Cauchy Goursat Theorem

We let C denote a simple closed contour $z = z(t)$ $(a \le t \le b)$, positively oriented, and we suppose that f is analytic at each point interior to and on C (*means that the derivative of the function exist at each point on and inside C).
Now since, we read earlier (in Sec. 44)

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt; \quad \text{— (1)}$$

Moreover, if

$$f(z) = u(x,y) + i v(x,y) \quad \text{and}$$

$$z(t) = x(t) + i y(t).$$

The integrand $f[z(t)]\,z'(t)$ in (1) is the product of the functions

$$u(x(t), y(t)) + i v(x(t), y(t)) \quad \text{and} \quad x'(t) + i y'(t)$$

of real variable t.
Thus

$$f[z(t)]\,z'(t) = (u + iv)(x'(t) + i y'(t))$$

$$= ux' - vy' + i(uy' + vx')$$

$$\int_C f(z)\,dz = \int_a^b (ux' - vy') + i(uy' + vx')\,dt \quad \text{— (2)}$$

In terms of line integrals of real-valued functions of two real variables (*using def in (1)), then.

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C v\,dx + u\,dy. \quad — (3)$$

Equation (3) can also be obtained simply by replacing $f(z)$ and dz on the left with the binomials

$$u + iv \quad \text{and} \quad dx + idy$$

| * $f(z) = u + iv$
| $dz = dx + idy$

respectively. and then expanding their products.

Expression (3) is also valid when C is any contour, not necessarily a simple closed one (*although in the start we consider C just as simple closed contour), and when $f[z(t)]$ is only piecewise continuous on it.

We now convert the line integrals on the right side of expression (3) into the double integrals by using a result from Calculus. Suppose that $P(x,y)$ and $Q(x,y)$, two real-valued functions, together with their first-order partial derivatives, are continuous throughout the closed region R. which is consisting of all points interior to and on a simple closed contour C.

According to Green's Theorem

$$\int_C (P\,dx + Q\,dy) = \iint_R (Q_x - P_y)\,dA$$

Now since f is analytic inside and on C (*according to the hypothesis of the theorem), so f is continuous on R (*which consist of interior and boundary of C). Hence the function u and v are also continuous on R. Similarly, if the derivative f' of f is continuous on R, so are the first-order partial derivatives of u and v. Since u and v are fullfilling the hypothesis of Green's theorem, so Green's theorem enables us to write equation (3) as

$$\int_C f(z)\,dz = \iint_R (-v_x - u_y)\,dA + i \iint_R (u_x - v_y)\,dA \quad \text{—— (4)}$$

But in view of the Cauchy Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x,$$

the integrands of these two double integrals in equation (4) are zero throughout R. So "when f is analytic in R and f' is continuous there,

$$\int_C f(z)\,dz = 0 \quad \text{—— (5)}$$

Once it has been established that value of integral is zero, the orientation of C doesn't matter.

The result in equation (5) was obtained by Cauchy in early part of nineteenth century. Later, Goursat ommit "the condition of continuity on f'". Since the derivative f' of an analytic function f is also analytic and the analyticity implies the continuity. We now state the revised form of Cauchy's result, known as Cauchy-Goursat theorem.

Theorem

If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

(*Skip the proof in Sec. 51)

Example

$$f(z) = \sin z^2$$

Since $f(z)$ is a composite function of two entire functions $\sin z$ and z^2, which are analytic throughout the entire complex plane. Considering C any simple closed contour lying in D

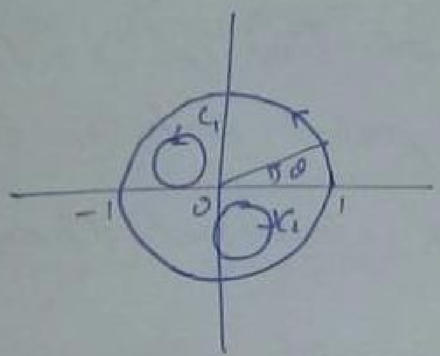
$$\int_C \sin z^2 = 0$$ according to Cauchy-Goursat theorem

Simply Connected Domains

A simply connected domain is a domain such that every simple closed contour within it encloses only points of D. The set of points interior to a simple closed contour is an example

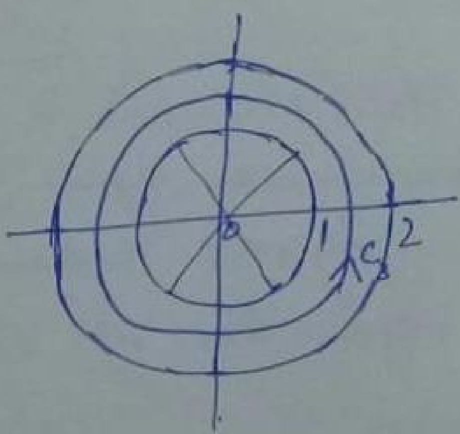
Consider

$z = e^{i\theta}$ $\quad (0 \le \theta \le 2\pi)$



The contours C_1 and C_2 lying in this domain contains only the points of this domain.

