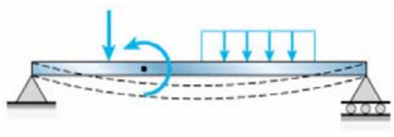
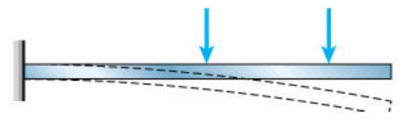
Chapter 4   Shear Forces and Bending Moments

4.1 Introduction

Consider a beam subjected to transverse loads as shown in figure, the deflections occur in the plane same as the loading plane, is called the plane of bending. In this chapter we discuss shear forces and bending moments in beams related to the loads.

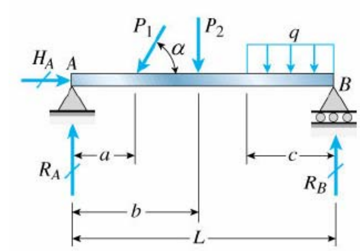




4.2 Types of Beams, Loads, and Reactions

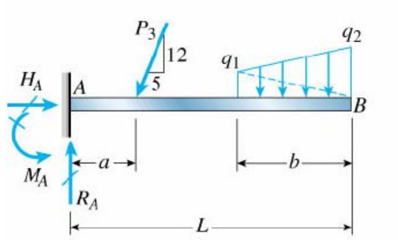
Type of beams

a. simply supported beam (simple beam)



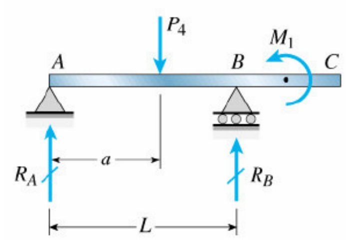
(a)

b. cantilever beam (fixed end beam)



(b)

c. beam with an overhang



(c)

Type of loads

a. concentrated load (single force)

b. distributed load (measured by their intensity) :

uniformly distributed load (uniform load)

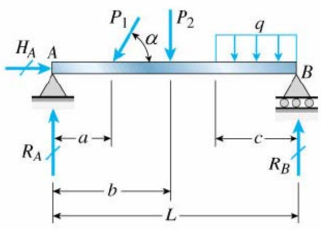
linearly varying load

c. couple

Reactions

consider the loaded beam in figure

equation of equilibrium in horizontal direction



(a)

$$\Sigma F_x = 0 \qquad H_A - P_1 \cos \alpha = 0$$

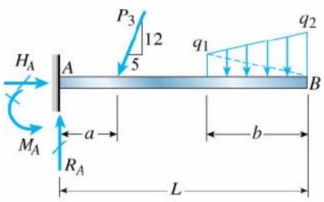
$$H_A = P_1 \cos \alpha$$

$$\Sigma M_B = 0 \qquad - R_A L + (P_1 \sin \alpha)(L - a) + P_2 (L - b) + q\, c^2 / 2 = 0$$

$$R_A = \frac{(P_1 \sin \alpha)(L - a)}{L} + \frac{P_2 (L - b)}{L} + \frac{q\, c^2}{2\, L}$$

$$R_B = \frac{(P_1 \sin \alpha)\, a}{L} + \frac{P_2\, b}{L} + \frac{q\, c^2}{2\, L}$$

for the cantilever beam



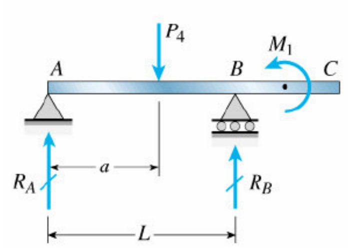
(b)

$$\Sigma F_x = 0 \qquad H_A = 5\, P_3 / 13$$

$$\Sigma F_y = 0 \qquad R_A = \frac{12\, P_3}{13} + \frac{(q_1 + q_2)\, b}{2}$$

$$\Sigma M_A = 0 \qquad M_A = \frac{12\,P_3}{13} + \frac{q_1\,b}{2}(L - 2b/3) + \frac{q_1\,b}{2}(L - b/3)$$

for the overhanging beam



(c)

