Hence

$$(z_2 z_3)^i = \left[e^{\pi/4} e^{i(\ln 2)/2}\right]\left[e^{3\pi/4} e^{i(\ln 2)/2}\right] e^{-2\pi},$$

or

$$(z_2 z_3)^i = z_2^i z_3^i \, e^{-2\pi}. \tag{2}$$

EXERCISES

1. Show that

(*a*) $(1+i)^i = \exp\left(-\dfrac{\pi}{4} + 2n\pi\right)\exp\left(i\dfrac{\ln 2}{2}\right) \quad (n = 0, \pm 1, \pm 2, \ldots);$

(*b*) $\dfrac{1}{i^{2i}} = \exp[(4n+1)\pi] \quad (n = 0, \pm 1, \pm 2, \ldots).$

2. Find the principal value of

(*a*) $(-i)^i$; (*b*) $\left[\dfrac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$; (*c*) $(1-i)^{4i}$.

Ans. (*a*) $\exp(\pi/2)$; (*b*) $-\exp(2\pi^2)$; (*c*) $e^{\pi}[\cos(2\ln 2) + i\sin(2\ln 2)]$.

3. Use definition (1), Sec. 35, of z^c to show that $(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}$.

4. Show that the result in Exercise 3 could have been obtained by writing

(*a*) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^{1/2}]^3$ and first finding the square roots of $-1+\sqrt{3}i$;

(*b*) $(-1+\sqrt{3}i)^{3/2} = [(-1+\sqrt{3}i)^3]^{1/2}$ and first cubing $-1+\sqrt{3}i$.

5. Show that the *principal* nth root of a nonzero complex number z_0 that was defined in Sec. 10 is the same as the principal value of $z_0^{1/n}$ defined by equation (3), Sec. 35.

6. Show that if $z \neq 0$ and a is a real number, then $|z^a| = \exp(a \ln|z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

7. Let $c = a + bi$ be a fixed complex number, where $c \neq 0, \pm 1, \pm 2, \ldots$, and note that i^c is multiple-valued. What additional restriction must be placed on the constant c so that the values of $|i^c|$ are all the same?

Ans. c is real.

8. Let c, c_1, c_2, and z denote complex numbers, where $z \neq 0$. Prove that if all of the powers involved are principal values, then

(*a*) $z^{c_1} z^{c_2} = z^{c_1+c_2}$; (*b*) $\dfrac{z^{c_1}}{z^{c_2}} = z^{c_1-c_2}$;

(*c*) $(z^c)^n = z^{cn} \quad (n = 1, 2, \ldots)$.

9. Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.

37. THE TRIGONOMETRIC FUNCTIONS $\sin z$ AND $\cos z$

Euler's formula (Sec. 7) tells us that

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x$$

for every real number x. Hence

$$e^{ix} - e^{-ix} = 2i \sin x \quad \text{and} \quad e^{ix} + e^{-ix} = 2\cos x.$$

That is,

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

It is, therefore, natural to define ***the sine and cosine functions*** of a complex variable z as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{1}$$

These functions are entire since they are linear combinations (Exercise 3, Sec. 26) of the entire functions e^{iz} and e^{-iz}. Knowing the derivatives

$$\frac{d}{dz} e^{iz} = ie^{iz} \quad \text{and} \quad \frac{d}{dz} e^{-iz} = -ie^{-iz}$$

of those exponential functions, we find from equations (1) that

$$\frac{d}{dz} \sin z = \cos z \quad \text{and} \quad \frac{d}{dz} \cos z = -\sin z. \tag{2}$$

It is easy to see from definitions (1) that the sine and cosine functions remain odd and even, respectively:

$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z. \tag{3}$$

Also,

$$e^{iz} = \cos z + i \sin z. \tag{4}$$

This is, of course, Euler's formula (Sec. 7) when z is real.

