CHAPTER

6

RESIDUES AND POLES

The Cauchy–Goursat theorem (Sec. 50) states that if a function is analytic at all points interior to and on a simple closed contour C, then the value of an integral of the function around that contour is zero. If, however, the function fails to be analytic at a finite number of points interior to C, there is, as we shall see in this chapter, a specific number, called a residue, which each of those points contributes to the value of the integral. We develop here the theory of residues; and, in Chap. 7, we shall illustrate their use in certain areas of applied mathematics.

74. ISOLATED SINGULAR POINTS

We saw in Sec. 25 that a function f is analytic at a point z_0 if it has a derivative at each point in some neighborhood of z_0. If, on the other hand, f fails to be analytic at z_0 but is analytic at some point in every neighborhood of it, we also saw in Sec. 25 that z_0 is a singular point of f.

The theory of residues in this chapter centers around a special type of singular point. Namely, a singular point z_0 is said to be ***isolated*** if there is a deleted ε neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout which f is analytic.

EXAMPLE 1. The function

$$f(z) = \frac{z-1}{z^5(z^2+9)}$$

has the three isolated singular points $z = 0$ and $z = \pm 3i$. In fact, the singular points of a rational function, or quotient of two polynomials, are always isolated. This because the zeros of the polynomial in the denominator are finite in number (Sec. 58).

EXAMPLE 2. The origin $z = 0$ is a singular point of the principal branch (Sec. 33)

$$F(z) = \operatorname{Log} z = \ln r + i\Theta \qquad (r > 0, -\pi < \Theta < \pi)$$

of the logarithmic function. It is *not*, however, an isolated singular point since every deleted ε neighborhood of it contains points on the negative real axis (see Fig. 88) and the branch is not even defined there. Similar remarks can be made regarding *any* branch

$$f(z) = \log z = \ln z + i\theta \qquad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

of the logarithmic function.

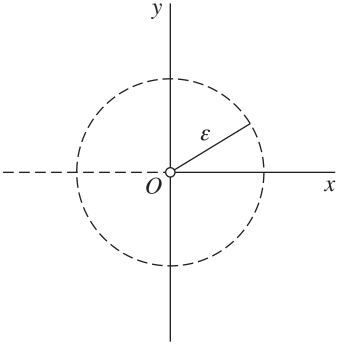

FIGURE 88

EXAMPLE 3. The function

$$f(z) = \frac{1}{\sin(\pi/z)}$$

clearly does not have a derivative at the origin $z = 0$; and because $\sin(\pi/z) = 0$ when $\pi/z = n\pi$ $(n = \pm 1, \pm 2, \ldots)$, the derivative of f also fails to exist at each of the points $z = 1/n$ $(n = \pm 1, \pm 2, \ldots)$. Inasmuch as the derivative of f does exist at every point that is not on the real axis, it follows that f is analytic at some point in every neighborhood of each of the points

(1) $$z = 0 \quad \text{and} \quad z = 1/n \ \ (n = \pm 1, \pm 2, \ldots).$$

Hence each of the points (1) is a singularity of f.

The singularity $z = 0$ is *not* isolated because every deleted ε neighborhood of it contains other singular points. More precisely, when a positive number ε is specified and m is any positive integer such that $m > 1/\varepsilon$, the fact that $0 < 1/m < \varepsilon$ means that the singularity $z = 1/m$ lies in the deleted ε neighborhood $0 < |z| < \varepsilon$.

The remaining points $z = 1/n$ $(n = \pm1, \pm2, \ldots)$ *are* in fact, isolated. In order to see this, let m denote any fixed positive integer and observe that f is analytic in the deleted neighborhood of $z = 1/m$ whose radius is

$$\varepsilon = \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}.$$

(See Fig. 89.) A similar observation can be made when m is a negative integer.

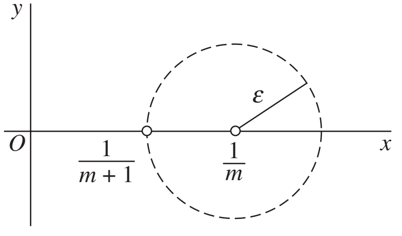

FIGURE 89

In this chapter, it will be important to keep in mind that if a function is analytic everywhere inside a simple closed contour C except for a *finite* number of singular points $z_1, z_2, \ldots, z_n$, those points must all be isolated and the deleted neighborhoods about them can be made small enough to lie entirely inside C. To see that this is so, consider any one of the points z_k. The radius ε of the needed deleted neighborhood can be any positive number that is smaller than the distances to the other singular points and also smaller than the distance from z_k to the closest point on C.

