7. Apply inequality (1), Sec. 47, to show that for all values of x in the interval $-1 \le x \le 1$, the functions*

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x + i\sqrt{1-x^2}\cos\theta)^n \, d\theta \qquad (n = 0, 1, 2, \ldots)$$

satisfy the inequality $|P_n(x)| \le 1$.

8. Let C_N denote the boundary of the square formed by the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer and the orientation of C_N is counterclockwise.

(a) With the aid of the inequalities

$$|\sin z| \ge |\sin x| \quad \text{and} \quad |\sin z| \ge |\sinh y|,$$

obtained in Exercises 8(a) and 9(a) of Sec. 38, show that $|\sin z| \ge 1$ on the vertical sides of the square and that $|\sin z| > \sinh(\pi/2)$ on the horizontal sides. Thus show that there is a positive constant A, *independent of* N, such that $|\sin z| \ge A$ for all points z lying on the contour C_N.

(b) Using the final result in part (a), show that

$$\left|\int_{C_N} \frac{dz}{z^2 \sin z}\right| \le \frac{16}{(2N+1)\pi A}$$

and hence that the value of this integral tends to zero as N tends to infinity.

48. ANTIDERIVATIVES

Although the value of a contour integral of a function $f(z)$ from a fixed point z_1 to another fixed point z_2 depends, in general, on the path that is taken, there are certain functions whose integrals from z_1 to z_2 have values that are ***independent of path***.

Recall statements (a) and (b) at the end of Sec. 45. Those statements also remind us of the fact that the values of integrals around closed paths are sometimes, but not always, zero. Our next theorem is useful in determining when integration is independent of path and, moreover, when an integral around a closed path has value zero.

The theorem contains an extension of the fundamental theorem of calculus that simplifies the evaluation of many contour integrals. The extension involves the concept of an ***antiderivative*** of a continuous function $f(z)$ on a domain D, or a function $F(z)$ such that $F'(z) = f(z)$ for all z in D. Note that an antiderivative is, of necessity, an analytic function. Note, too, that *an antiderivative of a given function* $f(z)$ *is unique except for an additive constant*. This is because the derivative of the difference $F(z) - G(z)$ of any two such antiderivatives is zero; and, according to the theorem

*These functions are actually polynomials in x. The are known as ***Legendre polynomials*** and are important in applied mathematics. See, for example, the authors' book (2012) that is listed in Appendix 1. The expression for $P_n(x)$ used in Exercise 7 is sometimes called ***Laplace's first integral form***.

in Sec. 25, an analytic function is constant in a domain D when its derivative is zero throughout D.

Theorem. *Suppose that a function $f(z)$ is continuous in a domain D. If any one of the following statements is true, then so are the others:*

(a) *$f(z)$ has an antiderivative $F(z)$ throughout D;*

(b) *the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely*

$$\int_{z_1}^{z_2} f(z)\,dz = F(z)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative in statement (a);

(c) *the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.*

It should be emphasized that the theorem does *not* claim that any of these statements is true for a given function $f(z)$. It says only that all of them are true or that none of them is true. The next section is devoted to the proof of the theorem and can be easily skipped by a reader who wishes to get on with other important aspects of integration theory. But we include here a number of examples illustrating how the theorem can be used.

EXAMPLE 1. The continuous function $f(z) = e^{\pi z}$ evidently has an antiderivative $F(z) = e^{\pi z}/\pi$ throughout the finite plane. Hence

$$\int_{i}^{i/2} e^{\pi z} dz = \frac{e^{\pi z}}{\pi}\Big]_{i}^{i/2} = \frac{1}{\pi}\left(e^{i\pi/2} - e^{i\pi}\right) = \frac{1}{\pi}(i+1) = \frac{1}{\pi}(1+i).$$

EXAMPLE 2. The function $f(z) = 1/z^2$, which is continuous everywhere except at the origin, has an antiderivative $F(z) = -1/z$ in the domain $|z| > 0$, consisting of the entire plane with the origin deleted. Consequently,

$$\int_C \frac{dz}{z^2} = 0$$

when C is the positively oriented unit circle $z = e^{i\theta}\,(-\pi \le \theta \le \pi)$ about the origin.

