4. Use the theorem in Sec. 77, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

$$(a)\ \frac{z^5}{1-z^3}; \qquad (b)\ \frac{1}{1+z^2}; \qquad (c)\ \frac{1}{z}.$$

Ans. $(a)\ -2\pi i$; $(b)\ 0$; $(c)\ 2\pi i$.

5. Let C denote the circle $|z| = 1$, taken counterclockwise, and use the following steps to show that

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)!}.$$

(a) By using the Maclaurin series for e^z and referring to Theorem 1 in Sec. 71, which justifies the term by term integration that is to be used, write the above integral as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(b) Apply the theorem in Sec. 76 to evaluate the integrals appearing in part (a) to arrive at the desired result.

6. Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points $z_1, z_2, \ldots, z_n$. Show that

$$\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

7. Let the degrees of the polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m \qquad (b_m \neq 0)$$

be such that $m \geq n + 2$. Use the theorem in Sec. 77 to show that if all of the zeros of $Q(z)$ are interior to a simple closed contour C, then

$$\int_C \frac{P(z)}{Q(z)}\, dz = 0.$$

[Compare with Exercise 4(b).]

78. THE THREE TYPES OF ISOLATED SINGULAR POINTS

We saw in Sec. 75 that the theory of residues is based on the fact that if f has an isolated singular point at z_0, then $f(z)$ has a Laurent series representation

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots$$

in a punctured disk $0 < |z - z_0| < R_2$. The portion

$$\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots \tag{2}$$

of the series, involving negative powers of $z - z_0$, is called the ***principal part*** of f at z_0. We now use the principal part to identify the isolated singular point z_0 as one of three special types. This classification will aid us in the development of residue theory that appears in following sections.

There are two extremes, the case in which every coefficient in the principal part (2) is zero and the case in which an infinite number of them are nonzero.

(*a*) *Removable Singular Points*

When every b_n is zero, so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (0 < |z - z_0| < R_2), \tag{3}$$

z_0 is known as a ***removable singular point***. Note that the residue at a removable singular point is always zero. If we define, or possibly redefine, f at z_0 so that $f(z_0) = a_0$, expansion (3) becomes valid throughout the entire disk $|z - z_0| < R_2$. Since a power series always represents an analytic function interior to its circle of convergence (Sec. 71), it follows that f is analytic at z_0 when it is assigned the value a_0 there. The singularity z_0 is, therefore, *removed*.

(*b*) *Essential Singular Points*

If an infinite number of the coefficients b_n in the principal part (2) are nonzero, z_0 is said to be an ***essential singular point*** of f.

(*c*) *Poles of Order m*

If the principal part of f at z_0 contains at least one nonzero term but the number of such terms is only finite, then there exists a positive integer m $(m \geq 1)$ such that

$$b_m \neq 0 \quad \text{and} \quad b_{m+1} = b_{m+2} = \cdots = 0.$$

That is, expansion (1) takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad (0 < |z - z_0| < R_2), \tag{4}$$

where $b_m \neq 0$. In this case, the isolated singular point z_0 is called a ***pole of order m***.* A pole of order $m = 1$ is usually referred to as a ***simple pole***.

In the next section, we shall give examples of these three types of isolated singular points; and in the remaining sections of the chapter, we shall examine in greater depth the theory of the three types of isolated singular points just described. The emphasis will be on useful and efficient methods for identifying poles and finding the corresponding residues.

The final section (Sec. 84) of the chapter includes three theorems that point out fundamental differences in the behavior of functions at the three types of isolated singular points.

79. EXAMPLES

The examples in this section illustrate the three types of isolated singularities described in Sec. 78.

EXAMPLE 1. The point $z_0 = 0$ is a removable singular point of the function

$$f(z) = \frac{1 - \cosh z}{z^2} \tag{1}$$

because

$$f(z) = \frac{1}{z^2}\left[1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right)\right] = -\frac{1}{2!} - \frac{z^2}{4!} - \frac{z^4}{6!} - \cdots \qquad (0 < |z| < \infty).$$

When the value $f(0) = -1/2$ is assigned, f becomes entire.