Finally, we mention that it is sometimes convenient to consider the point at infinity (Sec. 17) as an isolated singular point. To be specific, if there is a positive number R_1 such that f is analytic for $R_1 < |z| < \infty$, then f is said to have an ***isolated singular point at $z_0 = \infty$***. Such a singular point will be used in Sec. 77.

75. RESIDUES

When z_0 is an isolated singular point of a function f, there is a positive number R_2 such that f is analytic at each point z for which $0 < |z - z_0| < R_2$. Consequently, $f(z)$ has a Laurent series representation

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots$$
$$(0 < |z - z_0| < R_2),$$

where the coefficients a_n and b_n have certain integral representations (Sec. 66). In particular,

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}} \qquad (n = 1, 2, \ldots)$$

where C is any positively oriented simple closed contour around z_0 that lies in the punctured disk $0 < |z - z_0| < R_2$ (Fig. 90). When $n = 1$, this expression for b_n becomes

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz$$

or

(2) $$\int_C f(z)\,dz = 2\pi i b_1.$$

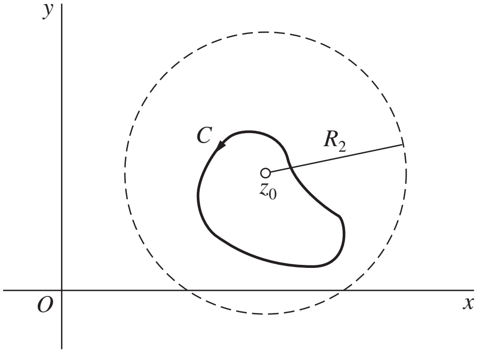

FIGURE 90

The complex number b_1, which is the coefficient of $1/(z - z_0)$ in expansion (1), is called the ***residue*** of f at the isolated singular point z_0, and we shall often write

$$b_1 = \operatorname*{Res}_{z=z_0} f(z).$$

Equation (2) then becomes

(3) $$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z).$$

Sometimes we simply use B to denote the residue when the function f and the point z_0 are clearly indicated.

Equation (3) provides a powerful method for evaluating certain integrals around simple closed contours.

EXAMPLE 1. Consider the integral

(4) $$\int_C \frac{e^z - 1}{z^4}\,dz$$

where C is the positively oriented unit circle $|z| = 1$ (Fig. 91). Since the integrand is analytic everywhere in the finite plane except at $z = 0$, it has a Laurent series representation that is valid when $0 < |z| < \infty$. Thus, according to equation (3), the value of integral (4) is $2\pi i$ times the residue of its integrand at $z = 0$.

To determine that residue, we recall (Sec. 64) the Maclaurin series representation

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (|z| < \infty)$$

and use it to write

$$\frac{e^z - 1}{z^5} = \frac{1}{z^5} \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-5}}{n!} \quad (0 < |z| < \infty).$$

The coefficient of $1/z$ in this last series occurs when $n - 5 = -1$, or when $n = 4$. Hence

$$\operatorname*{Res}_{z=0} \frac{e^z - 1}{z^5} = \frac{1}{4!} = \frac{1}{24};$$

and so

$$\int_C \frac{e^z - 1}{z^4}\, dz = 2\pi i \left(\frac{1}{24}\right) = \frac{\pi i}{12}.$$

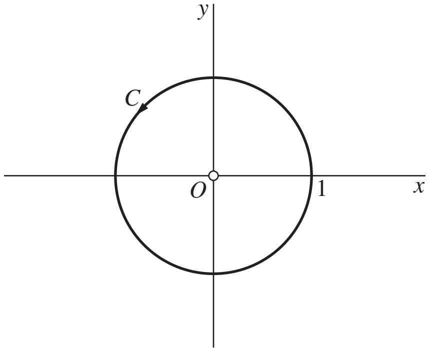

FIGURE 91

EXAMPLE 2. Let us show that

(5) $$\int_C \cosh\left(\frac{1}{z^2}\right) dz = 0$$

where C is the same positively oriented unit circle $|z| = 1$ as in Example 1. The composite function $\cosh(1/z^2)$ is analytic everywhere except at the origin since the same is true of $1/z^2$ and since $\cosh z$ is entire. The isolated singular point $z = 0$ is interior to C, and Fig. 91 in Example 1 can be used here as well. With the help of the Maclaurin series expansion (Sec. 64)

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \quad (|z| < \infty),$$

one can write the Laurent series expansion

$$\cosh\left(\frac{1}{z}\right) = 1 + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{4!} \cdot \frac{1}{z^4} + \frac{1}{6!} \cdot \frac{1}{z^6} \cdots \qquad (0 < |z| < \infty).$$

The residue of the integrand at its isolated singular point $z = 0$ is, therefore, zero ($b_1 = 0$), and the value of integral (5) is established.

