CHAPTER

4

# INTEGRALS

Integrals are extremely important in the study of functions of a complex variable. The theory of integration, to be developed in this chapter, is noted for its mathematical elegance. The theorems are generally concise and powerful, and many of the proofs are short.

## 41. DERIVATIVES OF FUNCTIONS $w(t)$

In order to introduce integrals of $f(z)$ in a fairly simple way, we need to first consider derivatives of complex-valued functions $w$ of a *real* variable $t$. We write

$$w(t) = u(t) + iv(t), \tag{1}$$

where the functions $u$ and $v$ are *real-valued* functions of $t$. The derivative

$$w'(t), \text{ or } \frac{d}{dt}w(t),$$

of the function (1) at a point $t$ is defined as

$$w'(t) = u'(t) + iv'(t), \tag{2}$$

provided each of the derivatives $u'$ and $v'$ exists at $t$.

Various rules learned in calculus, such as the ones for differentiating sums and products, apply just as they do for real-valued functions of a real variable $t$. Verifications can often be based on corresponding rules in calculus.

**EXAMPLE 1.** Assuming that the functions $u(t)$ and $v(t)$ in expression (1) are differentiable at $t$, let us prove that

$$\frac{d}{dt}[w(t)]^2 = 2w(t)w'(t). \tag{3}$$

To do this, we begin by writing

$$[w(t)]^2 = (u + iv)^2 = u^2 - v^2 + i2uv.$$

Then

$$\begin{aligned}\frac{d}{dt}[w(t)]^2 &= (u^2 - v^2)' + i(2uv)' \\ &= 2uu' - 2vv' + i2(uv' + u'v) \\ &= 2(u + iv)(u' + iv'),\end{aligned}$$

and we arrive at expression (3).

**EXAMPLE 2.** Another expected rule for differentiation that we shall often use is

$$\frac{d}{dt}e^{z_0 t} = z_0 e^{z_0 t}, \tag{4}$$

where $z_0 = x_0 + iy_0$. To verify this, we write

$$e^{z_0 t} = e^{x_0 t} e^{iy_0 t} = e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t$$

and refer to definition (2) to see that

$$\frac{d}{dt}e^{z_0 t} = (e^{x_0 t} \cos y_0 t)' + i(e^{x_0 t} \sin y_0 t)'.$$

Familiar rules from calculus and some simple algebra then lead us to the expression

$$\frac{d}{dt}e^{z_0 t} = (x_0 + iy_0)(e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t),$$

or

$$\frac{d}{dt}e^{z_0 t} = (x_0 + iy_0)e^{x_0 t} e^{iy_0 t}.$$

This is, of course, the same as equation (4).

While many rules in calculus carry over to functions of the type (1), not all of them do. The following example illustrates this.

**EXAMPLE 3.** Suppose that $w(t)$ is continuous on an interval $a \le t \le b$; that is, its component functions $u(t)$ and $v(t)$ are continuous there. Even if $w'(t)$ exists when $a < t < b$, the mean value theorem for derivatives no longer applies. To be precise, it is not necessarily true that there is a number $c$ in the interval $a < t < b$ such that

$$w'(c) = \frac{w(b) - w(a)}{b - a}. \tag{5}$$

To see this, consider the function $w(t) = e^{it}$ on the interval $0 \le t \le 2\pi$. When that function is used, $|w'(t)| = |ie^{it}| = 1$ (see Example 2); and this means that the derivative $w'(c)$ on the left in equation (5) is never zero. As for the quotient on the right in equation (5),

$$\frac{w(b) - w(a)}{b - a} = \frac{w(2\pi) - w(0)}{2\pi - 0} = \frac{e^{i2\pi} - e^{i0}}{2\pi} = \frac{1 - 1}{2\pi} = 0.$$

So there is no number $c$ such that equation (5) holds.

## 42. DEFINITE INTEGRALS OF FUNCTIONS *w(t)*

When $w(t)$ is a complex-valued function of a real variable $t$ and is written

$$w(t) = u(t) + iv(t), \tag{1}$$

where $u$ and $v$ are real-valued, the ***definite integral*** of $w(t)$ over an interval $a \le t \le b$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt, \tag{2}$$

provided the individual integrals on the right exist. Thus

$$\text{Re}\int_a^b w(t)\,dt = \int_a^b \text{Re}[w(t)]\,dt \quad \text{and} \quad \text{Im}\int_a^b w(t)\,dt = \int_a^b \text{Im}[w(t)]\,dt. \tag{3}$$

**EXAMPLE 1.** For an illustration of definition (2),

$$\int_0^{\pi/4} e^{it}\,dt = \int_0^{\pi/4} (\cos t + i\sin t)\,dt = \int_0^{\pi/4} \cos t\,dt + i\int_0^{\pi/4} \sin t\,dt$$
$$= [\sin t]_0^{\pi/4} + i[-\cos t]_0^{\pi/4} = \frac{1}{\sqrt{2}} + i\left(-\frac{1}{\sqrt{2}} + 1\right).$$

Improper integrals of $w(t)$ over unbounded intervals are defined in a similar way. [See Exercise 2(*d*).]

The existence of the integrals of $u$ and $v$ in definition (2) is ensured if those functions are ***piecewise continuous*** on the interval $a \le t \le b$. Such a function is continuous everywhere in the stated interval except possibly for a finite number of points where, although discontinuous, it has one-sided limits. Of course, only the right-hand limit is required at $a$; and only the left-hand limit is required at $b$. When both $u$ and $v$ are piecewise continuous, the function $w$ is said to have that property.

Anticipated rules for integrating a complex constant times a function $w(t)$, for integrating sums of such functions, and for interchanging limits of integration are all valid. Those rules, as well as the property

$$\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt,$$

are easy to verify by recalling corresponding results in calculus.

The *fundamental theorem of calculus,* involving antiderivatives, can, moreover, be extended so as to apply to integrals of the type (2). To be specific, suppose that the functions

$$w(t) = u(t) + iv(t) \quad \text{and} \quad W(t) = U(t) + iV(t)$$

are continuous on the interval $a \le t \le b$. If $W'(t) = w(t)$ when $a \le t \le b$, then $U'(t) = u(t)$ and $V'(t) = v(t)$. Hence, in view of definition (2),

$$\int_a^b w(t)\,dt = [U(t)]_a^b + i[V(t)]_a^b = [U(b) + iV(b)] - [U(a) + iV(a)].$$

That is,

$$\int_a^b w(t)\,dt = W(b) - W(a) = W(t)\Big]_a^b. \tag{4}$$

We now have another way to evaluate the integral of $e^{it}$ in Example 1.