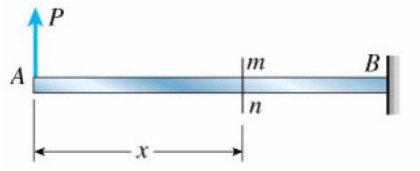
$$\Sigma M_B = 0 \qquad - R_A\,L + P_4\,(L - a) + M_1 = 0$$

$$\Sigma M_A = 0 \qquad - P_4\,a + R_B\,L + M_1 = 0$$

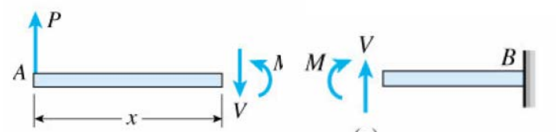
$$R_A = \frac{P_4\,(L - a) + M_1}{L} \qquad R_B = \frac{P_4\,a - M_1}{L}$$

4.3 Shear Forces and Bending Moments

Consider a cantilever beam with a concentrated load P applied at the end A, at the cross section mn, the shear force and bending moment are found



(a)



(b) (c)

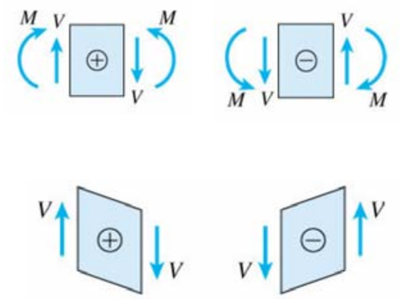
$$\Sigma F_y = 0 \qquad V = P$$

$$\Sigma M = 0 \qquad M = P\,x$$

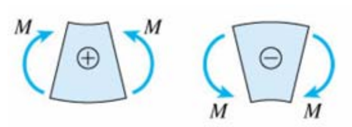
sign conventions (deformation sign conventions)

the shear force tends to rotate the material clockwise is defined as positive

the bending moment tends to compress the upper part of the beam and elongate the lower part is defined as positive



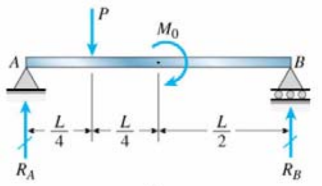
(a)



(b)

Example 4-1

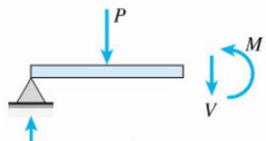
a simple beam AB supports a force P and a couple M_0, find the shear V and bending moment M at



(a)

(a) at $x = (L/2)_-$ (b) at $x = (L/2)_+$

$$R_A = \frac{3P}{4} - \frac{M_0}{L} \qquad R_B = \frac{P}{4} + \frac{M_0}{L}$$



(b)

(a) at $x = (L/2)_-$

$\Sigma F_y = 0 \qquad R_A - P - V = 0$

$V = R_A - P = -P/4 - M_0/L$

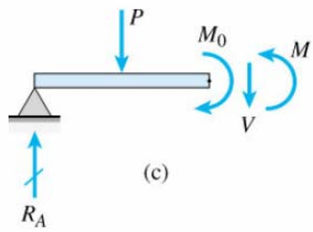
$\Sigma M = 0 \qquad -R_A(L/2) + P(L/4) + M = 0$

$M = R_A(L/2) + P(L/4) = PL/8 - M_0/2$

(b) at $x = (L/2)_+$ [similarly as (a)]

$V = -P/4 - M_0/L$

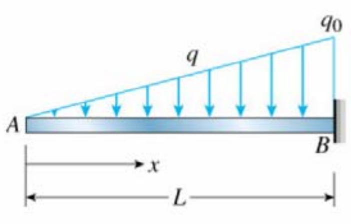
$M = PL/8 + M_0/2$



(c)

Example 4-2

a cantilever beam AB subjected to a linearly varying distributed load as shown, find the shear force V and the bending moment M



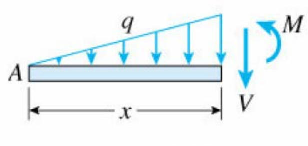
(a)