EXAMPLE 2. We recall from Example 3 in Sec. 68 that

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n} = 1 + \frac{1}{1!} \cdot \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \cdots \qquad (0 < |z| < \infty), \tag{2}$$

and it follows that $e^{1/z}$ has an essential singularity at $z_0 = 0$, where the residue b_1 is unity.

This example can be used to illustrate an important result known as ***Picard's theorem***. It concerns the behavior of a function near an essential singular point and states that *in each neighborhood of an essential singular point, a function assumes every finite value, with one possible exception, an infinite number of times.*†

*The reason for the terminology *pole* is pointed out on pp. 348–349 of the book (2005) by A. D. Wunsch as well as on p. 62 of the one (2010) by R. P. Boas, both of which are listed in Appendix 1. Also, the reason will be touched on in Sec. 84.

†For a proof of Picard's theorem, see Sec. 51 in Vol. III of the book by Markushevich, cited in Appendix 1.

It is easy to see, for instance, that $e^{1/z}$ assumes the value -1 an infinite number of times in each neighborhood of the origin. More precisely, since $e^z = -1$ when

$$z = (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

(see Sec. 30), it follows that $e^{1/z} = -1$ when

$$z = \frac{1}{(2n+1)\pi i} \cdot \frac{i}{i} = -\frac{i}{(2n+1)\pi} \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

So if n is large enough, an infinite number of such points lie in any given ε neighborhood of the origin. Zero is evidently the exceptional value when Picard's theorem is applied to $e^{1/z}$ at the origin.

EXAMPLE 3. From the representation

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}(1 + z + z^2 + z^3 + z^4 + \cdots) \tag{3}$$

$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots \quad (0 < |z| < 1),$$

one can see that f has a pole of order $m = 2$ at the origin and that

$$\operatorname*{Res}_{z=0} f(z) = 1.$$

From the limit

$$\lim_{z \to 0} \frac{1}{f(z)} = \lim_{z \to 0} [z^2(1-z)] = 0,$$

it follows that (see Sec. 17)

$$\lim_{z \to 0} f(z) = \infty. \tag{4}$$

Such a limit always occurs at poles, as will be shown in Sec. 84.

EXAMPLE 4. Finally, we observe that the function

$$f(z) = \frac{z^2 + z - 2}{z+1} = \frac{z(z+1) - 2}{z+1} = z - \frac{2}{z+1} = -1 + (z+1) - \frac{2}{z+1}$$
$$(0 < |z+1| < \infty)$$

has a simple pole at $z_0 = -1$. The residue there is -2. Moreover, since

$$\lim_{z \to -1} \frac{1}{f(z)} = \lim_{z \to -1} \frac{z+1}{z^2 + z - 2} = \frac{0}{-2} = 0,$$

we find that

$$\lim_{z \to -1} f(z) = \infty. \tag{5}$$

[Compare with limit (4) in Example 3.]

In the remaining sections of this chapter, we shall develop in greater depth the theory of the three types of isolated singular points just illustrated. The emphasis will be on useful and efficient methods for identifying poles and finding the corresponding residues.

EXERCISES

1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

$$(a)\ z\exp\left(\frac{1}{z}\right);\qquad (b)\ \frac{z^2}{1+z};\qquad (c)\ \frac{\sin z}{z};\qquad (d)\ \frac{\cos z}{z};\qquad (e)\ \frac{1}{(2-z)^3}.$$

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

$$(a)\ \frac{1-\cosh z}{z^3};\qquad (b)\ \frac{1-\exp(2z)}{z^4};\qquad (c)\ \frac{\exp(2z)}{(z-1)^2}.$$

Ans. (a) $m=1,\ B=-1/2$; (b) $m=3,\ B=-4/3$; (c) $m=2,\ B=2e^2$.

3. Suppose that a function f is analytic at z_0, and write $g(z)=f(z)/(z-z_0)$. Show that

(a) if $f(z_0)\neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

(b) if $f(z_0)=0$, then z_0 is a removable singular point of g.