**EXAMPLE 2.** Since (see Example 2 in Sec. 41)

$$\frac{d}{dt}\left(\frac{e^{it}}{i}\right) = \frac{1}{i}\frac{d}{dt}e^{it} = \frac{1}{i}\,ie^{it} = e^{it},$$

one can see that

$$\int_0^{\pi/4} e^{it}\,dt = \frac{e^{it}}{i}\Big]_0^{\pi/4} = \frac{e^{i\pi/4}}{i} - \frac{1}{i} = \frac{1}{i}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - 1\right)$$
$$= \frac{1}{i}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - 1\right) = \frac{1}{\sqrt{2}} + \frac{1}{i}\left(\frac{1}{\sqrt{2}} - 1\right).$$

Then, because $1/i = -i$,

$$\int_0^{\pi/4} e^{it}\,dt = \frac{1}{\sqrt{2}} + i\left(-\frac{1}{\sqrt{2}} + 1\right).$$

We recall from Example 3 in Sec. 41 how the mean value theorem for derivatives in calculus does not carry over to complex-valued functions $w(t)$. Our final example here shows that the mean value theorem for *integrals* does not carry over either. Thus special care must continue to be used in applying rules from calculus.

**EXAMPLE 3.** Let $w(t)$ be a continuous complex-valued function of $t$ defined on an interval $a \le t \le b$. In order to show that it is not necessarily true that there is a number $c$ in the interval $a < t < b$ such that

$$\int_a^b w(t)\,dt = w(c)(b - a), \tag{5}$$

we write $a = 0$, $b = 2\pi$ and use the same function $w(t) = e^{it}(0 \le t \le 2\pi)$ as in Example 3, Sec. 41. It is easy to see that

$$\int_a^b w(t)\,dt = \int_0^{2\pi} e^{it}\,dt = \frac{e^{it}}{i}\bigg]_0^{2\pi} = 0.$$

But, for any number $c$ such that $0 < c < 2\pi$,

$$|w(c)(b-a)| = |e^{ic}|\,2\pi = 2\pi;$$

and we find that the left-hand side of equation (5) is zero but that the right-hand side is not.

## EXERCISES

**1.** Use rules in calculus to establish the following rules when

$$w(t) = u(t) + iv(t)$$

is a complex-valued function of a real variable $t$ and $w'(t)$ exists:

($a$) $\dfrac{d}{dt}[z_0 w(t)] = z_0 w'(t)$, where $z_0 = x_0 + iy_0$ is a complex constant;

($b$) $\dfrac{d}{dt} w(-t) = -w'(-t)$ where $w'(-t)$ denotes the derivative of $w(t)$ with respect to $t$, evaluated at $-t$;

*Suggestion:* In part ($a$). show that each side of the identity to be verified can be written

$$(x_0 u' - y_0 v') + i(y_0 u' + x_0 v').$$

**2.** Evaluate the following integrals:

($a$) $\displaystyle\int_0^1 (1+it)^2 dt;$ ($b$) $\displaystyle\int_1^2 \left(\frac{1}{t} - i\right)^2 dt;$

($c$) $\displaystyle\int_0^{\pi/6} e^{i2t}\,dt;$ ($d$) $\displaystyle\int_0^{\infty} e^{-zt}\,dt$ $(\operatorname{Re} z > 0)$.

*Ans.* ($a$) $\dfrac{2}{3} + i$; ($b$) $-\dfrac{1}{2} - i\ln 4$; ($c$) $\dfrac{\sqrt{3}}{4} + \dfrac{i}{4}$; ($d$) $\dfrac{1}{z}$.

**3.** Show that if $m$ and $n$ are integers,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\,d\theta = \begin{cases} 0 & \text{when } m \ne n, \\ 2\pi & \text{when } m = n. \end{cases}$$

**4.** According to definition (2), Sec. 42, of definite integrals of complex-valued functions of a real variable,

$$\int_0^{\pi} e^{(1+i)x}\,dx = \int_0^{\pi} e^x \cos x\,dx + i\int_0^{\pi} e^x \sin x\,dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

*Ans.* $-(1+e^{\pi})/2, \quad (1+e^{\pi})/2.$

**5.** Let $w(t) = u(t) + iv(t)$ denote a continuous complex-valued function defined on an interval $-a \le t \le a$.

(*a*) Suppose that $w(t)$ is *even*; that is, $w(-t) = w(t)$ for each point $t$ in the given interval. Show that

$$\int_{-a}^{a} w(t)\,dt = 2\int_{0}^{a} w(t)\,dt.$$

(*b*) Show that if $w(t)$ is an *odd* function, one where $w(-t) = -w(t)$ for each point $t$ in the given interval, then

$$\int_{-a}^{a} w(t)\,dt = 0.$$

*Suggestion:* In each part of this exercise, use the corresponding property of integrals of *real-valued* functions of $t$, which is graphically evident.

## 43. CONTOURS

Integrals of complex-valued functions of a *complex* variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are introduced in this section.

A set of points $z = (x, y)$ in the complex plane is said to be an ***arc*** if

(1) $$x = x(t), \quad y = y(t) \qquad (a \le t \le b),$$

where $x(t)$ and $y(t)$ are continuous functions of the real parameter $t$. This definition establishes a continuous mapping of the interval $a \le t \le b$ into the $xy$, or $z$, plane; and the image points are ordered according to increasing values of $t$. It is convenient to describe the points of $C$ by means of the equation

(2) $$z = z(t) \qquad (a \le t \le b),$$

where

(3) $$z(t) = x(t) + iy(t).$$

The arc $C$ is a ***simple arc***, or a Jordan arc,* if it does not cross itself; that is, $C$ is simple if $z(t_1) \ne z(t_2)$ when $t_1 \ne t_2$. When the arc $C$ is simple except for the fact that $z(b) = z(a)$, we say that $C$ is a ***simple closed curve***, or a Jordan curve. Such a curve is ***positively oriented*** when it is in the counterclockwise direction.

The geometric nature of a particular arc often suggests different notation for the parameter $t$ in equation (2). This is, in fact, the case in the following examples.