A variety of identities carry over from trigonometry. For instance (see Exercises 2 and 3),

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \tag{5}$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2. \tag{6}$$

From these, it follows readily that

$$\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z, \tag{7}$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z, \quad \sin\left(z - \frac{\pi}{2}\right) = -\cos z, \tag{8}$$

and [Exercise 4(a)]

$$\sin^2 z + \cos^2 z = 1. \tag{9}$$

The periodic character of $\sin z$ and $\cos z$ is also evident:

$$\sin(z + 2\pi) = \sin z, \quad \sin(z + \pi) = -\sin z, \tag{10}$$

$$\cos(z + 2\pi) = \cos z, \quad \cos(z + \pi) = -\cos z. \tag{11}$$

When y is any real number, definitions (1) and the hyperbolic functions

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2}$$

from calculus can be used to write

$$\sin(iy) = i \sinh y \quad \text{and} \quad \cos(iy) = \cosh y. \tag{12}$$

Also, the real and imaginary components of $\sin z$ and $\cos z$ can be displayed in terms of those hyperbolic functions:

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \tag{13}$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y, \tag{14}$$

where $z = x + iy$. To obtain expressions (13) and (14), we write

$$z_1 = x \quad \text{and} \quad z_2 = iy$$

in identities (5) and (6) and then refer to relations (12). Observe that once expression (13) is obtained, relation (14) also follows from the fact (Sec. 21) that if the derivative of a function

$$f(z) = u(x, y) + iv(x, y)$$

exists at a point $z = (x, y)$, then

$$f'(z) = u_x(x, y) + iv_x(x, y).$$

Expressions (13) and (14) can be used (Exercise 7, Sec. 38), to show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \tag{15}$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y. \tag{16}$$

Inasmuch as $\sinh y$ tends to infinity as y tends to infinity, it is clear from these two equations that $\sin z$ and $\cos z$ are *not bounded* on the complex plane, whereas the absolute values of $\sin x$ and $\cos x$ are less than or equal to unity for all values of x. (See the definition of a bounded function at the end of Sec. 18.)

38. ZEROS AND SINGULARITIES OF TRIGONOMETRIC FUNCTIONS

A ***zero*** of a given function f is a number z_0 such that $f(z_0) = 0$. It is possible that a function of a real variable can have more zeros when the domain of definition is enlarged.

EXAMPLE. The function $f(x) = x^2 + 1$, defined on the real line, has no zeros. But the function $f(z) = z^2 + 1$, defined on the complex plane, has the zeros $z = \pm i$.

Consider now the sine function $f(z) = \sin z$ that was introduced in Sec. 37. Since $\sin z$ becomes the usual sine function $\sin x$ in calculus when z is real, we know that the real numbers

$$z = n\pi \qquad (n = 0, \pm 1, 2, \ldots)$$

are zeros of $\sin z$. One might ask if there are other zeros in the entire plane, and a similar question can be asked regarding the cosine function.

Theorem. *The zeros of* $\sin z$ *and* $\cos z$ *in the complex plane are the same as the zeros of* $\sin x$ *and* $\cos x$ *on the real line. That is,*

$$\sin z = 0 \qquad \text{if and only if} \qquad z = n\pi \quad (n = 0, \pm 1, 2, \ldots)$$

and

$$\cos z = 0 \qquad \text{if and only if} \qquad z = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

In order to prove this theorem, we consider first the sine function and assume that $\sin z = 0$. Since $\sin z$ becomes the usual sine function in calculus when z is real, we know that the real numbers $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$ are all zeros of $\sin z$. To show that *there are no other zeros,* we assume that $\sin z = 0$ and note how it follows from equation (15), Sec. 37, that

$$\sin^2 x + \sinh^2 y = 0.$$

This sum of two squares reveals that

$$\sin x = 0 \quad \text{and} \quad \sinh y = 0.$$

Evidently, then, $x = n\pi$ $(n = 0, \pm 1, 2, \ldots)$ and $y = 0$. Hence the zeros of $\sin z$ are as stated in the theorem.

As for the cosine function, the second of relations (8) in Sec. 37 tells us that

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right);$$

and it follows that the zeros of $\cos z$ are also the ones in the statement of the theorem.