Suggestion: As pointed out in Sec. 62, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

4. Write the function

$$f(z)=\frac{8a^3z^2}{(z^2+a^2)^3}\qquad (a>0)$$

as

$$f(z)=\frac{\phi(z)}{(z-ai)^3}\quad\text{where}\quad \phi(z)=\frac{8a^3z^2}{(z+ai)^3}.$$

Point out why $\phi(z)$ has a Taylor series representation about $z=ai$, and then use it to show that the principal part of f at that point is

$$\frac{\phi''(ai)/2}{z-ai}+\frac{\phi'(ai)}{(z-ai)^2}+\frac{\phi(ai)}{(z-ai)^3}=-\frac{i/2}{z-ai}-\frac{a/2}{(z-ai)^2}-\frac{a^2i}{(z-ai)^3}.$$

80. RESIDUES AT POLES

When a function f has an isolated singularity at a point z_0, the basic method for identifying z_0 as a pole and finding the residue there is to write the appropriate Laurent series and to note the coefficient of $1/(z-z_0)$. The following theorem provides an alternative characterization of poles and a way of finding residues at poles that is often more convenient.

Theorem. *Let z_0 be an isolated singular point of a function f. The following two statements are equivalent:*

(a) *z_0 is a pole of order m $(m = 1, 2, \ldots)$ of f;*

(b) *$f(z)$ can be written in the form*

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \qquad (m = 1, 2, \ldots),$$

where $\phi(z)$ is analytic and nonzero at z_0.

Moreover, if statements (a) and (b) are true,

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) \quad \textit{when } m = 1$$

and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \textit{when } m = 2, 3, \ldots .$$

Observe that these two expressions for residues need not have been written separately since, with the conventions that $\phi^{(0)}(z_0) = \phi(z_0)$ and $0! = 1$, the second expression reduces to the first when $m = 1$.

To prove the theorem, we first assume that statement (a) is true. That is, $f(z)$ has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \frac{b_m}{(z - z_0)^m} \qquad (b_m \neq 0),$$

which is valid in a punctured disk $0 < |z - z_0| < R_2$. Now a function $\phi(z)$ defined by means of the equations

$$\phi(z) = \begin{cases} (z - z_0)^m f(z) & \text{when } z \neq z_0, \\ b_m & \text{when } z = z_0 \end{cases}$$

evidently has the power series representation

$$\phi(z) = b_m + b_{m-1}(z - z_0) + \cdots + b_2 (z - z_0)^{m-2} + b_1 (z - z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{m+n}$$

throughout the entire disk $|z - z_0| < R_2$. Consequently, $\phi(z)$ is analytic in that disk (Sec. 71) and, in particular, at z_0. Inasmuch as $\phi(z_0) = b_m \neq 0$, the expression for $f(z)$ in statement (b) follows.

Suppose, on the other hand, that we know only that $f(z)$ has the form in statement (b) and recall (Sec. 62) that since $\phi(z)$ is analytic at z_0, it has a Taylor series representation

$$\phi(z) = \phi(z_0) + \frac{\phi'(z_0)}{1!}(z-z_0) + \frac{\phi''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n$$

in some neighborhood $|z-z_0| < \varepsilon$ of z_0. The quotient in statement (b) then tells us that

$$f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)/1!}{(z-z_0)^{m-1}} + \frac{\phi''(z_0)/2!}{(z-z_0)^{m-2}} + \cdots + \frac{\phi^{(m-1)}(z_0)/(m-1)!}{z-z_0} + \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$

when $0 < |z-z_0| < \varepsilon$. This Laurent series representation, together with the fact that $\phi(z_0) \neq 0$, reveals that z_0 is, indeed, a pole of order m of $f(z)$. The coefficient of $1/(z-z_0)$ tells us, of course, that the residue of $f(z)$ at z_0 is as stated in the theorem, whose proof is now complete.

81. EXAMPLES

The following examples serve to illustrate the use of the theorem in Sec. 80.