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*Named for C. Jordan (1838–1922), pronounced *jor-don′*.

**EXAMPLE 1.** The polygonal line (Sec. 12) defined by means of the equations

$$z = \begin{cases} x + ix & \text{when } 0 \le x \le 1, \\ x + i & \text{when } 1 \le x \le 2 \end{cases} \tag{4}$$

and consisting of a line segment from 0 to $1 + i$ followed by one from $1 + i$ to $2 + i$ (Fig. 36) is a simple arc.

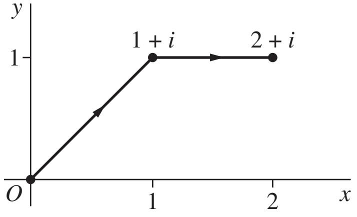


**FIGURE 36**

**EXAMPLE 2.** The unit circle

$$z = e^{i\theta} \qquad (0 \le \theta \le 2\pi) \tag{5}$$

about the origin is a simple closed curve, oriented in the counterclockwise direction. So is the circle

$$z = z_0 + Re^{i\theta} \qquad (0 \le \theta \le 2\pi), \tag{6}$$

centered at the point $z_0$ and with radius $R$ (see Sec. 7).

The same set of points can make up different arcs.

**EXAMPLE 3.** The arc

$$z = e^{-i\theta} \qquad (0 \le \theta \le 2\pi) \tag{7}$$

is not the same as the arc described by equation (5). The set of points is the same, but now the circle is traversed in the *clockwise* direction.

**EXAMPLE 4.** The points on the arc

$$z = e^{i2\theta} \qquad (0 \le \theta \le 2\pi) \tag{8}$$

are the same as those making up the arcs (5) and (7). The arc here differs, however, from each of those arcs since the circle is traversed *twice* in the counterclockwise direction.

The parametric representation used for any given arc $C$ is, of course, not unique. It is, in fact, possible to change the interval over which the parameter ranges to any other interval. To be specific, suppose that

$$t = \phi(\tau) \qquad (\alpha \le \tau \le \beta), \tag{9}$$

where $\phi$ is a real-valued function mapping an interval $\alpha \le \tau \le \beta$ onto the interval $a \le t \le b$ in representation (2). (See Fig. 37.) We assume that $\phi$ is continuous with a continuous derivative. We also assume that $\phi'(\tau) > 0$ for each $\tau$; this ensures that $t$ increases with $\tau$. Representation (2) is then transformed by equation (9) into

$$z = Z(\tau) \qquad (\alpha \le \tau \le \beta), \tag{10}$$

where

$$Z(\tau) = z[\phi(\tau)]. \tag{11}$$

This is illustrated in Exercise 3.

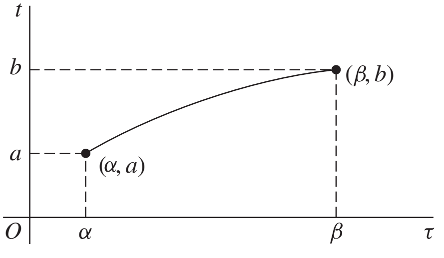


**FIGURE 37**
$t = \phi(\tau)$

Suppose now that the components $x'(t)$ and $y'(t)$ of the derivative (Sec. 41)

$$z'(t) = x'(t) + iy'(t) \tag{12}$$

of the function (3), used to represent $C$, are continuous on the entire interval $a \le t \le b$. The arc is then called a ***differentiable arc***, and the real-valued function

$$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable over the interval $a \le t \le b$. In fact, according to the definition of arc length in calculus, the length of $C$ is the number

$$L = \int_a^b |z'(t)|\, dt. \tag{13}$$

The value of $L$ is invariant under certain changes in the representation for $C$ that is used, as one would expect. More precisely, with the change of variable indicated in equation (9), expression (13) takes the form [see Exercise 1(*b*)]

$$L = \int_\alpha^\beta |z'[\phi(\tau)]|\phi'(\tau)\, d\tau.$$

So, if representation (10) is used for $C$, the derivative (Exercise 4)

$$Z'(\tau) = z'[\phi(\tau)]\phi'(\tau) \tag{14}$$

enables us to write expression (13) as

$$L = \int_{\alpha}^{\beta} |Z'(\tau)|\, d\tau.$$

Thus the same length of $C$ would be obtained if representation (10) were to be used.

If equation (2) represents a differentiable arc and if $z'(t) \neq 0$ anywhere in the interval $a < t < b$, then the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

is well defined for all $t$ in that open interval, with angle of inclination $\arg z'(t)$. Also, when $\mathbf{T}$ turns, it does so continuously as the parameter $t$ varies over the entire interval $a < t < b$. This expression for $\mathbf{T}$ is the one learned in calculus when $z(t)$ is interpreted as a radius vector. Such an arc is said to be ***smooth***. In referring to a smooth arc $z = z(t)$ $(a \leq t \leq b)$, then, we agree that the derivative $z'(t)$ is continuous on the closed interval $a \leq t \leq b$ and nonzero throughout the open interval $a < t < b$.

A ***contour***, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. Hence if equation (2) represents a contour, $z(t)$ is continuous, whereas its derivative $z'(t)$ is piecewise continuous. The polygonal line (4) is, for example, a contour. When only the initial and final values of $z(t)$ are the same, a contour $C$ is called a ***simple closed contour***. Examples are the circles (5) and (6), as well as the boundary of a triangle or a rectangle taken in a specific direction. The length of a contour or a simple closed contour is the sum of the lengths of the smooth arcs that make up the contour.

The points on any simple closed curve or simple closed contour $C$ are boundary points of two distinct domains, one of which is the interior of $C$ and is bounded. The other, which is the exterior of $C$, is unbounded. It will be convenient to accept this statement, known as the ***Jordan curve theorem***, as geometrically evident; the proof is not easy.*

## EXERCISES

**1.** Show that if $w(t) = u(t) + iv(t)$ is continuous on an interval $a \leq t \leq b$, then

(*a*) $\displaystyle\int_{-b}^{-a} w(-t)\, dt = \int_{a}^{b} w(\tau)\, d\tau;$

(*b*) $\displaystyle\int_{a}^{b} w(t)\, dt = \int_{\alpha}^{\beta} w[\phi(\tau)]\phi'(\tau)\, d\tau$, where $\phi(\tau)$ is the function in equation (9), Sec. 43.

