if the series of real numbers $\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}$ converges.

Corollary 2:

The absolute convergence of a series of complex numbers implies the convergence of that series.

Exercise

Q1. Use def. (1), Sec. 60 of limits of seq. to show that $\lim_{n \to \infty} \left(\frac{1}{n^2} + i\right) = i$

Soln.
Using def (1), Sec. 60

$$|z_n - z| < \varepsilon \quad \text{whenever} \quad n > n_0$$

$$\left|\frac{1}{n^2} + i - i\right| = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \varepsilon$$

Now from $\frac{1}{n^2} < \varepsilon \Rightarrow \frac{1}{\varepsilon} < n^2 \Rightarrow \frac{1}{\sqrt{\varepsilon}} < n$

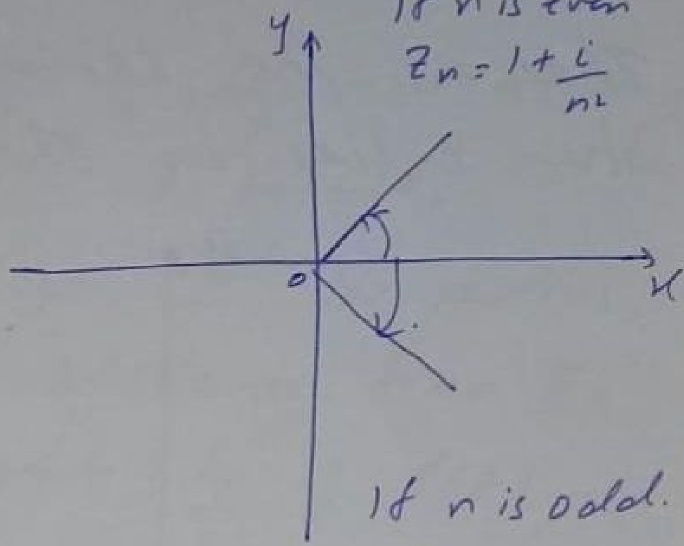
Thus

$$\left|\left(\frac{1}{n^2} + i\right) - i\right| < \varepsilon \quad \text{whenever} \quad n > \frac{1}{\sqrt{\varepsilon}}$$

$$\Rightarrow \lim_{n \to \infty} \left(\frac{1}{n^2} + i\right) = i$$

Q.2. Let $z_n = 1 + i\frac{(-1)^n}{n^2}$ $(n = 1, 2, \ldots)$

Let $\Theta_n = Arg(z_n)$ $(-\pi < \Theta_n \leq \pi)$



When n is even

$z_n = 1 + \frac{i}{n^2}$.

$\lim_{n \to \infty} z_n = 1.$

$\lim_{n \to \infty} \Theta_{2n} = 0$

when n is odd.

$z_n = 1 - \frac{i}{n^2}$

$\lim_{n \to \infty} z_n = 1.$ & $\lim_{n \to \infty} \Theta_{2n-1} = 0$

Thus $\lim_{n \to \infty} \Theta_n = 0$

62. Taylor Series

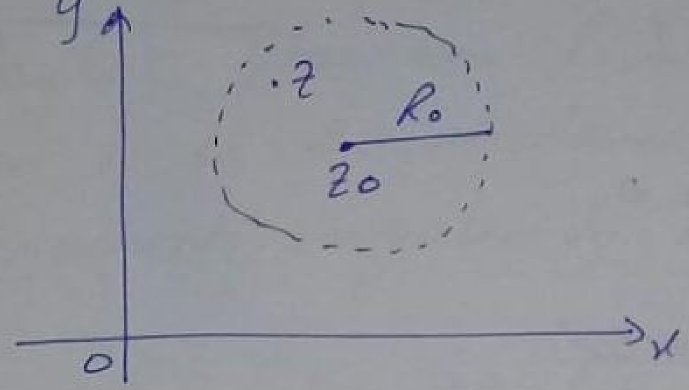
Theorem: Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$ centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < R_0),$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, 3 \ldots) \qquad (2)$$

That is, the series (1) converges to $f(z)$ when z lies in the stated open disk.