EXAMPLE 1. The function

$$f(z) = \frac{z+4}{z^2+1}$$

has an isolated singular point at $z = i$ and can be written

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{z+4}{z+i}.$$

Since $\phi(z)$ is analytic at $z = i$ and $\phi(i) \neq 0$, that point is a simple pole of f; and the residue there is

$$B_1 = \phi(i) = \frac{i+4}{2i} \cdot \frac{i}{i} = \frac{-1+4i}{-2} = \frac{1}{2} - 2i.$$

The point $z = -i$ is also a simple pole of f, with residue

$$B_2 = \frac{1}{2} + 2i.$$

EXAMPLE 2. If

$$f(z) = \frac{z^3+2z}{(z-i)^3},$$

then

$$f(z) = \frac{\phi(z)}{(z-i)^3} \quad \text{where} \quad \phi(z) = z^3 + 2z.$$

The function $\phi(z)$ is entire, and $\phi(i) = i \neq 0$. Hence f has a pole of order 3 at $z = i$, with residue

$$B = \frac{\phi''(i)}{2!} = \frac{6i}{2!} = 3i.$$

The theorem can, of course, be used when branches of multiple-valued functions are involved.

EXAMPLE 3. Suppose that

$$f(z) = \frac{(\log z)^3}{z^2 + 1},$$

where the branch

$$\log z = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

of the logarithmic function is to be used. To find the residue of f at the singularity $z = i$, we write

$$f(z) = \frac{\phi(z)}{z - i} \quad \text{where} \quad \phi(z) = \frac{(\log z)^3}{z + i}.$$

The function $\phi(z)$ is clearly analytic at $z = i$; and, since

$$\phi(i) = \frac{(\log i)^3}{2i} = \frac{(\ln 1 + i\pi/2)^3}{2i} = -\frac{\pi^3}{16} \neq 0,$$

f has a simple pole there. The residue is

$$B = \phi(i) = -\frac{\pi^3}{16}.$$

While the theorem in Sec. 80 can be extremely useful, the identification of an isolated singular point as a pole of a certain order is sometimes done most efficiently by appealing directly to a Laurent series.

EXAMPLE 4. If, for instance, the residue of the function

$$f(z) = \frac{1 - \cos z}{z^3}$$

is needed at the singularity $z = 0$, it would be incorrect to write

$$f(z) = \frac{\phi(z)}{z^3} \quad \text{where} \quad \phi(z) = 1 - \cos z$$

and to attempt an application of the theorem in Sec. 80 with $m = 3$. For it is necessary that $\phi(0) \neq 0$ if the theorem is to be used here. In this case, the simplest way to obtain the desired residue is to write out a few terms in the Laurent series

$$f(z) = \frac{1}{z^3}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right] = \frac{1}{z^3}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right)$$
$$= \frac{1}{2!}\cdot\frac{1}{z} - \frac{z}{4!} + \frac{z^3}{6!} - \cdots \qquad (0 < |z| < \infty).$$

This shows that $f(z)$ has a *simple pole* at $z = 0$, not a pole of order 3, the residue at $z = 0$ being $B = 1/2$.

EXAMPLE 5. Since $z^2 \sinh z$ is entire and its zeros are (Sec. 39)

$$z = n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

the point $z = 0$ is clearly an isolated singularity of the function

$$f(z) = \frac{1}{z^2 \sinh z}.$$

Here it would be a mistake to write

$$f(z) = \frac{\phi(z)}{z^2} \quad \text{where} \quad \phi(z) = \frac{1}{\sinh z}$$

and try to use the theorem in Sec. 80 with $m = 2$. This is because the function $\phi(z)$ is not even defined at $z = 0$. The needed residue, namely $B = -1/6$, follows at once from the Laurent series

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6}\cdot\frac{1}{z} + \frac{7}{360}z + \cdots \qquad (0 < |z| < \pi)$$

that was obtained in Exercise 5, Sec.73. The singularity at $z = 0$ is, of course, a pole of the *third* order, not the second order.

EXERCISES

1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z+1}{z^2+9}$; (b) $\dfrac{z^2+2}{z-1}$; (c) $\left(\dfrac{z}{2z+1}\right)^3$; (d) $\dfrac{e^z}{z^2+\pi^2}$.