$q = q_0 x / L$

$\Sigma F_y = 0 \qquad -V - \frac{1}{2}(q_0 x / L)(x) = 0$

$V = -q_0 x^2 / (2L)$

$V_{max} = -q_0 L / 2$



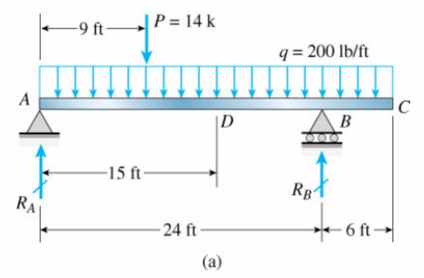
(b)

$\Sigma M = 0 \qquad M + ½ (q_0 x / L) (x) (x / 3) = 0$

$M = - q_0 x^3 / (6 L)$

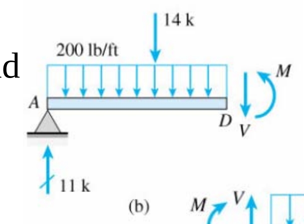
$M_{max} = - q_0 L^2 / 6$

Example 4-3



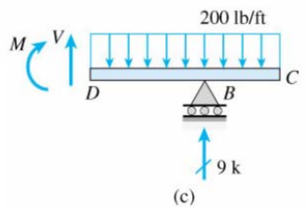
an overhanging beam ABC is supported to an uniform load of intensity q and a concentrated load P, calculate the shear force V and the bending moment M at D

from equations of equilibrium, it is found



$R_A = 40 \text{ kN} \qquad R_B = 48 \text{ kN}$

at section D



$\Sigma F_y = 0 \qquad 40 - 28 - 6 \text{ x } 5 - V = 0$

$V = - 18 \text{ kN}$

$\Sigma M = 0$

$- 40 \text{ x } 5 + 28 \text{ x } 2 + 6 \text{ x } 5 \text{ x } 2.5 + M = 0$

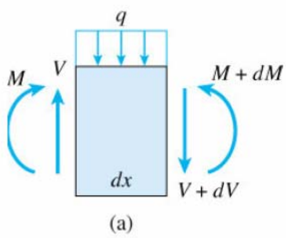
$M = 69 \text{ kN-m}$

from the free body diagram of the right-hand part, same results can be obtained

4.4 **Relationships Between Loads, Shear Forces, and Bending Moments**

consider an element of a beam of length dx subjected to distributed loads q

equilibrium of forces in vertical direction



$\Sigma F_y = 0 \qquad V - q\,dx - (V + dV) = 0$

or $\qquad \boldsymbol{dV / dx = -q}$

integrate between two points A and B

$$\int_A^B dV = -\int_A^B q\,dx$$

i.e.

$$V_B - V_A = -\int_A^B q\,dx$$

$= -$ (area of the loading diagram between A and B)

the area of loading diagram may be positive or negative

moment equilibrium of the element

$\Sigma M = 0 \qquad -M - q\,dx\,(dx/2) - (V + dV)\,dx + M + dM = 0$

or $\qquad \boldsymbol{dM / dx = V}$

maximum (or minimum) bending-moment occurs at $dM / dx = 0$, i.e. at the point of shear force $V = 0$

integrate between two points A and B

$$\int_A^B dM = \int_A^B V\,dx$$

i.e.

$$M_B - M_A = \int_A^B V\,dx$$

$=$ (area of the shear-force diagram between A and B)

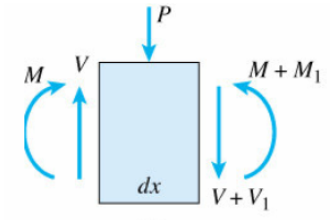
this equation is valid even when concentrated loads act on the beam between A and B, but it is not valid if a couple acts between A and B

concentrated loads

equilibrium of force

$$V \quad - \quad P \quad - \quad (V + V_1) \quad = \quad 0$$

or $\qquad V_1 \quad = \quad - P$



(b)

i.e. an abrupt change in the shear force occurs at any point where a concentrated load acts