Series (1) can also be written as

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \quad \text{— (3)} \quad (|z - z_0| < R_0)$$

We usually refer this as Taylor series expansion of $f(z)$ about the point z_0.

Note that

(i) If f is analytic at a point z_0
$\Rightarrow$ f is analytic in the ε-nbhd of z_0
i.e $|z - z_0| < \varepsilon$
$\Rightarrow$ f has Taylor series expansion about z_0 and in that case ε is served as R_0 in the statement of Taylor's theorem

(ii) If <u>f is entire</u>, R_0 can be chosen arbitrarily large and the condition of validity becomes $|z - z_0| < \infty$

(iii) No convergence test is required as long as f is analytic in $|z - z_0| < R_0$

(iv) In Taylor's theorem, R_0, the radius, is taken to be the distance from z_0 to the nearest singularity of $f(z)$.

(v) When $z_0 = 0$, f is analytic in $|z| < R_0$, the series become

(4) — $$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \qquad (|z| < R_0)$$

which is called the Maclaurin Series.

(63. No need of the proof)

64. <u>Examples</u>

In this section you will developed the Mac. Series, centered at $z_0 = 0$. for various fns. 6 Maclaurin series representation are given in this sec. I will derive some. The rest you will derive yourself.

Example 1

$$f(z) = \frac{1}{1-z}$$

The given fn. has only one singularity $z=1$.
So the desired Maclaurin series converges to $f(z)$ when $|z-0| < 1 \Rightarrow |z| < 1$.

Taking derivative of the given fn.

$$f'(z) = -1(1-z)^{-2}(-1) \qquad f'(0) = 1!$$

$$= \frac{1}{(1-z)^2} = \frac{1!}{(1-z)^{1+1}}$$

$$f''(z) = \frac{2}{(1-z)^3} = \frac{2!}{(1-z)^{2+1}} \qquad f''(0) = 2!$$

$$f'''(z) = \frac{2 \cdot 3}{(1-z)^4} = \frac{3!}{(1-z)^{3+1}} \qquad \vdots$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \qquad f^{(n)}(0) = n!$$

Substituting the values, we get

$$f(z) = f^{(0)}(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots$$

$$= 1 + \frac{1!}{1!} z + \frac{2!}{2!} z^2 + \frac{3!}{3!} z^3 + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} z^n$$

or by directly substituting the value of $f^{(n)}(0) = n!$ in expression (4) gives

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \sum_{n=0}^{\infty} z^n$$

$$(1) - \boxed{\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n} \quad (|z| < 1)$$

Maclaurin series of $f(z) = \frac{1}{1-z}$ centered at $z_0 = 0$.

Now to derive the Maclaurin series of $f(z) = \frac{1}{1+z}$, we can adopt the same method as we do for $f(z) = \frac{1}{1-z}$ or else we replace z by $-z$ in equation (1) and get

$$\frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n \qquad |-z| < 1$$

$$\boxed{\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad |z| < 1}$$

Since the function $f(z) = \frac{1}{1+z}$ has the singularity $z=-1$ and the distance from $z_0=0$ to $z=-1$ is 1 so $R_0 = 1$.
So the condition of validity will be
$|z| < 1$.

\# $f(z) = \frac{1}{z}$ singularity is $z=0$

Since the sing. pt of $f(z)$ is $z=0$, so we cannot develop its Mac. series about $z_0=0$. We developed its Taylor series representation.
For this replace z by $1-z$ in (1)

from (1) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

$$\frac{1}{1-(1-z)} = \sum_{n=0}^{\infty} (1-z)^n \qquad |1-z| < 1$$

$$\frac{1}{z} = \sum_{n=0}^{\infty} \left(-(z-1)\right)^n \qquad |(-1)(z-1)| < 1$$

Since for Taylor series, we need terms in the form of $z-z_0$, so

$$\boxed{\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \qquad (|z-1| < 1)}$$

$f(z) = \frac{1}{1-z}$

We now have to find the Taylor series representation of the given fn. about the point $z_0 = i$.