If we take annular domain between two concentric circles, e.g $1 \le |z| \le 2$, it is, however not simply connected.



The area that is crossed is not the part of annular domain $1 \le |z| \le 2$. If we take any simple closed contour C_3 in this annular domain, it consists of the points of annular domain, as well as the points not in the annular domain (the portion that is crossed)

Theorem (Cauchy - Goursat Theorem for Simply - Connected Domain)

If a function f is analytic throughout a simply connected domain D, then

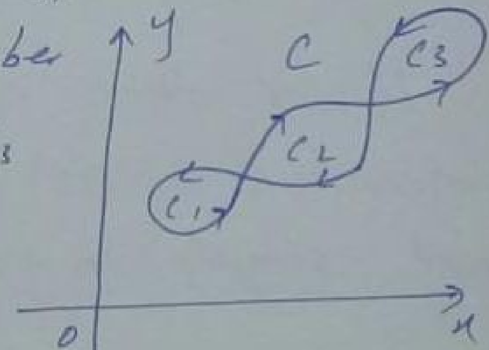
$$\int_C f(z)\,dz = 0 \qquad (1)$$

for every closed contour C lying in D.

Proof.

The proof is easy if C is a simple closed contour. For if C is simple and lies in D, the function f is analytic at each point interior to and on C, and according to Cauchy - Goursat theorem, expression (1) holds.

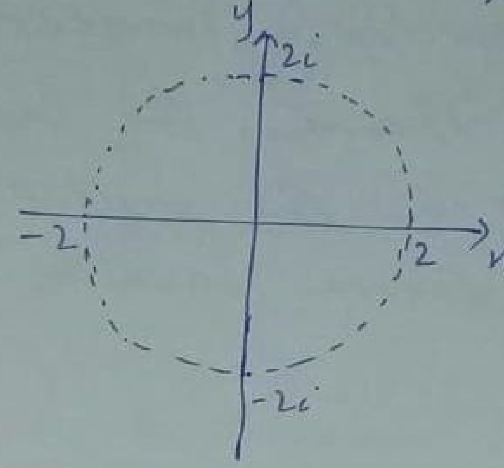
For if C is closed but not simple and intersects itself a finite number of times, then it consists of a finite number of simple closed contours C_1, C_2, C_3. Since f is analytic on C and interior to C, then f is analytic at on and interior to each simple closed contour C_1, C_2 & C_3, and the Cauchy - Goursat theorem can again be applied. Since the values of integrals around each C_1, C_2 and C_3 are zero, regardless of their orientation,



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz = 0$$

Example

Since $|z|<2$ is the simply connected domain



The function $f(z) = \dfrac{\sin z}{(z^2+9)^5}$ has singular points

$z = \pm 3i$, which lie outside this simply connected domain, so if we consider any simple closed contour lying totally in this domain, then according to above theorem

$(z^2+9)^5 = 0$

$z^2+9 = 0$

$z^2 = -9$

$z = \pm 3i$

$$\int_c \frac{\sin z}{(z^2+9)^5}\, dz = 0$$

Corollary 1:

A function that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.

Corollary 2:

Entire functions always possess antiderivatives.

3.1 Multiply Connected Domains

A domain that is not simply connected is said to be multiply connected domain. In the following theorem, the Cauchy-Goursat theorem is adapted to the multiply connected domain, where the theorem involves n contours C_k $(k=1, 2, \ldots n)$

Theorem.

Suppose that

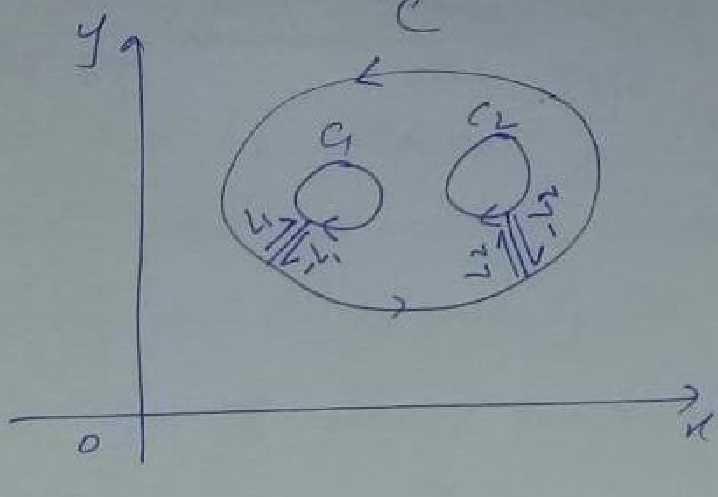
(a) C is a simple closed contour, described in the counter clockwise direction;

(b) C_k $(k=1, 2, \ldots n)$ are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no point in common.