*Suggestion:* These identities can be obtained by noting that they are valid for *real-valued* functions of $t$.

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*See pp. 115–116 of the book by Newman or Sec. 13 of the one by Thron, both of which are cited in Appendix 1. The special case in which $C$ is a simple closed polygon is proved on pp. 281–285 of Vol. 1 of the work by Hille, also cited in Appendix 1.

**2.** Let $C$ denote the right-hand half of the circle $|z| = 2$, in the counterclockwise direction, and note that two parametric representations for $C$ are

$$z = z(\theta) = 2\,e^{i\theta} \qquad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

and

$$z = Z(y) = \sqrt{4 - y^2} + iy \qquad (-2 \le y \le 2).$$

Verify that $Z(y) = z[\phi(y)]$, where

$$\phi(y) = \arctan \frac{y}{\sqrt{4 - y^2}} \qquad \left(-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}\right).$$

Also, show that this function $\phi$ has a positive derivative, as required in the conditions following equation (9), Sec. 43.

**3.** Derive the equation of the line through the points $(\alpha, a)$ and $(\beta, b)$ in the $\tau t$ plane that are shown in Fig. 37. Then use it to find the linear function $\phi(\tau)$ which can be used in equation (9), Sec. 43, to transform representation (2) in that section into representation (10) there.

*Ans.* $\phi(\tau) = \dfrac{b - a}{\beta - \alpha}\,\tau + \dfrac{a\beta - b\alpha}{\beta - \alpha}.$

**4.** Verify expression (14), Sec. 43, for the derivative of $Z(\tau) = z[\phi(\tau)]$.

*Suggestion:* Write $Z(\tau) = x[\phi(\tau)] + iy[\phi(\tau)]$ and apply the chain rule for real-valued functions of a real variable.

**5.** Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ lying on a smooth arc $z = z(t)$ $(a \le t \le b)$. Show that if $w(t) = f[z(t)]$, then

$$w'(t) = f'[z(t)]z'(t)$$

when $t = t_0$.

*Suggestion:* Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$, so that

$$w(t) = u[x(t), y(t)] + iv[x(t), y(t)].$$

Then apply the chain rule in calculus for functions of two real variables to write

$$w' = (u_x x' + u_y y') + i(v_x x' + v_y y'),$$

and use the Cauchy–Riemann equations.

**6.** Let $y(x)$ be a real-valued function defined on the interval $0 \le x \le 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin(\pi/x) & \text{when } 0 < x \le 1, \\ 0 & \text{when } x = 0. \end{cases}$$

(*a*) Show that the equation

$$z = x + iy(x) \qquad (0 \le x \le 1)$$

represents an arc $C$ that intersects the real axis at the points $z = 1/n$ $(n = 1, 2, \ldots)$ and $z = 0$, as shown in Fig. 38.

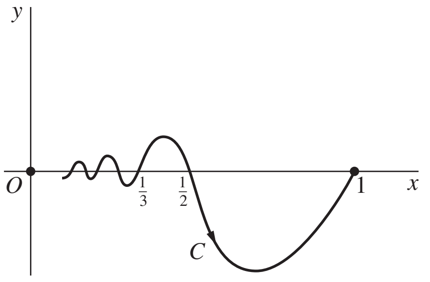


FIGURE 38

(*b*) Verify that the arc $C$ in part (*a*) is, in fact, a *smooth* arc.

*Suggestion:* To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \le \left| x^3 \sin\left(\frac{\pi}{x}\right) \right| \le x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

## 44. CONTOUR INTEGRALS

We turn now to integrals of complex-valued functions $f$ of the complex variable $z$. Such an integral is defined in terms of the values $f(z)$ along a given contour $C$, extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane. It is, therefore, a line integral; and its value depends, in general, on the contour $C$ as well as on the function $f$. It is written

$$\int_C f(z)\,dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)\,dz,$$

the latter notation often being used when the value of the integral is independent of the choice of the contour taken between two fixed end points. While the integral can be defined directly as the limit of a sum,* we choose to define it in terms of a definite integral of the type introduced in Sec. 42.

Definite integrals in calculus can be interpreted as areas, and they have other interpretations as well. Except in special cases, no corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane.

Suppose that the equation

$$z = z(t) \quad (a \le t \le b) \tag{1}$$

represents a contour $C$, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. We assume that $f[z(t)]$ is *piecewise continuous* (Sec. 42) on the interval $a \le t \le b$ and

*See, for instance, pp. 245ff in Vol. I of the book by Markushevich that is listed in Appendix 1.

refer to the function $f(z)$ as being piecewise continuous on $C$. We then define the line integral, or ***contour integral***, of $f$ along $C$ in terms of the parameter $t$:

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt. \tag{2}$$

Note that since $C$ is a contour, $z'(t)$ is also piecewise continuous on $a \le t \le b$; and so the existence of integral (2) is ensured.

The value of a contour integral is invariant under a change in the representation of its contour when the change is of the type (11), Sec. 43. This can be seen by following the same general procedure that was used in Sec. 43 to show the invariance of arc length.

We mention here some important and expected properties of contour integrals; and we begin with the agreement that when a contour $C$ is given, $-C$ denotes the same set of points on $C$ but with the order of those points reversed (Fig. 39). Observe that if $C$ has the representation (1), a representation for $-C$ is

$$z = z(-t) \qquad (-b \le t \le -a). \tag{3}$$

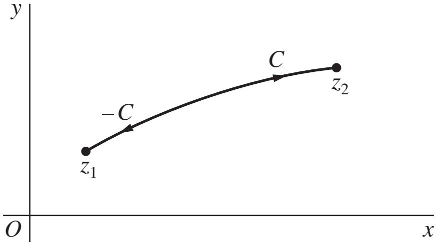


**FIGURE 39**

Also, if $C_1$ is a contour from $z_1$ to $z_2$ and $C_2$ is a contour from $z_2$ to $z_3$, the resulting contour is called a ***sum*** and we write $C = C_1 + C_2$ (see Fig. 40). Note, too, that the sum of contours $C_1$ and $-C_2$ is well defined when $C_1$ and $C_2$ have the same final points. It is denoted by $C = C_1 - C_2$.