The other four trigonometric functions are defined in terms of the sine and cosine functions by the expected relations:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \tag{1}$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}. \tag{2}$$

Observe that the quotients $\tan z$ and $\sec z$ are analytic everywhere except at the singularities (Sec. 25)

$$z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

which are the zeros of $\cos z$. Likewise, $\cot z$ and $\csc z$ have singularities at the zeros of $\sin z$, namely

$$z = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

By differentiating the right-hand sides of equations (1) and (2), we obtain the anticipated differentiation formulas

$$\frac{d}{dz} \tan z = \sec^2 z, \qquad \frac{d}{dz} \cot z = -\csc^2 z, \tag{3}$$

$$\frac{d}{dz} \sec z = \sec z \tan z, \qquad \frac{d}{dz} \csc z = -\csc z \cot z. \tag{4}$$

The periodicity of each of the trigonometric functions defined by equations (1) and (2) follows readily from equations (10) and (11) in Sec. 37. For example,

$$\tan(z + \pi) = \tan z. \tag{5}$$

Mapping properties of the transformation $w = \sin z$ are especially important in the applications later on. A reader who wishes at this time to learn some of those properties is sufficiently prepared to read Secs. 104 and 105 (Chap. 8), where they are discussed.

EXERCISES

1. Give details in the derivation of expressions (2), Sec. 37, for the derivatives of $\sin z$ and $\cos z$.

2. (*a*) With the aid of expression (4), Sec. 37, show that

$$e^{iz_1} e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Then use relations (3), Sec. 37, to show how it follows that

$$e^{-iz_1} e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

(*b*) Use the results in part (*a*) and the fact that

$$\sin(z_1 + z_2) = \frac{1}{2i}\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right] = \frac{1}{2i}\left(e^{iz_1} e^{iz_2} - e^{-iz_1} e^{-iz_2}\right)$$

to obtain the identity

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

in Sec. 37.

3. According to the final result in Exercise 2(*b*),

$$\sin(z + z_2) = \sin z \cos z_2 + \cos z \sin z_2.$$

By differentiating each side here with respect to z and then setting $z = z_1$, derive the expression

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

that was stated in Sec. 37.

4. Verify identity (9) in Sec. 37 using

(a) identity (6) and relations (3) in that section;

(b) the lemma in Sec. 28 and the fact that the entire function

$$f(z) = \sin^2 z + \cos^2 z - 1$$

has zero values along the x axis.

5. Use identity (9) in Sec. 37 to show that

(a) $1 + \tan^2 z = \sec^2 z$; (b) $1 + \cot^2 z = \csc^2 z$.

6. Establish differentiation formulas (3) and (4) in Sec. 38.

7. In Sec. 37, use expressions (13) and (14) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.

Suggestion: Recall the identities $\sin^2 x + \cos^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

8. Point out how it follows from expressions (15) and (16) in Sec. 37 for $|\sin z|^2$ and $|\cos z|^2$ that

(a) $|\sin z| \geq |\sin x|$; (b) $|\cos z| \geq |\cos x|$.

9. With the aid of expressions (15) and (16) in Sec. 37 for $|\sin z|^2$ and $|\cos z|^2$, show that

(a) $|\sinh y| \leq |\sin z| \leq \cosh y$; ($b$) $|\sinh y| \leq |\cos z| \leq \cosh y$.

10. (a) Use definitions (1), Sec. 37, of $\sin z$ and $\cos z$ to show that

$$2\sin(z_1 + z_2)\sin(z_1 - z_2) = \cos 2z_2 - \cos 2z_1.$$

(b) With the aid of the identity obtained in part (a), show that if $\cos z_1 = \cos z_2$, then at least one of the numbers $z_1 + z_2$ and $z_1 - z_2$ is an integral multiple of 2π.

11. Use the Cauchy–Riemann equations and the theorem in Sec. 21 to show that neither $\sin \overline{z}$ nor $\cos \overline{z}$ is an analytic function of z anywhere.

12. Use the reflection principle (Sec. 29) to show that for all z,

(a) $\overline{\sin z} = \sin \overline{z}$; ($b$) $\overline{\cos z} = \cos \overline{z}$.

13. With the aid of expressions (13) and (14) in Sec. 37, give direct verifications of the relations obtained in Exercise 12.

