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Cauchy Integral Formula.

Theorem: Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0} \qquad (1)$$

Expression (1) is called the Cauchy Integral formula.

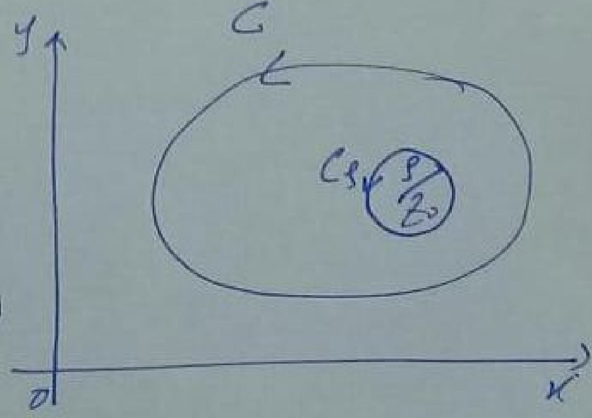
Proof:

Since the integrand $\frac{f(z)}{z-z_0}$ is analytic everywhere inside C and on C except at the point z_0 which lies inside C. Let C_ρ be a positively oriented circle with radius ρ, and centered at z_0. Suppose further that ρ is sufficiently small that C_ρ lies inside C. Equation for C_ρ is

$$|z - z_0| = \rho$$

or $z = z_0 + \rho e^{i\theta}$

$(0 \le \theta \le 2\pi)$



Now since $\frac{f(z)}{z-z_0}$ is analytic on C and C_ρ and between the contours C and C_ρ, it follows from the principle of deformation of paths

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)\,dz}{z-z_0} \quad \text{— (2)}$$

Considering R.H.S of (2)

$$\int_{C_\rho} \frac{f(z)\,dz}{z-z_0} = \int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})}{\rho e^{i\theta}}\,\rho i e^{i\theta}\,d\theta \quad \Big| \; z = z_0+\rho e^{i\theta}, \; dz = \rho i e^{i\theta} d\theta$$

$$= i\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta$$

Thus equation (2) takes the form

$$\int_C \frac{f(z)}{z-z_0}\,dz = i\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta$$

Since we can reduce ρ to zero so

$$\int_C \frac{f(z)}{z-z_0}\,dz = i \lim_{\rho\to 0}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta$$

$$= i\int_0^{2\pi} f(z_0)\,d\theta$$

$$= i\, f(z_0)\int_0^{2\pi} d\theta$$

$$\int_C \frac{f(z)\,dz}{z-z_0} = 2\pi i\, f(z_0)$$

Thus (1) is established.