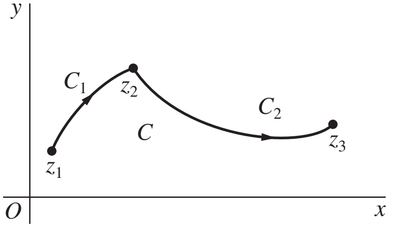


**FIGURE 40**
$C = C_1 + C_2$

In stating properties of contour integrals, we assume that all functions $f(z)$ and $g(z)$ are piecewise continuous on any contour used.

The first property is

$$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz, \tag{4}$$

where $z_0$ is any complex constant. This follows from definition (2) and properties of integrals of complex-valued functions $w(t)$ mentioned in Sec. 42, and the same is true of the property

$$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz. \tag{5}$$

By using representation (3) and referring to Exercise 1(*b*), Sec. 42, one can see that

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f\,[z(-t)]\,\frac{d}{dt}z(-t)\,dt = -\int_{-b}^{-a} f\,[z(-t)]\,z'(-t)\,dt$$

where $z'(-t)$ denotes the derivative of $z(t)$ with respect to $t$, evaluated at $-t$. Then, by making the substitution $\tau = -t$ in this last integral and referring to Exercise 1(*a*), Sec. 43, we obtain the expression

$$\int_{-C} f(z)\,dz = -\int_a^b f[z(\tau)]z'(\tau)\,d\tau,$$

which is the same as

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz. \tag{6}$$

Finally, consider a path $C$, with representation (1), that consists of a contour $C_1$ from $z_1$ to $z_2$ followed by a contour $C_2$ from $z_2$ to $z_3$, the initial point of $C_2$ being the final point of $C_1$ (Fig. 40). There is a value $c$ of $t$, where $a < c < b$, such that $z(c) = z_2$. Consequently, $C_1$ is represented by

$$z = z(t) \qquad (a \le t \le c)$$

and $C_2$ is represented by

$$z = z(t) \qquad (c \le t \le b).$$

Also, by a rule for integrals of functions $w(t)$ that was noted in Sec. 42,

$$\int_a^b f[z(t)]z'(t)\,dt = \int_a^c f[z(t)]z'(t)\,dt + \int_c^b f[z(t)]z'(t)\,dt.$$

Evidently, then,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz. \tag{7}$$

## 45. SOME EXAMPLES

The purpose of this and the next section is to illustrate how contour integrals are to be evaluated when definition (2), Sec. 44, of such integrals is used and to illustrate some of the properties of contour integrals that were mentioned in Sec. 44. We defer development of antiderivatives until Sec. 48.

**EXAMPLE 1.** Let us evaluate the contour integral

$$\int_{C_1} \frac{dz}{z}$$

where $C_1$ is the top half

$$z = e^{i\theta} \qquad (0 \le \theta \le \pi)$$

of the circle $|z| = 1$ from $z = 1$ to $z = -1$ (see Fig. 41). According to definition (2), Sec. 44,

$$\int_{C_1} \frac{dz}{z} = \int_0^{\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = i \int_0^{\pi} d\theta = \pi i. \tag{1}$$

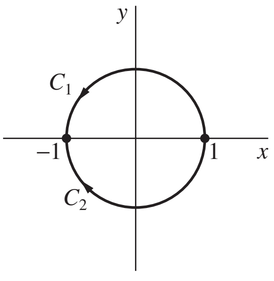


**FIGURE 41**
$C = C_1 - C_2$

Now let us evaluate the integral

$$\int_{C_2} \frac{dz}{z}$$

over the *bottom* half of the same circle $|z| = 1$ from $z = 1$ to $z = -1$, also shown in Fig. 41. To evaluate this integral, we use the parametric representation

$$z = e^{i\theta} \qquad (\pi \le \theta \le 2\pi)$$

of the contour $-C_2$. Then

$$\int_{C_2} \frac{dz}{z} = -\int_{-C_2} \frac{dz}{z} = -\int_{\pi}^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = -i \int_{\pi}^{2\pi} d\theta = -\pi i. \tag{2}$$

Note that the values of integrals (1) and (2) are not the same. Note, too, that if $C$ is the closed curve $C = C_1 - C_2$, then

$$\int_{C} \frac{dz}{z} = \int_{C_1} \frac{dz}{z} - \int_{C_2} \frac{dz}{z} = \pi i - (-\pi i) = 2\pi i. \tag{3}$$

**EXAMPLE 2.** We begin here by letting $C$ denote an arbitrary smooth arc (Sec. 43)

$$z = z(t) \qquad (a \le t \le b)$$

from a fixed point $z_1$ to a fixed point $z_2$ (Fig. 42). In order to evaluate the integral

$$\int_C z\, dz = \int_a^b z(t) z'(t)\, dt,$$

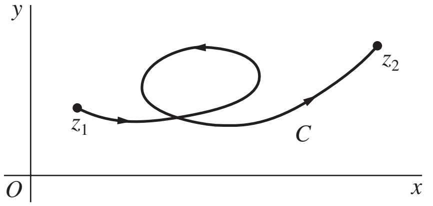


FIGURE 42

we note that according to Example 1, Sec. 41,

$$\frac{d}{dt}\frac{[z(t)]^2}{2} = z(t)z'(t).$$

Then, because $z(a) = z_1$ and $z(b) = z_2$, we have

$$\int_C z\,dz = \frac{[z(t)]^2}{2}\Bigg]_a^b = \frac{[z(b)]^2 - [z(a)]^2}{2} = \frac{z_2^2 - z_1^2}{2}.$$

Inasmuch as the value of this integral depends only on the end points of $C$ and is otherwise independent of the arc that is taken, we may write

$$\int_{z_1}^{z_2} z\,dz = \frac{z_2^2 - z_1^2}{2}. \tag{4}$$

Expression (4) is also valid when $C$ is a contour that is not necessarily smooth since a contour consists of a finite number of smooth arcs $C_k$ $(k = 1, 2, \ldots, n)$, joined end to end. More precisely, suppose that each $C_k$ extends from $z_k$ to $z_{k+1}$. Then

$$\int_C z\,dz = \sum_{k=1}^{n} \int_{C_k} z\,dz = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} z\,dz = \sum_{k=1}^{n} \frac{z_{k+1}^2 - z_k^2}{2} = \frac{z_{n+1}^2 - z_1^2}{2}, \tag{5}$$

where this last summation has telescoped and $z_1$ is the initial point of $C$ and $z_{n+1}$ is its final point.