We are reminded in this example that although the analyticity of a function within and on a simple closed contour C is a sufficient condition for the value of the integral around C to be zero, it is not a *necessary* condition.

EXAMPLE 3. A residue can be used to evaluate the integral

$$\int_C \frac{dz}{z(z-2)^5} \tag{6}$$

where C is the positively oriented circle $|z-2| = 1$ (Fig. 92). Since the integrand is analytic everywhere in the finite plane except at the points $z = 0$ and $z = 2$, it has a Laurent series representation that is valid in the punctured disk $0 < |z-2| < 2$, which is shown in Fig. 92. Thus, according to equation (3), the value of integral (6) is $2\pi i$ times the residue of its integrand at $z = 2$. The nature of that integrand suggests that we might use the geometric series (Sec. 64)

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

to determine the residue. We write

$$\frac{1}{z(z-2)^5} = \frac{1}{(z-2)^5} \cdot \frac{1}{2+(z-2)} = \frac{1}{2(z-2)^5} \cdot \frac{1}{1-\left(-\dfrac{z-2}{2}\right)},$$

and then use the geometric series:

$$\frac{1}{z(z-2)^5} = \frac{1}{2(z-2)^5} \sum_{n=0}^{\infty} \left(-\frac{z-2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^{n-5} \qquad (0 < |z-2| < 2).$$

In this Laurent series, which could be written in the form (1), the coefficient of $1/(z-2)$ is the desired residue, namely $1/32$. Consequently,

$$\int_C \frac{dz}{z(z-2)^5} = 2\pi i \left(\frac{1}{32}\right) = \frac{\pi i}{16}.$$

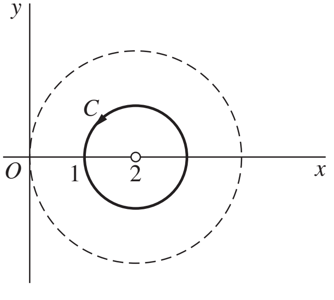

FIGURE 92

76. CAUCHY'S RESIDUE THEOREM

If, except for a *finite* number of singular points, a function f is analytic inside a simple closed contour C, those singular points must all be isolated (Sec. 74). The following theorem, which is known as ***Cauchy's residue theorem,*** is a precise statement of the fact that if f is also analytic on C and if C is positively oriented, then the value of the integral of f around C is $2\pi i$ times the sum of the residues of f at the singular points inside C.

Theorem. *Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, \ldots, n)$ inside C* (Fig. 93)*, then*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \tag{1}$$

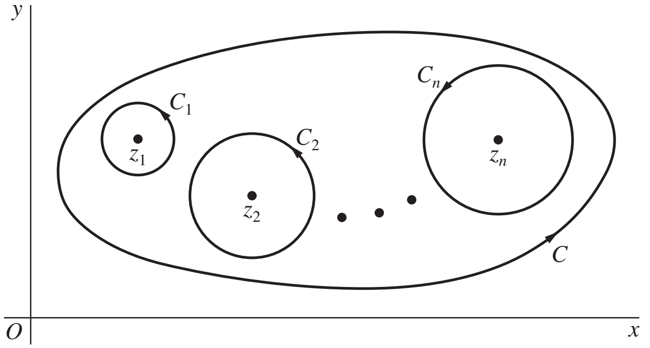

FIGURE 93

To prove the theorem, let the points z_k $(k = 1, 2, \ldots, n)$ be centers of positively oriented circles C_k which are interior to C and are so small that no two of them have points in common. The circles C_k, together with the simple closed contour C, form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain consisting of the points inside C and exterior to each C_k. Hence, according to the adaptation of the Cauchy–Goursat theorem to such domains (Sec. 53),

$$\int_C f(z)\,dz - \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0.$$

This reduces to equation (1) because (Sec. 75)

$$\int_{C_k} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z) \quad (k = 1, 2, \ldots, n),$$

and the proof is complete.