Ans. (a) $m = 1, B = \dfrac{3 \pm i}{6}$; (b) $m = 1, B = 3$; (c) $m = 3, B = -\dfrac{3}{16}$;
(d) $m = 1, B = \pm\dfrac{i}{2\pi}$.

2. Show that

(a) $\displaystyle\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}}$ $\quad(|z| > 0, 0 < \arg z < 2\pi)$;

(b) $\displaystyle\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi + 2i}{8}$;

(c) $\displaystyle\operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}}$ $\quad(|z| > 0, 0 < \arg z < 2\pi)$.

3. In each case, find the order m of the pole and the corresponding residue B at the singularity $z = 0$:

$$(a)\ \frac{\sinh z}{z^4}; \qquad (b)\ \frac{1}{z(e^z - 1)}.$$

Ans. $(a)\ m = 3,\ B = \dfrac{1}{6}$; $\quad (b)\ m = 2,\ B = -\dfrac{1}{2}$.

4. Find the value of the integral

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2+9)}\, dz,$$

taken counterclockwise around the circle $(a)\ |z - 2| = 2$; $(b)\ |z| = 4$.

Ans. $(a)\ \pi i$; $\quad (b)\ 6\pi i$.

5. Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)},$$

taken counterclockwise around the circle $(a)\ |z| = 2$; $(b)\ |z + 2| = 3$.

Ans. $(a)\ \pi i/32$; $\quad (b)\ 0$.

6. Evaluate the integral

$$\int_C \frac{\cosh \pi z}{z(z^2+1)}\, dz$$

when C is the circle $|z| = 2$, described in the positive sense.

Ans. $4\pi i$.

7. Use the theorem in Sec. 77, involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

$$(a)\ f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}; \qquad (b)\ f(z) = \frac{z^3 e^{1/z}}{1 + z^3}.$$

Ans. $(a)\ 9\pi i$; $\quad (b)\ 2\pi i$.

8. Let z_0 be an isolated singular point of a function f and suppose that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where m is a positive integer and $\phi(z)$ is analytic and nonzero at z_0. By applying the extended form (3), Sec. 55, of the Cauchy integral formula to the function $\phi(z)$, show that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!},$$

as stated in the theorem of Sec. 80.

Suggestion: Since there is a neighborhood $|z - z_0| < \varepsilon$ throughout which $\phi(z)$ is analytic (see Sec. 25), the contour used in the extended Cauchy integral formula can be the positively oriented circle $|z - z_0| = \varepsilon/2$.

82. ZEROS OF ANALYTIC FUNCTIONS

Zeros and poles of functions are closely related. In fact, we shall see in the next section how zeros can be a source of poles. We need, however, some preliminary results regarding zeros of analytic functions.

Suppose that a function f is analytic at a point z_0. We know from Sec. 57 that all of the derivatives $f^{(n)}(z)$ $(n = 1, 2, \ldots)$ exist at z_0. If $f(z_0) = 0$ and if there is a positive integer m such that

$$(1) \qquad f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \quad \text{and} \quad f^{(m)}(z_0) \neq 0,$$

where m is a positive integer, f is said to have a ***zero of order m*** at z_0. We agree, of course, that $f^{(0)}(z_0) = f(z_0)$ when $m = 1$. Our first theorem here provides a useful alternative definition of zeros of order m.

Theorem 1. *Let f denote a function that is analytic at a point z_0. The following two statements are equivalent:*

(*a*) *f has a zero of order m at z_0;*

(*b*) *there is a function g, which is analytic and nonzero at z_0, such that*

$$f(z) = (z - z_0)^m g(z).$$

Our proof of this theorem has two parts. First, we need to show that the truth of statement (*a*) implies the truth of statement (*b*). Once that is accomplished, we need to show that if statement (*b*) is true, then so is statement (*a*). Both parts use the fact (Sec. 62) that if a given function is analytic at a point z_0, then it must have a Taylor series representation in powers of $(z - z_0)$ that is valid throughout some neighborhood $|z - z_0| < \varepsilon$ of z_0.