If $f(z)$ is given in the form $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, one can sometimes apply definition (2), Sec. 44, using one of the variables $x$ and $y$ as the parameter.

**EXAMPLE 3.** Here we first let $C_1$ denote the polygonal line $OAB$ shown in Fig. 43 and evaluate the integral

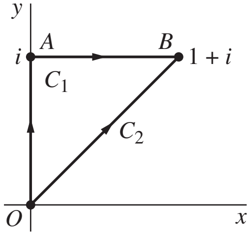


FIGURE 43
$C = C_1 - C_2$

(6) $$I_1 = \int_{C_1} f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz,$$

where

$$f(z) = y - x - i3x^2 \quad (z = x + iy).$$

The leg *OA* may be represented parametrically as $z = 0 + iy$ $(0 \le y \le 1)$; and, since $x = 0$ at points on that line segment, the values of $f$ there vary with the parameter $y$ according to the equation $f(z) = y$ $(0 \le y \le 1)$. Consequently,

$$\int_{OA} f(z)\,dz = \int_0^1 yi\,dy = i\int_0^1 y\,dy = \frac{i}{2}.$$

On the leg $AB$, the points are $z = x + i$ $(0 \le x \le 1)$; and, since $y = 1$ on this segment,

$$\int_{AB} f(z)\,dz = \int_0^1 (1 - x - i3x^2)\cdot 1\,dx = \int_0^1 (1-x)\,dx - 3i\int_0^1 x^2\,dx = \frac{1}{2} - i.$$

In view of equation (6), we now see that

(7) $$I_1 = \frac{1-i}{2}.$$

If $C_2$ denotes the segment $OB$ of the line $y = x$ in Fig. 43, with parametric representation $z = x + ix$ $(0 \le x \le 1)$, the fact that $y = x$ on *OB* enables us to write

$$I_2 = \int_{C_2} f(z)\,dz = \int_0^1 -i3x^2(1+i)\,dx = 3(1-i)\int_0^1 x^2\,dx = 1 - i.$$

Evidently, then, the integrals of $f(z)$ along the two paths $C_1$ and $C_2$ have *different values* even though those paths have the same initial and the same final points.

Observe how it follows that the integral of $f(z)$ over the simple closed contour *OABO*, or $C_1 - C_2$, has the *nonzero value*

$$I_1 - I_2 = \frac{-1+i}{2}.$$

These three examples serve to illustrate the following important facts about contour integrals:

(*a*) the value of a contour integral of a given function from one fixed point to another might be independent of the path taken (Example 2), but that is not always the case (Examples 1 and 3);

(*b*) contour integrals of a given function around every closed contour might all have value zero (Example 2), but that is not always the case (Examples 1 and 3).

The question of predicting when contour integrals are independent of path or always have value zero when the path is closed will be taken up in Secs. 48, 50, and 52.

**EXAMPLE 2.** Using the principal branch

$$f(z) = z^{-1+i} = \exp[(-1+i)\mathrm{Log}z] \qquad (|z| > 0, -\pi < \mathrm{Arg}\, z < \pi)$$

of the power function $z^{-1+i}$, let us evaluate the integral

$$I = \int_C z^{-1+i}\, dz \tag{3}$$

where $C$ is the positively oriented unit circle (Fig. 45)

$$z = e^{i\theta} \qquad (-\pi \le \theta \le \pi)$$

about the origin.

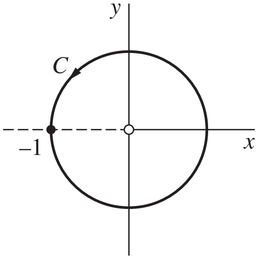


**FIGURE 45**

When $z(\theta) = e^{i\theta}$, it is easy to see that

$$f[z(\theta)]z'(\theta) = e^{(-1+i)(\ln 1 + i\theta)} i e^{i\theta} = i e^{-\theta}. \tag{4}$$

Inasmuch as the function (4) is piecewise continuous on $-\pi < \theta < \pi$, integral (3) exists. In fact,

$$I = i\int_{-\pi}^{\pi} e^{-\theta} d\theta = i\,[-\,e^{-\theta}]_{-\pi}^{\pi} = i(-e^{-\pi} + e^{\pi}),$$

or

$$I = i\,2\frac{e^{\pi} - e^{-\pi}}{2} = i\,2\sinh\pi.$$

## EXERCISES

For the functions $f$ and contours $C$ in Exercises 1 through 8, use parametric representations for $C$, or legs of $C$, to evaluate

$$\int_C f(z)\, dz.$$

**1.** $f(z) = (z+2)/z$ and $C$ is

($a$) the semicircle $z = 2\,e^{i\theta}$ $(0 \le \theta \le \pi)$;

($b$) the semicircle $z = 2\,e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

($c$) the circle $z = 2\,e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

*Ans.* ($a$) $-4 + 2\pi i$; ($b$) $4 + 2\pi i$; ($c$) $4\pi i$.

**2.** $f(z) = z - 1$ and $C$ is the arc from $z = 0$ to $z = 2$ consisting of

($a$) the semicircle $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

($b$) the segment $z = x$ $(0 \le x \le 2)$ of the real axis.

*Ans.* ($a$) 0; ($b$) 0.

**3.** $f(z) = \pi \exp(\pi \bar{z})$ and $C$ is the boundary of the square with vertices at the points 0, 1, $1 + i$, and $i$, the orientation of $C$ being in the counterclockwise direction.

*Ans.* $4(e^{\pi} - 1)$.

**4.** $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and $C$ is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

*Ans.* $2 + 3i$.

**5.** $f(z) = 1$ and $C$ is an arbitrary contour from any fixed point $z_1$ to any fixed point $z_2$ in the $z$ plane.

*Ans.* $z_2 - z_1$.

**6.** $f(z)$ is the principal branch

$$z^i = \exp(i \operatorname{Log} z) \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of the power function $z^i$, and $C$ is the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

*Ans.* $-\dfrac{1 + e^{-\pi}}{2}(1 - i)$.

**7.** $f(z)$ is the principal branch

$$z^{-1-2i} = \exp[(-1 - 2i)\operatorname{Log} z] \qquad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of the indicated power function, and $C$ is the contour

$$z = e^{i\theta} \qquad \left(0 \le \theta \le \frac{\pi}{2}\right).$$

*Ans.* $i\dfrac{e^{\pi} - 1}{2}$.