Note that the integral of the function $f(z) = 1/z$ around the same circle *cannot* be evaluated in a similar way. For, although the derivative of any branch $F(z)$ of $\log z$ is $1/z$ (Sec. 33), $F(z)$ is not differentiable, or even defined, along its branch cut. In particular, if a ray $\theta = \alpha$ from the origin is used to form the branch cut, $F'(z)$ fails to exist at the point where that ray intersects the circle C (see Fig. 51). So C does not lie in any domain throughout which $F'(z) = 1/z$, and one cannot make direct use of an

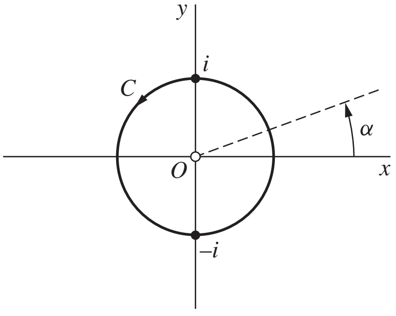

FIGURE 51

antiderivative. But Example 3, just below, illustrates how a combination of *two* different antiderivatives can be used to evaluate $f(z) = 1/z$ around C.

EXAMPLE 3. Let C_1 denote the *right* half

$$z = e^{i\theta} \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

of the circle C in Fig. 51. The principal branch

$$\operatorname{Log} z = \ln r + i\Theta \qquad (r > 0, -\pi < \Theta < \pi)$$

of the logarithmic function serves as an antiderivative of the function $1/z$ in the evaluation of the integral of $1/z$ along C_1 (Fig. 52):

$$\int_{C_1} \frac{dz}{z} = \int_{-i}^{i} \frac{dz}{z} = \operatorname{Log} z\Big]_{-i}^{i} = \operatorname{Log} i - \operatorname{Log}(-i)$$
$$= \left(\ln 1 + i\frac{\pi}{2}\right) - \left(\ln 1 - i\frac{\pi}{2}\right) = \pi i.$$

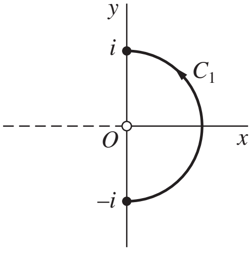

FIGURE 52

Next let C_2 denote the *left* half

$$z = e^{i\theta} \qquad \left(\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right)$$

of the same circle C and consider the branch

$$\log z = \ln r + i\theta \qquad (r > 0, 0 < \theta < 2\pi)$$

of the logarithmic function (Fig. 53). One can write

$$\int_{C_2} \frac{dz}{z} = \int_{i}^{-i} \frac{dz}{z} = \log z \Big]_{i}^{-i} = \log(-i) - \log i$$
$$= \left(\ln 1 + i\frac{3\pi}{2} \right) - \left(\ln 1 + i\frac{\pi}{2} \right) = \pi i.$$

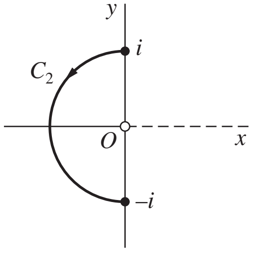

FIGURE 53

The value of the integral of $1/z$ around the entire circle $C = C_1 + C_2$ is thus obtained:

$$\int_{C} \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = \pi i + \pi i = 2\pi i.$$

EXAMPLE 4. Let us use an antiderivative to evaluate the integral

$$\int_{C_1} z^{1/2}\, dz, \tag{1}$$

where the integrand is the branch

$$f(z) = z^{1/2} = \exp\left(\frac{1}{2}\log z\right) = \sqrt{r}e^{i\theta/2} \qquad (r > 0, 0 < \theta < 2\pi) \tag{2}$$

of the square root function and where C_1 is any contour from $z = -3$ to $z = 3$ that, except for its end points, lies above the x axis (Fig. 54). Although the integrand is piecewise continuous on C_1, and the integral therefore exists, the branch (2) of $z^{1/2}$ is