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k, then

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0 \qquad (1)$$

Proof



Let L_k be polygonal line joining C to C_k, $k=1,2,\dots n$ in the multiply connected domain such that L_k has no self intersection and all L_k are disjoint. Thus a simple closed contour Γ can be formed, such that

$$\Gamma = C + L_1 + C_1 + (-L_1) + L_2 + C_2 + (-L_2) + \dots + L_n + C_n + (-L_n)$$

Then by Cauchy Goursat theorem.

$$\int_\Gamma f(z)\,dz = 0$$

$$\Rightarrow \left[\int_C + \int_{L_1} + \int_{C_1} + \int_{-L_1} + \dots + \int_{L_n} + \int_{C_n} + \int_{-L_n}\right] f(z)\,dz = 0$$

$$\Rightarrow \left[\int_C + \int_{C_1} + \int_{C_2} + \dots + \int_{C_n}\right] f(z)\,dz = 0$$

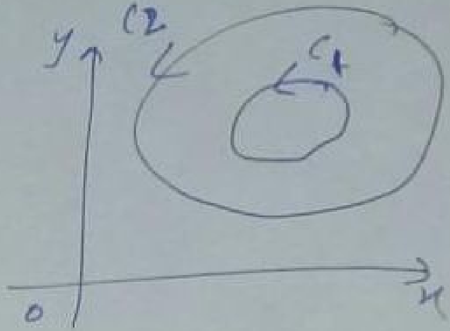
$$\Rightarrow \int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$$

* $\int_{L_1} f(z)\,dz$ * $\int_{-L_1} f(z)\,dz$ cancels out each other.

Corollary: Principal of Deformation of Paths

Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If a function 'f' is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \qquad (2)$$



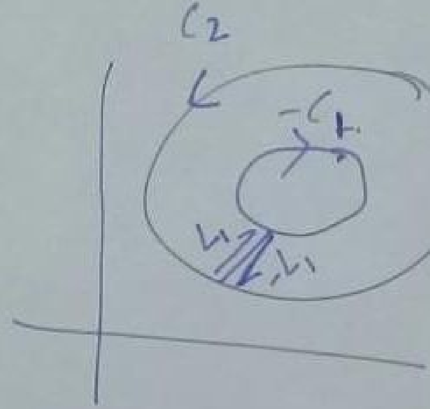
Proof: By the above theorem

$$\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Hence proved (2).



$$\Gamma = C_1 + L_1 + (-C_2) + (-L_1)$$

$$\int_{\Gamma} f(z)\,dz = 0$$

$$\Rightarrow \left[\int_{C_1} + \int_{L_1} + \int_{-C_2} + \int_{-L_1}\right] f(z)\,dz = 0$$

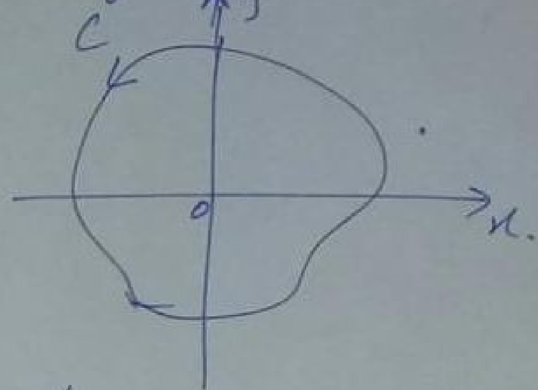
$$\Rightarrow \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

Example

C - any positively oriented simple closed contour surrounding the origin.

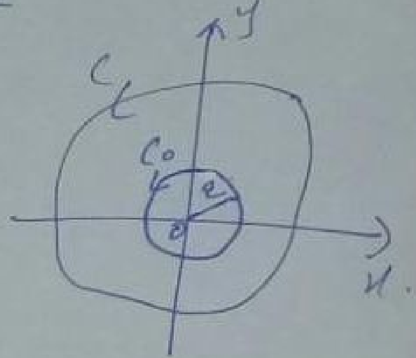
Then show that

$$\int_C \frac{dz}{z} = 2\pi i$$



Solution

Choose C_0 be a positively oriented circle with center at the origin with radius so small $\varepsilon > 0$ such that C_0 lies entirely in C.



Let the parametric equation of C_0

$$z = \varepsilon e^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

Then by the principal of deformation of paths.

$$\int_C \frac{dz}{z} = \int_{C_0} \frac{dz}{z}$$

$$= \int_0^{2\pi} \frac{1}{\varepsilon e^{i\theta}} \cdot i\varepsilon e^{i\theta}\, d\theta$$

$$= i\int_0^{2\pi} d\theta = i(2\pi)$$

$\because f(z) = \frac{1}{z}$ is analytic on both C and C_0 and the region between them

Ex Q1, 2, 3