14. Show that

(a) $\overline{\cos(iz)} = \cos(i\overline{z})$ for all z;

(b) $\overline{\sin(iz)} = \sin(i\overline{z})$ if and only if $z = n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

15. Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts and then the imaginary parts of $\sin z$ and $\cosh 4$.

Ans. $\left(\dfrac{\pi}{2} + 2n\pi\right) \pm 4i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

16. With the aid of expression (14), Sec. 37, show that the roots of the equation $\cos z = 2$ are

$$z = 2n\pi + i\cosh^{-1} 2 \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Then express them in the form

$$z = 2n\pi \pm i\ln(2 + \sqrt{3}) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

39. HYPERBOLIC FUNCTIONS

The ***hyperbolic sine and cosine functions*** of a complex variable z are defined as they are with a real variable:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{1}$$

Since e^z and e^{-z} are entire, it follows from definitions (1) that $\sinh z$ and $\cosh z$ are entire. Furthermore,

$$\frac{d}{dz}\sinh z = \cosh z, \quad \frac{d}{dz}\cosh z = \sinh z. \tag{2}$$

Because of the way in which the exponential function appears in definitions (1) and in the definitions (Sec. 37)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

of $\sin z$ and $\cos z$, the hyperbolic sine and cosine functions are closely related to those trigonometric functions:

$$-i\sinh(iz) = \sin z, \quad \cosh(iz) = \cos z, \tag{3}$$

$$-i\sin(iz) = \sinh z, \quad \cos(iz) = \cosh z. \tag{4}$$

Note how it follows readily from relations (4) and the periodicity of $\sin z$ and $\cos z$ that $\sinh z$ and $\cosh z$ are *periodic with period* $2\pi i$.

Some of the most frequently used identities involving hyperbolic sine and cosine functions are

$$\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z, \tag{5}$$

$$\cosh^2 z - \sinh^2 z = 1, \tag{6}$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2, \tag{7}$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \tag{8}$$

and

$$\sinh z = \sinh x \cos y + i \cosh x \sin y, \tag{9}$$

$$\cosh z = \cosh x \cos y + i \sinh x \sin y, \tag{10}$$

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y, \tag{11}$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y, \tag{12}$$

where $z = x + iy$. While these identities follow directly from definitions (1), they are often more easily obtained from related trigonometric identities, with the aid of relations (3) and (4).

EXAMPLE 1. To illustrate the method of proof just suggested, let us verify identity (6), starting with the relation

$$\sin^2 z + \cos^2 z = 1 \tag{13}$$

in Sec. 37. Using relations (3) to replace $\sin z$ and $\cos z$ in relation (13) here, we have

$$-\sinh^2(iz) + \cosh^2(iz) = 1.$$

Then, replacing z by $-iz$ in this last equation, we arrive at identity (6).

EXAMPLE 2. Let us verify expression (12) using the second of relations (4). We begin by writing

$$|\cosh z|^2 = |\cos(iz)|^2 = |\cos(-y + ix)|^2. \tag{14}$$

Now we already know from relation (16) in Sec. 37 that

$$|\cos(x + iy)|^2 = \cos^2 x + \sinh^2 y,$$

and this tells us that

$$|\cos(-y + ix)|^2 = \cos^2 y + \sinh^2 x. \tag{15}$$

Expressions (14) and (15) now combine to yield relation (12).

We turn now to the zeros of $\sinh z$ and $\cosh z$. We present the results as a theorem in order to emphasize their importance in later chapters and in order to provide easy comparison with the theorem in Sec. 38, regarding the zeros of $\sin z$ and $\cos z$. In fact, the theorem here is an immediate consequence of relations (4) and that earlier theorem.