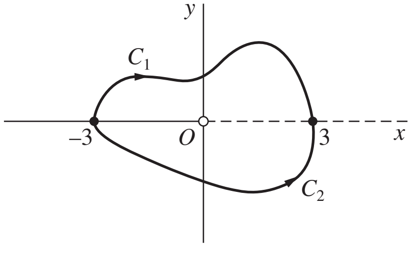

FIGURE 54

50. CAUCHY–GOURSAT THEOREM

In Sec. 48, we saw that when a continuous function f has an antiderivative on a domain D, the integral of $f(z)$ around any given closed contour C lying entirely in D has value zero. In this section, we present a theorem giving other conditions on a function f which ensure that the value of the integral of $f(z)$ around a ***simple closed contour*** (Sec. 43) is zero. The theorem is central to the theory of functions of a complex variable; and some modifications of it, involving certain special types of domains, will be given in Secs. 52 and 53.

We let C denote a simple closed contour $z = z(t)$ $(a \le t \le b)$, described in the ***positive sense*** (counterclockwise), and we assume that f is analytic at each point interior to and on C. According to Sec. 44,

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt; \tag{1}$$

and if

$$f(z) = u(x, y) + iv(x, y) \quad \text{and} \quad z(t) = x(t) + iy(t),$$

the integrand $f[z(t)]z'(t)$ in expression (1) is the product of the functions

$$u[x(t), y(t)] + iv[x(t), y(t)], \quad x'(t) + iy'(t)$$

of the real variable t. Thus

$$\int_C f(z)\,dz = \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt. \tag{2}$$

In terms of line integrals of real-valued functions of two real variables, then,

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy. \tag{3}$$

Observe that expression (3) can be obtained formally by replacing $f(z)$ and dz on the left with the binomials

$$u + iv \quad \text{and} \quad dx + i\,dy,$$

respectively, and expanding their product. Expression (3) is, of course, also valid when C is any contour, not necessarily a simple closed one, and when $f[z(t)]$ is only piecewise continuous on it.

We next recall a result from calculus that enables us to express the line integrals on the right in equation (3) as double integrals. Suppose that two real-valued functions $P(x, y)$ and $Q(x, y)$, together with their first-order partial derivatives, are continuous throughout the closed region R consisting of all points interior to and on the simple closed contour C. Green's theorem states that

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA.$$

Now f is continuous on R, since it is analytic there. Hence the functions u and v are also continuous on R. Likewise, if the derivative f' of f is continuous on R, so are the first-order partial derivatives of u and v. Green's theorem then enables us to rewrite equation (3) as

$$\int_C f(z)\,dz = \iint_R (-v_x - u_y)\,dA + i \iint_R (u_x - v_y)\,dA. \tag{4}$$

But, in view of the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x,$$

the integrands of these two double integrals are zero throughout R. So *when f is analytic in R and f' is continuous there*,

$$\int_C f(z)\,dz = 0. \tag{5}$$

This result was obtained by Cauchy in the early part of the nineteenth century.

Note that once it has been established that the value of this integral is zero, the orientation of C is immaterial. That is, statement (5) is also true if C is taken in the clockwise direction, since then

$$\int_C f(z)\,dz = -\int_{-C} f(z)\,dz = 0.$$

EXAMPLE. If C is any simple closed contour, in either direction, then

$$\int_C \sin(z^2)\,dz = 0.$$

This is because the composite function $f(z) = \sin(z^2)$ is analytic everywhere and its derivative $f'(z) = 2z\cos(z^2)$ is continuous everywhere.

Goursat* was the first to prove that *the condition of continuity on f' can be omitted*. Its removal is important and will allow us to show, for example, that the derivative f' of an analytic function f is analytic without having to assume the continuity of f', which follows as a consequence. We now state the revised form of Cauchy's result, which is known as the ***Cauchy-Goursat theorem***.

Theorem. *If a function f is analytic at all points interior to and on a simple closed contour C, then*

$$\int_C f(z)\,dz = 0.$$

The proof is presented in the next section, where, to be specific, we assume that C is positively oriented. The reader who wishes to accept this theorem without proof may pass directly to Sec. 52.

*E. Goursat (1858–1936), pronounced *gour-sah'*.