**8.** $f(z)$ is the principal branch

$$z^{a-1} = \exp[(a - 1)\operatorname{Log} z] \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of the power function $z^{a-1}$, where $a$ is a nonzero real number, and $C$ is the positively oriented circle of radius $R$ about the origin.

*Ans.* $i\dfrac{2R^a}{a}\sin a\pi$, where the positive value of $R^a$ is to be taken.

**9.** Let $C$ denote the positively oriented unit circle $|z| = 1$ about the origin.

($a$) Show that if $f(z)$ is the principal branch

$$z^{-3/4} = \exp\left[-\frac{3}{4}\operatorname{Log} z\right] \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

of $z^{-3/4}$, then

$$\int_C f(z)dz = 4\sqrt{2}\,i.$$

(*b*) Show that if $g(z)$ is the branch

$$z^{-3/4} = \exp\left[-\frac{3}{4}\log z\right] \quad (|z| > 0, 0 < \arg z < 2\pi)$$

of the same power function as in part (*a*), then

$$\int_C g(z)dz = -4 + 4i.$$

This exercise demonstrates how the value of an integral of a power function depends in general on the branch that is used.

**10.** With the aid of the result in Exercise 3, Sec. 42, evaluate the integral

$$\int_C z^m\,\overline{z}^n dz,$$

where $m$ and $n$ are integers and $C$ is the unit circle $|z| = 1$, taken counterclockwise.

**11.** Let $C$ denote the semicircular path shown in Fig. 46. Evaluate the integral of the function $f(z) = \overline{z}$ along $C$ using the parametric representation (see Exercise 2, Sec. 43)

(*a*) $z = 2e^{i\theta}\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$; (*b*) $z = \sqrt{4 - y^2} + iy \quad (-2 \le y \le 2)$.

*Ans.* $4\pi i$.

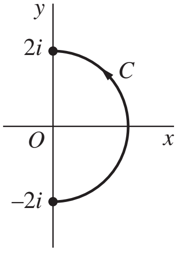


**FIGURE 46**

**12.** (*a*) Suppose that a function $f(z)$ is continuous on a smooth arc $C$, which has a parametric representation $z = z(t)$ $(a \le t \le b)$; that is, $f[z(t)]$ is continuous on the interval $a \le t \le b$. Show that if $\phi(\tau)$ $(\alpha \le \tau \le \beta)$ is the function described in Sec. 43, then

$$\int_a^b f[z(t)]z'(t)\,dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)\,d\tau$$

where $Z(\tau) = z[\phi(\tau)]$.

(*b*) Point out how it follows that the identity obtained in part (*a*) remains valid when $C$ is any contour, not necessarily a smooth one, and $f(z)$ is piecewise continuous on $C$. Thus show that the value of the integral of $f(z)$ along $C$ is the same when the representation $z = Z(\tau)$ $(\alpha \le \tau \le \beta)$ is used, instead of the original one.

*Suggestion:* In part (*a*), use the result in Exercise 1(*b*), Sec. 43, and then refer to expression (14) in that section.

**13.** Let $C_0$ denote the circle centered at $z_0$ with radius $R$, and use the parametrization

$$z = z_0 + Re^{i\theta} \qquad (-\pi \le \theta \le \pi)$$

to show that

$$\int_{C_0} (z - z_0)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots, \\ 2\pi i & \text{when } n = 0. \end{cases}$$

(Put $z_0 = 0$ and then compare the result with the one in Exercise 8 when the constant $a$ there is a nonzero integer.)

## 47. UPPER BOUNDS FOR MODULI OF CONTOUR INTEGRALS

We turn now to an inequality involving contour integrals that is extremely important in various applications. We present the result as a theorem but preface it with a needed lemma involving functions $w(t)$ of the type encountered in Secs. 41 and 42.

***Lemma.*** *If $w(t)$ is a piecewise continuous complex-valued function defined on an interval $a \le t \le b$, then*

$$\left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt. \tag{1}$$

This inequality clearly holds when the value of the integral on the left is zero. Thus, in the verification, we may assume that its value is a *nonzero* complex number and write

$$\int_a^b w(t)\,dt = r_0\,e^{i\theta_0}. \tag{2}$$

Solving for $r_0$, we have

$$r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt. \tag{3}$$

Now the left-hand side of this equation is a real number, and so the right-hand side is too. Thus, using the fact that the real part of a real number is the number itself, we find that

$$r_0 = \text{Re} \int_a^b e^{-i\theta_0} w(t)\,dt.$$

Hence, in view of the first of properties (3) in Sec. 42,

$$r_0 = \int_a^b \text{Re}[e^{-i\theta_0} w(t)]\,dt. \tag{4}$$

But

$$\text{Re}[e^{-i\theta_0} w(t)] \le |e^{-i\theta_0} w(t)| = |e^{-i\theta_0}|\,|w(t)| = |w(t)|,$$

and it follows from equation (4) that

$$r_0 \le \int_a^b |w(t)|\,dt.$$

Finally, equation (2) tells us that $r_0$ is the same as the left-hand side of inequality (1), and the verification of the lemma is complete.

***Theorem.*** *Let $C$ denote a contour of length $L$, and suppose that a function $f(z)$ is piecewise continuous on $C$. If $M$ is a nonnegative constant such that*

$$|f(z)| \le M \tag{5}$$

*for all points $z$ on $C$ at which $f(z)$ is defined, then*

$$\left| \int_C f(z)\,dz \right| \le ML. \tag{6}$$

To obtain inequality (6), we assume that inequality (5) holds and let

$$z = z(t) \qquad (a \le t \le b)$$

be a parametric representation of $C$. According to the lemma,

$$\left| \int_C f(z)\,dz \right| = \left| \int_a^b f[z(t)]z'(t)\,dt \right| \le \int_a^b |f[z(t)]z'(t)|\,dt.$$

Inasmuch as

$$|f[z(t)]z'(t)| = |f[z(t)]|\,|z'(t)| \le M\,|z'(t)|$$

when $a \le t \le b$, except possibly for a finite number of points, it follows that

$$\left| \int_C f(z)\,dz \right| \le M \int_a^b |z'(t)|\,dt.$$

Since the integral on the right here represents the length $L$ of $C$ (see Sec. 43), inequality (6) is established. It is, of course, a strict inequality if inequality (5) is strict.