Theorem. *The zeros of* $\sinh z$ *and* $\cosh z$ *in the complex plane all lie on the imaginary axis. To be specific,*

$$\sinh z = 0 \qquad \textit{if and only if} \qquad z = n\pi i \quad (n = 0, \pm 1, 2, \ldots)$$

and

$$\cosh z = 0 \qquad \textit{if and only if} \qquad z = \left(\frac{\pi}{2} + n\pi\right) i \quad (n = 0, \pm 1, \pm 2, \ldots).$$

The ***hyperbolic tangent*** of z is defined by means of the equation

(16) $$\tanh z = \frac{\sinh z}{\cosh z}$$

and is analytic in every domain in which $\cosh z \neq 0$. The functions $\coth z$, $\operatorname{sech} z$, and $\operatorname{csch} z$ are the reciprocals of $\tanh z$, $\cosh z$, and $\sinh z$, respectively. It is straightforward to verify the following differentiation formulas, which are the same as those established in calculus for the corresponding functions of a real variable:

(17) $$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z, \qquad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z,$$

(18) $$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z, \qquad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z.$$

EXERCISES

1. Verify that the derivatives of $\sinh z$ and $\cosh z$ are as stated in equations (2), Sec. 39.

2. Prove that $\sinh 2z = 2 \sinh z \cosh z$ by starting with

(*a*) definitions (1), Sec. 39, of $\sinh z$ and $\cosh z$;

(*b*) the identity $\sin 2z = 2 \sin z \cos z$ (Sec. 37) and using relations (3) in Sec. 39.

3. Show how identities (6) and (8) in Sec. 39 follow from identities (9) and (6), respectively, in Sec. 37.

4. Write $\sinh z = \sinh(x + iy)$ and $\cosh z = \cosh(x + iy)$, and then show how expressions (9) and (10) in Sec. 39 follow from identities (7) and (8), respectively, in that section.

5. Derive expression (11) in Sec. 39 for $|\sinh z|^2$.

6. Show that $|\sinh x| \le |\cosh z| \le \cosh x$ by using

(*a*) identity (12), Sec. 39;

(*b*) the inequalities $|\sinh y| \le |\cos z| \le \cosh y$, obtained in Exercise 9(*b*), Sec. 38.

7. Show that

(*a*) $\sinh(z + \pi i) = -\sinh z$; (*b*) $\cosh(z + \pi i) - \cosh z$;

(*c*) $\tanh(z + \pi i) = \tanh z$.

8. Give details showing that the zeros of $\sinh z$ and $\cosh z$ are as in the theorem in Sec. 39.

9. Using the results proved in Exercise 8, locate all zeros and singularities of the hyperbolic tangent function.

10. Show that $\tanh z = -i \tan(iz)$.
Suggestion: Use identities (4) in Sec. 39.

11. Derive differentiation formulas (17), Sec. 39.

12. Use the reflection principle (Sec. 29) to show that for all z,

(*a*) $\overline{\sinh z} = \sinh \overline{z}$; (*b*) $\overline{\cosh z} = \cosh \overline{z}$.

13. Use the results in Exercise 12 to show that $\overline{\tanh z} = \tanh \overline{z}$ at points where $\cosh z \neq 0$.

14. By accepting that the stated identity is valid when z is replaced by the real variable x and using the lemma in Sec. 28, verify that

(*a*) $\cosh^2 z - \sinh^2 z = 1$; (*b*) $\sinh z + \cosh z = e^z$.

[Compare with Exercise 4(*b*), Sec. 38.]

15. Why is the function $\sinh(e^z)$ entire? Write its real component as a function of x and y, and state why that function must be harmonic everywhere.

16. By using one of the identities (9) and (10) in Sec. 39 and then proceeding as in Exercise 15, Sec. 38, find all roots of the equation

(*a*) $\sinh z = i$; (*b*) $\cosh z = \dfrac{1}{2}$.

Ans. (*a*) $z = \left(2n + \dfrac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(*b*) $z = \left(2n \pm \dfrac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

17. Find all roots of the equation $\cosh z = -2$. (Compare this exercise with Exercise 16, Sec. 38.)

Ans. $z = \pm \ln(2 + \sqrt{3}) + (2n + 1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$.

40. INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms.

In order to define the inverse sine function $\sin^{-1} z$, we write

$$w = \sin^{-1} z \quad \text{when} \quad z = \sin w.$$

That is, $w = \sin^{-1} z$ when

$$z = \frac{e^{iw} - e^{-iw}}{2i}.$$

If we put this equation in the form

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0,$$

which is quadratic in e^{iw}, and solve for e^{iw} [see Exercise 8(*a*), Sec. 11], we find that

$$e^{iw} = iz + (1 - z^2)^{1/2} \tag{1}$$

where $(1 - z^2)^{1/2}$ is, of course, a double-valued function of z. Taking logarithms of each side of equation (1) and recalling that $w = \sin^{-1} z$, we arrive at the expression

$$\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}]. \tag{2}$$

The following example emphasizes the fact that $\sin^{-1} z$ is a multiple-valued function, with infinitely many values at each point z.

EXAMPLE. Expression (2) tells us that

$$\sin^{-1}(-i) = -i \log(1 \pm \sqrt{2}).$$

But

$$\log(1+\sqrt{2}) = \ln(1+\sqrt{2}) + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\log(1-\sqrt{2}) = \ln(\sqrt{2}-1) + (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Since

$$\ln(\sqrt{2}-1) = \ln \frac{1}{1+\sqrt{2}} = -\ln(1+\sqrt{2}),$$

then, the numbers

$$(-1)^n \ln(1+\sqrt{2}) + n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

constitute the set of values of $\log(1 \pm \sqrt{2})$. Thus, in rectangular form,

$$\sin^{-1}(-i) = n\pi + i(-1)^{n+1} \ln(1+\sqrt{2}) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

One can apply the technique used to derive expression (2) for $\sin^{-1} z$ to show that

$$\cos^{-1} z = -i \log\left[z + i(1-z^2)^{1/2}\right] \tag{3}$$

and that

$$\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}. \tag{4}$$

The functions $\cos^{-1} z$ and $\tan^{-1} z$ are also multiple-valued. When specific branches of the square root and logarithmic functions are used, all three inverse functions become single-valued and analytic because they are then compositions of analytic functions.

The derivatives of these three functions are readily obtained from their logarithmic expressions. The derivatives of the first two depend on the values chosen for the square roots:

$$\frac{d}{dz} \sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}, \tag{5}$$

$$\frac{d}{dz} \cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}. \tag{6}$$

The derivative of the last one,

$$\frac{d}{dz} \tan^{-1} z = \frac{1}{1+z^2}, \tag{7}$$

does not, however, depend on the manner in which the function is made single-valued.

Inverse hyperbolic functions can be treated in a corresponding manner. It turns out that

$$\sinh^{-1} z = \log\left[z + (z^2 + 1)^{1/2}\right], \tag{8}$$

$$\cosh^{-1} z = \log\left[z + (z^2 - 1)^{1/2}\right], \tag{9}$$

and

$$\tanh^{-1} z = \frac{1}{2} \log \frac{1 + z}{1 - z}. \tag{10}$$

Finally, we remark that common alternative notation for all of these inverse functions is arcsin z, etc.

EXERCISES

1. Find all the values of

(a) $\tan^{-1}(2i)$; (b) $\tan^{-1}(1 + i)$; (c) $\cosh^{-1}(-1)$; (d) $\tanh^{-1} 0$.

Ans. (a) $\left(n + \frac{1}{2}\right)\pi + \frac{i}{2} \ln 3 \; (n = 0, \pm 1, \pm 2, \ldots)$;

(d) $n\pi i \; (n = 0, \pm 1, \pm 2, \ldots)$.

2. Solve the equation $\sin z = 2$ for z by

(a) equating real parts and then imaginary parts in that equation;

(b) using expression (2), Sec. 40, for $\sin^{-1} z$.

Ans. $z = \left(2n + \frac{1}{2}\right)\pi \pm i \ln(2 + \sqrt{3}) \; (n = 0, \pm 1, \pm 2, \ldots)$.

3. Solve the equation $\cos z = \sqrt{2}$ for z.

4. Derive expression (5), Sec. 40, for the derivative of $\sin^{-1} z$.

5. Derive expression (4), Sec. 40, for $\tan^{-1} z$.

6. Derive expression (7), Sec. 40, for the derivative of $\tan^{-1} z$.

7. Derive expression (9), Sec. 40, for $\cosh^{-1} z$.