equilibrium of moment

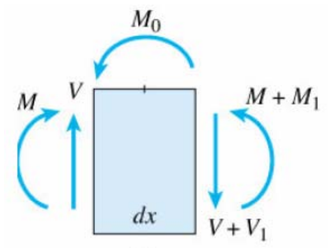
$$- M \quad - P\,(dx/2) \quad - \quad (V + V_1)\,dx \quad + \quad M \quad + \quad M_1 \quad = \quad 0$$

or $\qquad M_1 \quad = \quad P\,(dx/2) \quad + \quad V\,dx \quad + \quad V_1\,dx \quad \simeq \quad 0$

since the length dx of the element is infinitesimally small, i.e. M_1 is also infinitesimally small, thus, the bending moment does not change as we pass through the point of application of a concentrated load

loads in the form of couples

equilibrium of force $\quad V_1 \quad = \quad 0$



(c)

i.e. no change in shear force at the point of application of a couple

equilibrium of moment

$$- M + M_0 - (V + V_1)\,dx + M + M_1 \quad = \quad 0$$

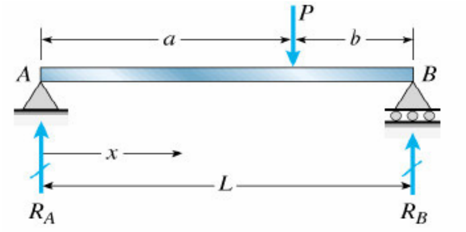
or $\qquad M_1 \quad = \quad - M_0$

the bending moment changes abruptly at a point of application of a couple

4.5 Shear-Force and Bending-Moment Diagrams

concentrated loads

consider a simply support beam AB with a concentrated load P



(a)

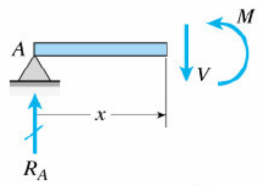
$R_A = P\,b\,/\,L \qquad R_B = P\,a\,/\,L$

for $0 < x < a$

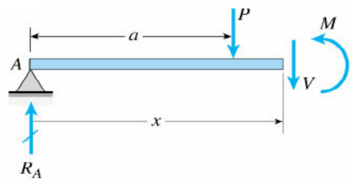
$V = R_A = P\,b\,/\,L$

$M = R_A\,x = P\,b\,x\,/\,L$

note that $\mathbf{dM\,/\,dx} = P\,b\,/\,L = V$



(b)



(c)

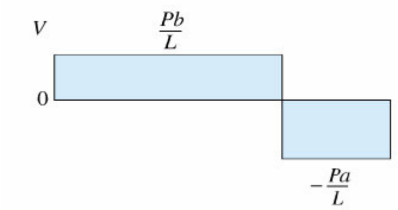
for $a < x < L$

$V = R_A - P = -P\,a\,/\,L$

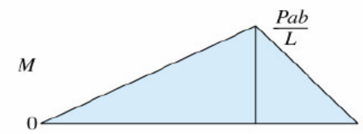
$M = R_A\,x - P\,(x - a) = P\,a\,(L - x)\,/\,L$

note that $\mathbf{dM\,/\,dx} = -P\,a\,/\,L = V$

with $M_{max} = P\,a\,b\,/\,L$



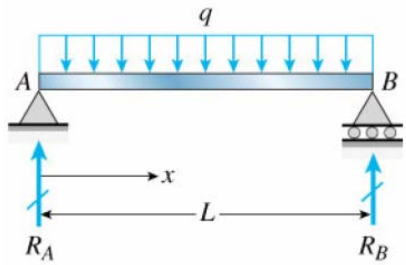
(d)



(e)

uniform load

consider a simple beam AB with a uniformly distributed load of constant intensity q

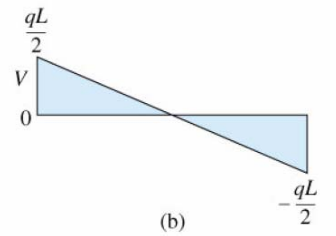


(a)

$$R_A = R_B = q\,L/2$$

$$V = R_A - q\,x = q\,L/2 - q\,x$$

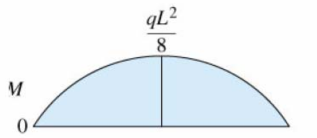
$$M = R_A\,x - q\,x\,(x/2) = q\,L\,x/2 - q\,x^2/2$$