(*a*) implies (*b*)

We start the first part of the proof by assuming that f has a zero of order m at z_0 and showing how statement (*b*) follows. The analyticity of f at z_0 and conditions (1) tell us that in some neighborhood $|z - z_0| < \varepsilon$ there is a Taylor series representation

$$\begin{aligned} f(z) &= \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0)^{m+1} + \frac{f^{(m+2)}(z_0)}{(m+2)!}(z - z_0)^{m+2} + \cdots \\ &= (z - z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \frac{f^{(m+2)}(z_0)}{(m+2)!}(z - z_0)^2 + \cdots \right]. \end{aligned}$$

Consequently, $f(z)$ has the form shown in statement (*b*), where

$$g(z) = \frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \frac{f^{(m+2)}(z_0)}{(m+2)!}(z - z_0)^2 + \cdots \qquad (|z - z_0| < \varepsilon).$$

The convergence of this last series when $|z - z_0| < \varepsilon$ ensures that g is analytic in that neighborhood and, in particular, at z_0 (Sec. 71). Moreover,

$$g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0.$$

This completes the proof of the first part of the theorem.

(*b*) implies (*a*)

Here we assume that the expression for $f(z)$ in part (b) holds; and we note that since the function $g(z)$ is analytic at z_0, it has a Taylor series representation

$$g(z) = g(z_0) + \frac{g'(z_0)}{1!}(z - z_0) + \frac{g''(z_0)}{2!}(z - z_0)^2 + \cdots$$

in some neighborhood $|z - z_0| < \varepsilon$ of z_0. The expression for $f(z)$ in part (b) thus takes the form

$$f(z) = g(z_0)(z - z_0)^m + \frac{g'(z_0)}{1!}(z - z_0)^{m+1} + \frac{g''(z_0)}{2!}(z - z_0)^{m+2} + \cdots$$

when $|z - z_0| < \varepsilon$. Since this is actually a Taylor series expansion for $f(z)$, according to Theorem 1 in Sec. 72, conditions (1) hold; in particular,

$$f^{(m)}(z_0) = m!g(z_0) \neq 0.$$

Hence z_0 is a zero of order m of f. The proof is now complete.

EXAMPLE. The polynomial $f(z) = z^3 - 1$ has a zero of order $m = 1$ at $z_0 = 1$ since

$$f(z) = (z - 1)g(z),$$

where $g(z) = z^2 + z + 1$, and because f and g are entire and $g(1) = 3 \neq 0$. Note how the fact that $z_0 = 1$ is a zero of order $m = 1$ of f also follows from the observations that

$$f(1) = 0 \quad \text{and} \quad f'(1) = 3 \neq 0.$$

Our next theorem is a precise statement of the fact that an analytic function $f(z)$ has only ***isolated zeros*** when is not identically equal to zero. This means that if z_0 is a zero of such a function $f(z)$, there is a deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 in which $f(z)$ is nonzero. (Compare with the definition of an isolated singularity in Sec. 74.)

Theorem 2. *Given a function f and a point z_0, suppose that*

(a) *f is analytic at z_0;*

(b) *$f(z_0) = 0$ but $f(z)$ is not identically equal to zero in any neighborhood of z_0.*

Then $f(z) \neq 0$ throughout some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.

To prove this, let f be as stated and observe that not all of the derivatives of f at z_0 are zero. If they were, all of the coefficients in the Taylor series for f about z_0 would be zero; and that would mean that $f(z)$ is identically equal to zero in some neighborhood of z_0. So it is clear from the definition of zeros of order m at the beginning of this section that f must have a zero of some finite order m at z_0. According to Theorem 1, then,

$$f(z) = (z - z_0)^m g(z) \tag{2}$$

where $g(z)$ is analytic and nonzero at z_0.

Now g is continuous, in addition to being nonzero, at z_0 because it is analytic there. Hence there is some neighborhood $|z - z_0| < \varepsilon$ in which equation (2) holds and in which $g(z) \neq 0$ (see Sec. 18). Consequently, $f(z) \neq 0$ in the *deleted* neighborhood $0 < |z - z_0| < \varepsilon$; and the proof is complete.