Conclusion

We now use the theorem in Sec. 47 to find an upper bound for each modulus on the right in inequality (8). To do this, we first recall that each C_j coincides either entirely or partially with the boundary of a square. In either case, we let s_j denote the length of a side of the square. Since, in the jth integral, both the variable z and the point z_j lie in that square,

$$|z - z_j| \leq \sqrt{2}s_j.$$

In view of inequality (5), then, we know that each integrand on the right in inequality (8) satisfies the condition

$$(9) \qquad |(z - z_j)\delta_j(z)| = |z - z_j|\,|\delta_j(z)| < \sqrt{2}s_j\varepsilon.$$

As for the length of the path C_j, it is $4s_j$ if C_j is the boundary of a square. In that case, we let A_j denote the area of the square and observe that

$$(10) \qquad \left|\int_{C_j}(z - z_j)\delta_j(z)\,dz\right| < \sqrt{2}s_j\varepsilon 4s_j = 4\sqrt{2}A_j\varepsilon.$$

If C_j is the boundary of a partial square, its length does not exceed $4s_j + L_j$, where L_j is the length of that part of C_j which is also a part of C. Again letting A_j denote the area of the full square, we find that

$$(11) \qquad \left|\int_{C_j}(z - z_j)\delta_j(z)\,dz\right| < \sqrt{2}s_j\varepsilon(4s_j + L_j) < 4\sqrt{2}A_j\varepsilon + \sqrt{2}SL_j\varepsilon,$$

where S is the length of a side of some square that encloses the entire contour C as well as all of the squares originally used in covering R (Fig. 59). Note that the sum of all the A_j's does not exceed S^2.

If L denotes the length of C, it now follows from inequalities (8), (10), and (11) that

$$\left|\int_C f(z)\,dz\right| < (4\sqrt{2}S^2 + \sqrt{2}SL)\varepsilon.$$

Since the value of the positive number ε is arbitrary, we can choose it so that the right-hand side of this last inequality is as small as we please. The left-hand side, which is independent of ε, must therefore be equal to zero; and statement (3) follows. This completes the proof of the Cauchy–Goursat theorem.

52. SIMPLY CONNECTED DOMAINS

A ***simply connected*** domain D is a domain such that every simple closed contour within it encloses only points of D. The set of points interior to a simple closed contour is an example. The annular domain between two concentric circles is, however, *not* simply connected. Domains that are not simply connected are discussed in the next section.

The closed contour in the Cauchy–Goursat theorem (Sec. 50) need not be simple when the theorem is adapted to simply connected domains. More precisely, the contour can actually cross itself. The following theorem allows for this possibility.

Theorem. *If a function f is analytic throughout a simply connected domain D, then*

$$\int_C f(z)\,dz = 0 \tag{1}$$

for every closed contour C lying in D.

The proof is easy if C is a *simple* closed contour or if it is a closed contour that intersects itself a *finite* number of times. For if C is simple and lies in D, the function f is analytic at each point interior to and on C; and the Cauchy–Goursat theorem ensures that equation (1) holds. Furthermore, if C is closed but intersects itself a finite number of times, it consists of a finite number of simple closed contours, and the Cauchy-Goursat theorem can again be applied. This is illustrated in Fig. 60, where two simple closed contours C_1 and C_2 make up C. Since the values of the integrals around C_1 and C_2 are zero, *regardless of their orientations*,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0.$$

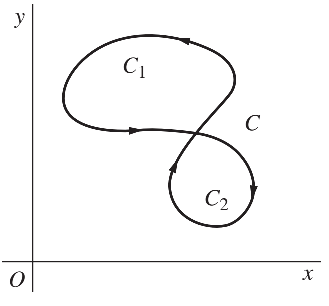

FIGURE 60

Subtleties arise if the closed contour has an *infinite* number of self-intersection points. One method that can sometimes be used to show that the theorem still applies is illustrated in Exercise 5, Sec. 53.*

EXAMPLE. If C denotes any closed contour lying in the open disk $|z| < 2$ (Fig. 61), then

$$\int_C \frac{\sin z}{(z^2+9)^5}\,dz = 0.$$

*For a proof of the theorem involving more general paths of finite length, see, for example, Secs. 63–65 in Vol. I of the book by Markushevich that is cited in Appendix 1.