(b)

note that $\quad \boldsymbol{dM/dx} = q\,L/2 - q\,x/2 = V$

$$M_{max} = q\,L^2/8 \quad \text{at} \quad x = L/2$$



(c)

several concentrated loads

for $\quad 0 < x < a_1 \qquad V = R_A \qquad M = R_A\,x$

$$M_1 = R_A\,a_1$$

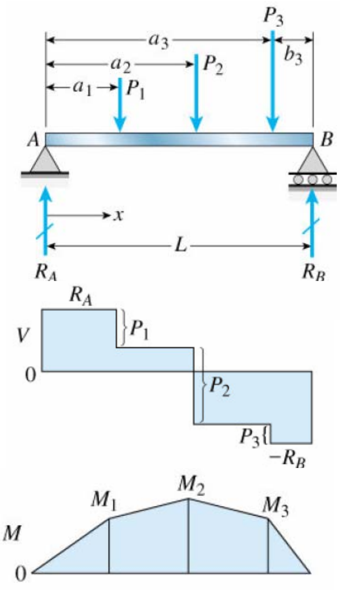
for $\quad a_1 < x < a_2 \qquad V = R_A - P_1$

$$M = R_A\,x - P_1\,(x - a_1)$$

$$M_2 - M_1 = (R_A - P_1)(a_2 - a_1)$$

similarly for others

$M_2 = M_{\max} \quad$ because $V = 0$ at that point



(c)

Example 4-4

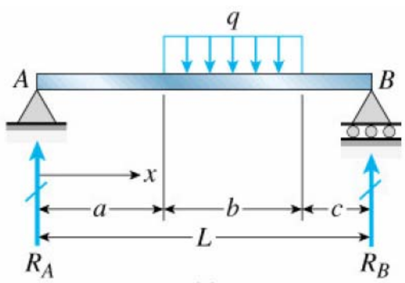
construct the shear-force and bending -moment diagrams for the simple beam *AB*

$$R_A = q\,b\,(b + 2c)/2L$$

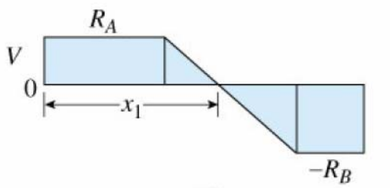
$$R_B = q\,b\,(b + 2a)/2L$$

for $\quad 0 < x < a$

$$V = R_A \qquad M = R_A\,x$$



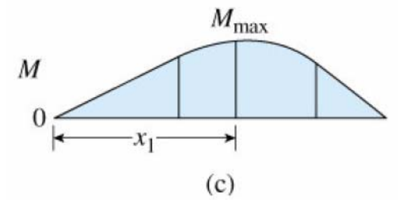
(a)



(b)

for $a < x < a + b$

$$V = R_A - q(x - a)$$
$$M = R_A x - q(x - a)^2 / 2$$



(c)

for $a + b < x < L$

$$V = -R_B \qquad M = R_B (L - x)$$

maximum moment occurs where $V = 0$

i.e. $x_1 = a + b(b + 2c) / 2L$

$$M_{max} = q\,b\,(b + 2c)(4\,a\,L + 2\,b\,c + b^2) / 8L^2$$

for $a = c$, $x_1 = L / 2$

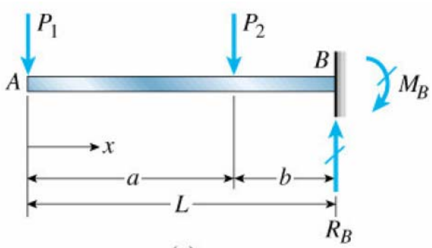
$$M_{max} = q\,b\,(2L - b) / 8$$

for $b = L$, $a = c = 0$ (uniform loading over the entire span)

$$M_{max} = q\,L^2 / 8$$

Example 4-5

construct the V- and M-dia for the cantilever beam supported to P_1 and P_2

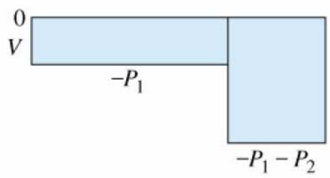


(a)