Our final theorem here concerns functions with zeros that are not all isolated. It was referred to earlier in Sec. 28 and makes an interesting contrast to Theorem 2 just above.

Theorem 3. *Given a function f and a point z_0, suppose that*

(*a*) *f is analytic throughout a neighborhood N_0 of z_0;*

(*b*) $f(z) = 0$ *at each point z of a domain D or line segment L containing z_0* (Fig. 96).

Then $f(z) \equiv 0$ *in* N_0; *that is, $f(z)$ is identically equal to zero throughout N_0.*

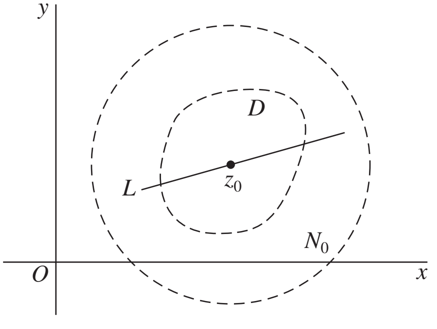

FIGURE 96

We begin the proof with the observation that under the stated conditions, $f(z) \equiv 0$ in some neighborhood N of z_0. For, otherwise, there would be a deleted neighborhood of z_0 throughout which $f(z) \neq 0$, according to Theorem 2; and that would be inconsistent with the condition that $f(z) = 0$ everywhere in a domain D or on a line segment L containing z_0. Since $f(z) \equiv 0$ in the neighborhood N, then, it follows that all of the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, 1, 2, \ldots)$$

in the Taylor series for $f(z)$ about z_0 must be zero. Thus $f(z) \equiv 0$ in the neighborhood N_0, since the Taylor series also represents $f(z)$ in N_0. This completes the proof.

83. ZEROS AND POLES

The following theorem establishes a connection between zeros of order m and poles of order m.

Theorem 1. *Suppose that*

(*a*) *two functions p and q are analytic at a point z_0;*

(*b*) *$p(z_0) \neq 0$ and q has a zero of order m at z_0.*

Then the quotient $p(z)/q(z)$ has a pole of order m at z_0.

The proof is easy. Let p and q be as in the statement of the theorem. Since q has a zero of order m at z_0, we know from Theorem 2 in Sec. 82 that there is a deleted neighborhood of z_0 throughout which $q(z) \neq 0$; and so z_0 is an isolated singular point of the quotient $p(z)/q(z)$. Theorem 1 in Sec. 82 tells us, moreover, that

$$q(z) = (z - z_0)^m g(z),$$

where $g(z)$ is analytic and nonzero at z_0. Consequently,

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m} \quad \text{where} \quad \phi(z) = \frac{p(z)}{g(z)}. \tag{1}$$

Since $\phi(z)$ is analytic and nonzero at z_0, it now follows from the theorem in Sec. 80 that z_0 is a pole of order m of $p(z)/q(z)$.

EXAMPLE 1. The two functions

$$p(z) = 1 \quad \text{and} \quad q(z) = 1 - \cos z$$

are entire, and we know from Exercise 2 that $q(z)$ has a zero of order $m = 2$ at the point $z_0 = 0$. Hence it follows from Theorem 1 that the quotient

$$\frac{p(z)}{q(z)} = \frac{1}{1 - \cos z}$$

has a pole of order $m = 2$ at that point.

Theorem 1 leads us to another method for identifying *simple* poles and finding the corresponding residues. This method, stated just below as Theorem 2, is sometimes easier to use than the theorem in Sec. 80.