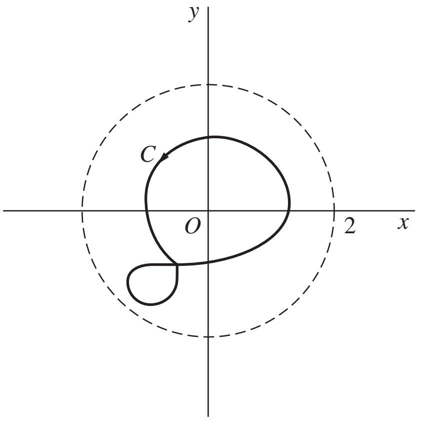

FIGURE 61

This is because the disk is a simply connected domain and the two singularities $z = \pm 3i$ of the integrand are exterior to the disk.

Corollary 1. *A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.*

We begin the proof of this corollary with the observation that a function f is continuous on a domain D when it is analytic there. Consequently, since equation (1) holds for the function in the hypothesis of this corollary and for each closed contour C in D, f has an antiderivative throughout D, according to the theorem in Sec. 48.

Corollary 2. *Entire functions always possess antiderivatives.*

This corollary is an immediate consequence of Corollary 1 and the fact that the finite plane is simply connected.

53. MULTIPLY CONNECTED DOMAINS

A domain that is not simply connected (Sec. 52) is said to be ***multiply connected***. The following theorem is an adaptation of the Cauchy–Goursat theorem to multiply connected domains. While the statement of the theorem involves n contours labeled $C_k (k = 1, 2, \ldots, n)$, we shall be guided in the proof by Fig. 62, where $n = 2$.

Theorem. *Suppose that*

(*a*) *C is a simple closed contour, described in the counterclockwise direction;*

(*b*) *C_k $(k = 1, 2, \ldots, n)$ are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common* (Fig. 62).

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k, then

(1)
$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0.$$

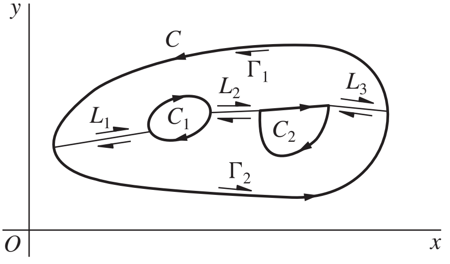

FIGURE 62

Note that in equation (1), the direction of each path of integration is such that the multiply connected domain lies to the *left* of that path.

To prove the theorem, we introduce a polygonal path L_1, consisting of a finite number of line segments joined end to end, to connect the outer contour C to the inner contour C_1. We introduce another polygonal path L_2 which connects C_1 to C_2; and we continue in this manner, with L_{n+1} connecting C_n to C. As indicated by the single-barbed arrows in Fig. 62, two simple closed contours Γ_1 and Γ_2 can be formed, each consisting of polygonal paths L_k or $-L_k$ and pieces of C and C_k and each described in such a direction that the points enclosed by them lie to the left. The Cauchy–Goursat theorem can now be applied to f on Γ_1 and Γ_2, and the sum of the values of the integrals over those contours is found to be zero. Since the integrals in opposite directions along each path L_k cancel, only the integrals along C and the C_k remain; and we arrive at statement (1).

Corollary. *Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2* (Fig. 63). *If a function f is analytic in the closed region consisting of those contours and all points between them, then*

(2)
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

This corollary is known as the ***principle of deformation of paths*** since it tells us that if C_1 is continuously deformed into C_2, always passing through points at which f is analytic, then the value of f over C_1 never changes. To verify this corollary. we need only observe how it follows from the theorem that

$$\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz = 0.$$

But this the same as equation (2).

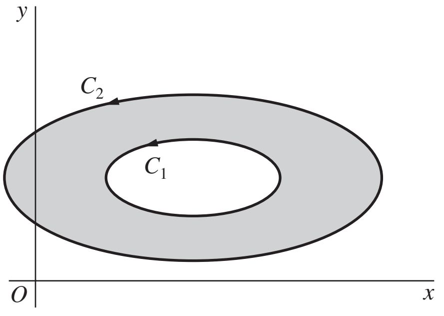

FIGURE 63

EXAMPLE. When C is any positively oriented simple closed contour surrounding the origin, the corollary can be used to show that

$$\int_C \frac{dz}{z} = 2\pi i.$$

This is done by constructing a positively oriented circle C_0 with center at the origin and radius so small that C_0 lies entirely inside C (Fig. 64). Since (see Exercise 13, Sec. 46)

$$\int_{C_0} \frac{dz}{z} = 2\pi i$$

and since $1/z$ is analytic everywhere except at $z = 0$, the desired result follows.