EXAMPLE. Let us use the theorem to evaluate the integral

$$\int_C \frac{4z-5}{z(z-1)}\,dz \tag{2}$$

where C is the circle $|z| = 2$, described in the counterclockwise direction (Fig. 94). The integrand has the two isolated singularities $z = 0$ and $z = 1$, both of which are interior to C. The corresponding residues B_1 at $z = 0$ and B_2 at $z = 1$ are readily found with the aid of the Maclaurin series representation (Sec. 64)

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad (|z| < 1).$$

We observe first that when $0 < |z| < 1$,

$$\frac{4z-5}{z(z-1)} = \frac{4z-5}{z}\cdot\frac{-1}{1-z} = \left(4 - \frac{5}{z}\right)(-1 - z - z^2 - \cdots);$$

and by identifying the coefficient of $1/z$ in the product on the right here, we find that

$$B_1 = 5. \tag{3}$$

Also, since

$$\begin{aligned}\frac{4z-5}{z(z-1)} &= \frac{4(z-1)-1}{z-1}\cdot\frac{1}{1+(z-1)}\\ &= \left(4 - \frac{1}{z-1}\right)[1 - (z-1) + (z-1)^2 - \cdots]\end{aligned}$$

when $0 < |z-1| < 1$, it follows that

$$B_2 = -1. \tag{4}$$

Thus

$$\int_C \frac{4z-5}{z(z-1)}\,dz = 2\pi i(B_1 + B_2) = 8\pi i. \tag{5}$$

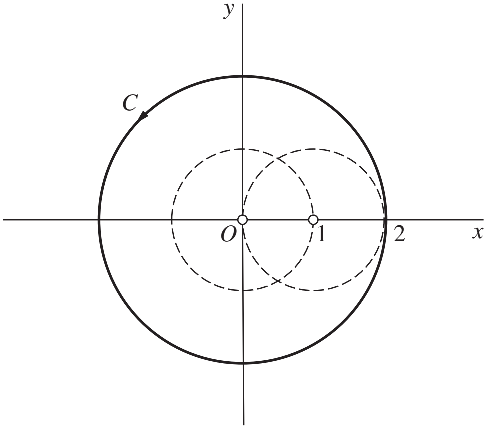

FIGURE 94

In this example, it is actually easier to start by writing the integrand in integral (2) as the sum of its partial fractions:

$$\frac{4z-5}{z(z-1)} = \frac{5}{z} + \frac{-1}{z-1}.$$

Then, since $5/z$ is already a Laurent series when $0 < |z| < 1$ and since $-1/(z-1)$ is a Laurent series when $0 < |z-1| < 1$, it follows that statement (5) is true.

77. RESIDUE AT INFINITY

Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C. Next, let R_1 denote a positive number which is large enough that C lies inside the circle $|z| = R_1$ (see Fig. 95). The function f is evidently analytic throughout the domain $R_1 < |z| < \infty$ and, as already mentioned at the end of Sec. 74, the point at infinity is then said to be an isolated singular point of f.

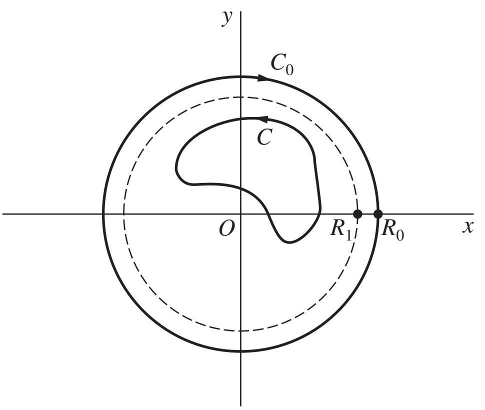

FIGURE 95

Now let C_0 denote a circle $|z| = R_0$, oriented in the *clockwise* direction, where $R_0 > R_1$. The ***residue of f at infinity*** is defined by means of the equation

$$\int_{C_0} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\infty} f(z). \tag{1}$$

Note that the circle C_0 keeps the point at infinity on the left, just as the singular point in the finite plane is on the left in equation (3), Sec. 75. Since f is analytic throughout the closed region bounded by C and C_0, the principle of deformation of paths (Sec. 53) tells us that

$$\int_C f(z)\,dz = \int_{-C_0} f(z)\,dz = -\int_{C_0} f(z)\,dz.$$

So, in view of definition (1),

$$\int_C f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z). \tag{2}$$

To find this residue, write the Laurent series (see Sec. 66)