$$R_B = P_1 + P_2 \qquad M_B = P_1 L + P_2 b$$

for $0 < x < a$

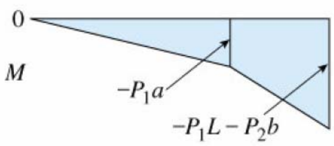
$$V = -P_1 \qquad M = -P_1 x$$



(b)

for $a < x < L$

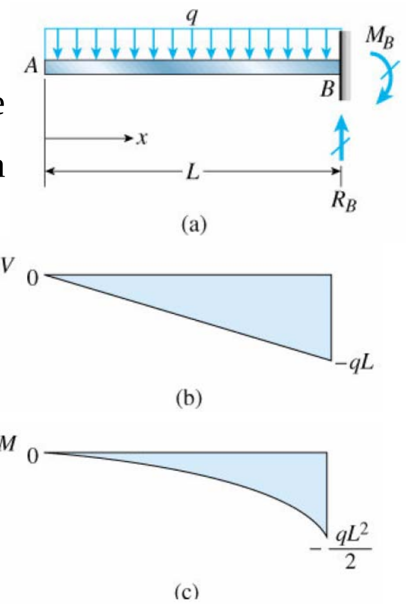
$$V = -P_1 - P_2 \qquad M = -P_1 x - P_2 (x - a)$$



(c)

Example 4-6

construct the V- and M-dia for the cantilever beam supporting to a constant uniform load of intensity q



$$R_B = q\,L \qquad M_B = q\,L^2 / 2$$

then $\quad V = -\,q\,x \qquad M = -\,q\,x^2 / 2$

$$V_{max} = -\,q\,L \qquad M_{max} = -\,q\,L^2 / 2$$

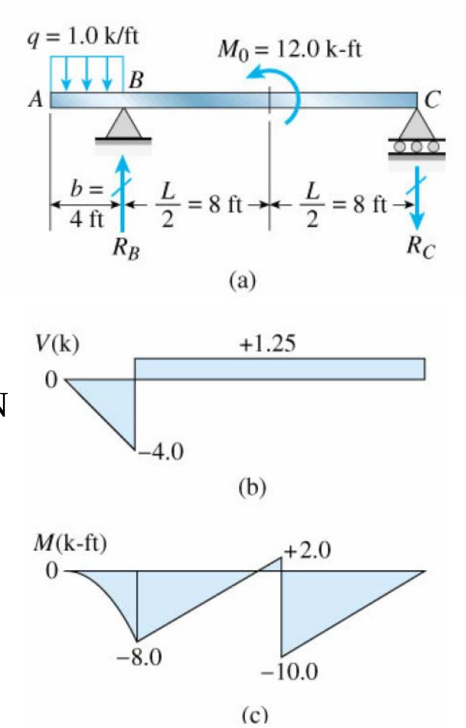
alternative method

$$V - V_A = V - 0 = V = -\int_0^x q\,dx = -\,q\,x$$

$$M - M_A = M - 0 = M = -\int_0^x V\,dx = -\,q\,x^2 / 2$$

Example 4-7

an overhanging beam is subjected to a uniform load of $q = 1$ kN/m on AB and a couple $M_0 = 12$ kN-m on midpoint of BC, construct the V- and M-dia for the beam



$$R_B = 5.25 \text{ kN} \qquad R_C = 1.25 \text{ kN}$$

shear force diagram

$$V = -\,q\,x \qquad \text{on } AB$$

$$V = \text{constant} \qquad \text{on } BC$$

bending moment diagram

$$M_B = - q\, b^2 / 2 = - 1 \text{ x } 4^2 / 2 = - 8 \text{ kN-m}$$

the slope of M on BC is constant (1.25 kN), the bending moment just to the left of M_0 is

$$M = - 8 + 1.25 \text{ x } 8 = 2 \text{ kN-m}$$

the bending moment just to the right of M_0 is

$$M = 2 - 12 = - \ 10 \text{ kN-m}$$

and the bending moment at point C is

$$M_C = - 10 + 1.25 \text{ x } 8 = 0 \qquad \text{as expected}$$