Note that the radius of C_0 could equally well have been so large that C lies entirely inside C_0.

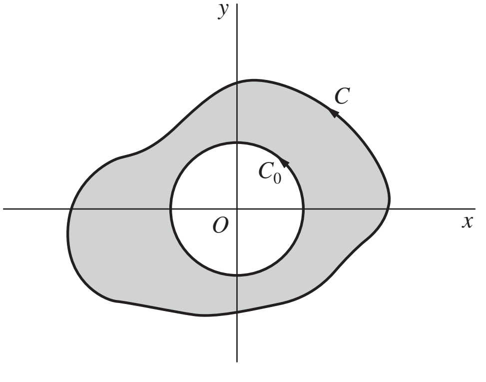

FIGURE 64

EXERCISES

1. Apply the Cauchy–Goursat theorem to show that

$$\int_C f(z)\,dz = 0$$

when the contour C is the unit circle $|z| = 1$, in either direction, and when

(*a*) $f(z) = \dfrac{z^2}{z+3}$; (*b*) $f(z) = z\,e^{-z}$; (*c*) $f(z) = \dfrac{1}{z^2+2z+2}$;

(*d*) $f(z) = \operatorname{sech} z$; (*e*) $f(z) = \tan z$; (*f*) $f(z) = \operatorname{Log}(z+2)$.

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$ (Fig. 65). With the aid of the corollary in Sec. 53, point out why

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

when

(*a*) $f(z) = \dfrac{1}{3z^2+1}$; (*b*) $f(z) = \dfrac{z+2}{\sin(z/2)}$; (*c*) $f(z) = \dfrac{z}{1-e^z}$.

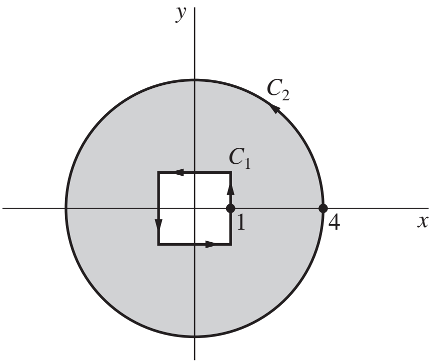

FIGURE 65

3. If C_0 denotes a positively oriented circle $|z - z_0| = R$, then

$$\int_{C_0} (z - z_0)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0, \end{cases}$$

according to Exercise 13, Sec. 46. Use that result and the corollary in Sec. 53 to show that if C is the boundary of the rectangle $0 \le x \le 3$, $0 \le y \le 2$, described in the positive sense, then

$$\int_C (z - 2 - i)^{n-1}\,dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0. \end{cases}$$

4. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \qquad (b > 0).$$

(*a*) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in Fig. 66 can be written

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy.$$

Thus, with the aid of the Cauchy–Goursat theorem, show that

$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy.$$

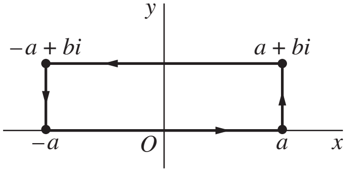

FIGURE 66

(*b*) By accepting the fact that*

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| \le \int_0^b e^{y^2}dy,$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part (*a*).

5. According to Exercise 6, Sec. 43, the path C_1 from the origin to the point $z = 1$ along the graph of the function defined by means of the equations

$$y(x) = \begin{cases} x^3\sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0 \end{cases}$$

is a smooth arc that intersects the real axis an infinite number of times. Let C_2 denote the line segment along the real axis from $z = 1$ back to the origin, and let C_3 denote any smooth arc from the origin to $z = 1$ that does not intersect itself and has only its end points in common with the arcs C_1 and C_2 (Fig. 67). Apply the Cauchy–Goursat

*The usual way to evaluate this integral is by writing its square as

$$\int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$$

and then evaluating this iterated integral by changing to polar coordinates. Details are given in, for example, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 680–681, 1983.