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad (R_1 < |z| < \infty), \tag{3}$$

where

$$c_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)\,dz}{z^{n+1}} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{4}$$

Replacing z by $1/z$ in equation (3) and then multiplying through the result by $1/z^2$, we see that

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n} \quad \left(0 < |z| < \frac{1}{R_1}\right)$$

and

$$c_{-1} = \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

Putting $n = -1$ in expression (4), we now have

$$c_{-1} = \frac{1}{2\pi i}\int_{-C_0} f(z)\,dz,$$

or

$$\int_{C_0} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]. \tag{5}$$

Note how it follows from this and definition (1) that

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]. \tag{6}$$

With equations (2) and (6), the following theorem is now established. This theorem is sometimes more efficient to use than Cauchy's residue theorem in Sec. 76 since it involves only one residue.

Theorem. *If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then*

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]. \tag{7}$$

EXAMPLE. It is easy to see that the singularities of the function

$$f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}$$

all lie inside the positively oriented circle C centered at the origin with radius 3. In order to use the theorem in this section, we write

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z} \cdot \frac{z-3}{(z+1)(z^4+2)}. \tag{8}$$

Inasmuch as the quotient

$$\frac{z-3}{(z+1)(z^4+2)}$$

is analytic at the origin, it has a Maclaurin series representation whose first term is the nonzero number $-3/2$. Hence, in view of expression (8),

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z}\left(-\frac{3}{2} + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right) = -\frac{3}{2}\cdot\frac{1}{z} + a_1 + a_2 z + a_3 z^2 + \cdots$$

for all z in some punctured disk $0 < |z| < R_0$. It is now clear that

$$\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -\frac{3}{2},$$

and so

$$\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)}dz = 2\pi i\left(-\frac{3}{2}\right) = -3\pi i. \tag{9}$$

EXERCISES

1. Find the residue at $z = 0$ of the function

$(a)\ \dfrac{1}{z+z^2}$; $(b)\ z\cos\left(\dfrac{1}{z}\right)$; $(c)\ \dfrac{z-\sin z}{z}$; $(d)\ \dfrac{\cot z}{z^4}$; $(e)\ \dfrac{\sinh z}{z^4(1-z^2)}$.

Ans. (a) 1; (b) $-1/2$; (c) 0; (d) $-1/45$; (e) $7/6$.

2. Use Cauchy's residue theorem (Sec. 76) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

$(a)\ \dfrac{\exp(-z)}{z^2}$; $(b)\ \dfrac{\exp(-z)}{(z-1)^2}$; $(c)\ z^2\exp\left(\dfrac{1}{z}\right)$; $(d)\ \dfrac{z+1}{z^2-2z}$.

Ans. (a) $-2\pi i$; (b) $-2\pi i/e$; (c) $\pi i/3$; (d) $2\pi i$.

3. In the example in Sec. 76, two residues were used to evaluate the integral

$$\int_C \frac{4z-5}{z(z-1)}dz$$

where C is the positively oriented circle $|z| = 2$. Evaluate this integral once again by using the theorem in Sec. 77 and finding only one residue.

4. Use the theorem in Sec. 77, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

$$(a)\ \frac{z^5}{1-z^3}; \qquad (b)\ \frac{1}{1+z^2}; \qquad (c)\ \frac{1}{z}.$$

Ans. $(a)\ -2\pi i$; $(b)\ 0$; $(c)\ 2\pi i$.

5. Let C denote the circle $|z| = 1$, taken counterclockwise, and use the following steps to show that

$$\int_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)!}.$$

(a) By using the Maclaurin series for e^z and referring to Theorem 1 in Sec. 71, which justifies the term by term integration that is to be used, write the above integral as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n \exp\left(\frac{1}{z}\right) dz.$$

(b) Apply the theorem in Sec. 76 to evaluate the integrals appearing in part (a) to arrive at the desired result.

6. Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points $z_1, z_2, \ldots, z_n$. Show that

$$\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

7. Let the degrees of the polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m \qquad (b_m \neq 0)$$

be such that $m \geq n + 2$. Use the theorem in Sec. 77 to show that if all of the zeros of $Q(z)$ are interior to a simple closed contour C, then

$$\int_C \frac{P(z)}{Q(z)}\, dz = 0.$$

[Compare with Exercise 4(b).]

78. THE THREE TYPES OF ISOLATED SINGULAR POINTS

We saw in Sec. 75 that the theory of residues is based on the fact that if f has an isolated singular point at z_0, then $f(z)$ has a Laurent series representation

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$