Note that since $C$ is a contour and $f$ is piecewise continuous on $C$, a number $M$ such as the one appearing in inequality (5) will always exist. This is because the real-valued function $|f[z(t)]|$ is continuous on the closed bounded interval $a \le t \le b$ when $f$ is continuous on $C$; and such a function always reaches a maximum value $M$ on that interval.* Hence $|f(z)|$ has a maximum value on $C$ when $f$ is continuous on it. The same is, then, true when $f$ is *piecewise* continuous on $C$.

*See, for instance A. E. Taylor and W. R. Mann, "*Advanced Calculus*," 3d ed., pp. 86–90, 1983.

**EXAMPLE 1.** Let $C$ be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant (Fig. 47). Inequality (6) can be used to show that

$$\left| \int_C \frac{z-2}{z^4+1}\, dz \right| \le \frac{4\pi}{15}. \tag{7}$$

This is done by noting first that if $z$ is a point on $C$, then

$$|z-2| = |z + (-2)| \le |z| + |-2| = 2 + 2 = 4$$

and

$$|z^4 + 1| \ge ||z|^4 - 1| = 15.$$

Thus, when $z$ lies on $C$,

$$\left| \frac{z-2}{z^4+1} \right| = \frac{|z-2|}{|z^4+1|} \le \frac{4}{15}.$$

By writing $M = 4/15$ and observing that $L = \pi$ is the length of $C$, we may now use inequality (6) to obtain inequality (7).

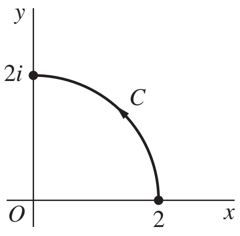


**FIGURE 47**

**EXAMPLE 2.** Let $C_R$ denote the semicircle

$$z = Re^{i\theta} \qquad (0 \le \theta \le \pi)$$

from $z = R$ to $z = -R$, where $R > 3$ (Fig. 48). It is easy to show that

$$\lim_{R\to\infty} \int_{C_R} \frac{(z+1)\, dz}{(z^2+4)(z^2+9)} = 0 \tag{8}$$

without actually evaluating the integral. To do this, we observe that if $z$ is a point on $C_R$,

$$|z+1| \le |z| + 1 = R + 1,$$
$$|z^2 + 4| \ge ||z|^2 - 4| = R^2 - 4,$$

and

$$|z^2 + 9| \ge ||z|^2 - 9| = R^2 - 9.$$

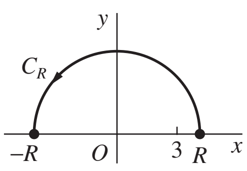


**FIGURE 48**

This means that if $z$ is on $C_R$ and $f(z)$ is the integrand in integral (8), then

$$|f(z)| = \left|\frac{z+1}{(z^2+4)\,(z^2+9)}\right| = \frac{|z+1|}{|z^2+4|\,|z^2+9|} \le \frac{R+1}{(R^2-4)\,(R^2-9)} = M_R,$$

where $M_R$ serves as an upper bound for $|f(z)|$ on $C_R$. Since the length of the semicircle is $\pi R$, we may refer to the theorem in this section, using

$$M_R = \frac{R+1}{(R^2-4)\,(R^2-9)} \quad \text{and} \quad L = \pi R,$$

to write

$$\left|\int_{C_R} \frac{(z+1)\,dz}{(z^2+4)(z^2+9)}\right| \le M_R L \tag{9}$$

where

$$M_R L = \frac{\pi(R^2+R)}{(R^2-4)\,(R^2-9)} \cdot \frac{\dfrac{1}{R^4}}{\dfrac{1}{R^4}} = \frac{\pi\left(\dfrac{1}{R^2}+\dfrac{1}{R^3}\right)}{\left(1-\dfrac{4}{R^2}\right)\left(1-\dfrac{9}{R^2}\right)}.$$

This shows that $M_R L \to 0$ as $R \to \infty$, and limit (8) follows from inequality (9).

## EXERCISES

**1.** Without evaluating the integral, show that

$$(a)\ \left|\int_C \frac{z+4}{z^3-1}dz\right| \le \frac{6\pi}{7}; \qquad (b)\ \left|\int_C \frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$$

when $C$ is the arc that was used in Example 1, Sec. 47.

**2.** Let $C$ denote the line segment from $z = i$ to $z = 1$ (Fig. 49), and show that

$$\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$$

without evaluating the integral.

*Suggestion:* Observe that of all the points on the line segment, the midpoint is closest to the origin, that distance being $d = \sqrt{2}/2$.

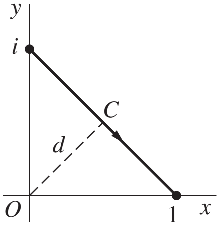


**FIGURE 49**

**3.** Show that if $C$ is the boundary of the triangle with vertices at the points $0$, $3i$, and $-4$, oriented in the counterclockwise direction (see Fig. 50), then

$$\left| \int_C (e^z - \bar{z})\, dz \right| \le 60.$$

*Suggestion:* Note that $|e^z - \bar{z}| \le e^x + \sqrt{x^2 + y^2}$ when $z = x + iy$.

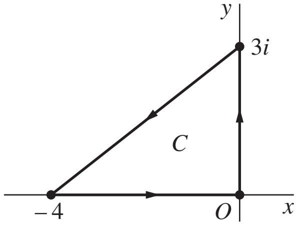


**FIGURE 50**

**4.** Let $C_R$ denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, by dividing the numerator and denominator on the right here by $R^4$, show that the value of the integral tends to zero as $R$ tends to infinity. (Compare with Example 2 in Sec. 47.)

**5.** Let $C_R$ be the circle $|z| = R$ $(R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left( \frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as $R$ tends to infinity.

**6.** Let $C_\rho$ denote a circle $|z| = \rho$ $(0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of $z$, then there is a nonnegative constant $M$, *independent of* $\rho$, such that

$$\left| \int_{C_\rho} z^{-1/2} f(z)\, dz \right| \le 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as $\rho$ tends to 0.

*Suggestion:* Note that since $f(z)$ is analytic, and therefore continuous, throughout the disk $|z| \le 1$, it is bounded there (Sec. 18).