Since the sing. pt is $z = 1$. The distance between $z_0 = i$ to $z = 1$ is

$$|i-1| = \sqrt{(1)^2+(1)^2} = \sqrt{2}$$

So we set $R_0 = \sqrt{2}$.

Thus the condition of validity is $|z-i| < \sqrt{2}$

To find the Taylor series about $z_0 = i$, the series must include the terms of form $z-i$. For this we write

$$\frac{1}{1-z} = \frac{1}{1-z+i-i} = \frac{1}{(1+i)-(z-i)}$$

$$= \frac{1}{1+i\left(1-\frac{z-i}{1+i}\right)} \quad (\because \text{ taking } 1+i \text{ common})$$

$$= \frac{1}{1+i} \cdot \frac{1}{1-\frac{z-i}{1+i}}$$

$$= \frac{1}{1+i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1+i}\right)^n \quad \left(\because^{*} \text{ Using } \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z|<1)\right)$$

$$\boxed{\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1+i)^{n+1}} \qquad |z-i| < \sqrt{2}}$$

The reason for using the Mac. series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

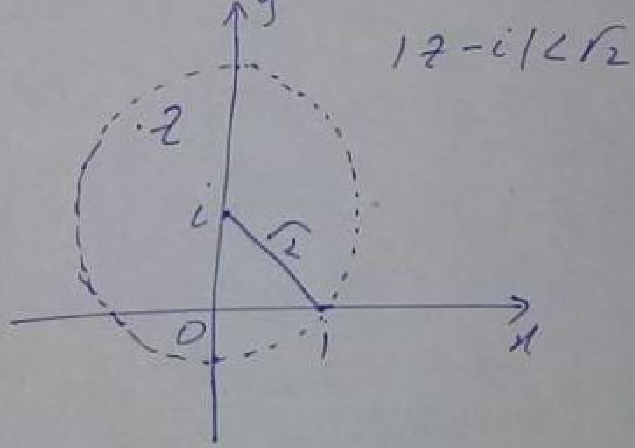
for the given fn. is that its condition $|z| < 1$ also holds for

$$\frac{1}{1 - \frac{z-i}{1-i}},$$

since

$$\left|\frac{z-i}{1-i}\right| = \frac{|z-i|}{|1-i|} = \frac{|z-i|}{\sqrt{2}} < 1.$$

$$\Rightarrow \left|\frac{z-i}{1-i}\right| < 1.$$



Example - 2

$f(z) = e^z$ is entire, so $R_0 = \infty$

the condition of validity is $|z| < \infty$ for deriving its Maclaurin series.

Since

$$f^{(n)}(z) = e^z \qquad (n = 0, 1, 2, \ldots)$$

& $\quad f^{(n)}(0) = e^0 = 1$

So the Mac series will be

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \qquad |z-0| < R_0$$

$$\boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty)}$$

To find Mac series of $z^3 e^{2z}$, we replace z by $2z$ and get

$$e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}$$

multiplying by z^3

$$z^3 e^{2z} = z^3 \sum \frac{2^n z^n}{n!} \qquad (|z| < \infty)$$

$$z^3 e^{2z} = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{n+3}$$

or by replacing n by $n-3$, we have.

$$z^3 e^{2z} = \sum_{n-3=0}^{\infty} \frac{2^{n-3}}{(n-3)!} z^{n-3+3}$$

$$\boxed{z^3 e^{2z} = \sum_{n=3}^{\infty} \frac{2^{n-3}}{(n-3)!} z^n \qquad (|z| < \infty)}$$

The condition of validity ($|z| < \infty$) remains the same because $z^3 e^{2z}$ is also entire.

Example 3, 4, 5, 6

Do yourself.

Exercise Pg # 195

Q1, 2, 3, 4, 5, 7, 8, 9