Theorem 2. *Let two functions p and q be analytic at a point z_0. If*

$$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \textit{and} \quad q'(z_0) \neq 0,$$

then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{2}$$

To show this, we assume that p and q are as stated and observe that because of the conditions on q, the point z_0 is a zero of order $m = 1$ of that function. According to Theorem 1 in Sec. 82, then,

$$q(z) = (z - z_0)g(z) \tag{3}$$

where $g(z)$ is analytic and nonzero at z_0. Furthermore, Theorem 1 in this section tells us that z_0 is a simple pole of $p(z)/q(z)$; and expression (1) for $p(z)/q(z)$ in the proof of that theorem becomes

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{z - z_0} \quad \text{where} \quad \phi(z) = \frac{p(z)}{g(z)}.$$

Since this $\phi(z)$ is analytic and nonzero at z_0, we know from the theorem in Sec. 80 that

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{g(z_0)}. \tag{4}$$

But $g(z_0) = q'(z_0)$, as is seen by differentiating each side of equation (3) and then setting $z = z_0$. Expression (4) thus takes the form (2).

EXAMPLE 2. Consider the function

$$f(z) = \cot z = \frac{\cos z}{\sin z},$$

which is a quotient of the entire functions $p(z) = \cos z$ and $q(z) = \sin z$. Its singularities occur at the zeros of q, or at the points

$$z = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Since

$$p(n\pi) = (-1)^n \neq 0, \quad q(n\pi) = 0, \quad \text{and} \quad q'(n\pi) = (-1)^n \neq 0,$$

Theorem 2 tells us that each singular point $z = n\pi$ of f is a simple pole, with residue

$$B_n = \frac{p(n\pi)}{q'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1.$$

EXAMPLE 3. The residue of the function

$$f(z) = \frac{z - \sinh z}{z^2 \sinh z}$$

at the zero $z = \pi i$ of $\sinh z$ (see Sec. 39) is readily found by writing

$$p(z) = z - \sinh z \quad \text{and} \quad q(z) = z^2 \sinh z.$$

Because

$$p(\pi i) = \pi i \neq 0, \quad q(\pi i) = 0, \quad \text{and} \quad q'(\pi i) = \pi^2 \neq 0,$$

Theorem 2 tells us that $z = \pi i$ is a simple pole of f and that the residue there is

$$B = \frac{p(\pi i)}{q'(\pi i)} = \frac{\pi i}{\pi^2} = \frac{i}{\pi}.$$

EXAMPLE 4. Since the point

$$z_0 = \sqrt{2}e^{i\pi/4} = 1 + i$$

is a zero of the polynomial $z^4 + 4$ (see Exercise 6, Sec. 11), it is also an isolated singularity of the function

$$f(z) = \frac{z}{z^4 + 4}.$$

Writing $p(z) = z$ and $q(z) = z^4 + 4$, we find that

$$p(z_0) = z_0 \neq 0, \quad q(z_0) = 0, \quad \text{and} \quad q'(z_0) = 4z_0^3 \neq 0.$$

Theorem 2 then reveals that z_0 is a simple pole of f. The residue there is, moreover,

$$B_0 = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{8i} = -\frac{i}{8}.$$

Although this residue can also be found by the method in Sec. 80, the computation is somewhat more involved.

There are expressions similar to expression (2) for residues at poles of higher order, but they are lengthier and, in general, not practical.

EXERCISES

1. Show that the point $z = 0$ is a simple pole of the function

$$f(z) = \csc z = \frac{1}{\sin z}$$

and that the residue there is unity by appealing to Theorem 2 in Sec. 83. (Compare with Exercise 3, Sec. 73, where this result is evident from a Laurent series.)

2. Use conditions (1) in Sec. 82 to show that the function

$$q(z) = 1 - \cos z$$

has a zero of order $m = 2$ at the point $z_0 = 0$.

3. Show that

(a) $\displaystyle \operatorname*{Res}_{z=\pi i/2} \frac{\sinh z}{z^2 \cosh z} = -\frac{4}{\pi^2}$;

(b) $\displaystyle \operatorname*{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -2\cos(\pi t)$.

4. Show that

(a) $\displaystyle \operatorname*{Res}_{z=z_n}(z \sec z) = (-1)^{n+1} z_n$ where $\displaystyle z_n = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\displaystyle \operatorname*{Res}_{z=z_n}(\tanh z) = 1$ where $\displaystyle z_n = \left(\frac{\pi}{2} + n\pi\right)i \quad (n = 0, \pm 1, \pm 2, \ldots)$.