CHAPTER

2

ANALYTIC FUNCTIONS

We now consider functions of a complex variable and develop a theory of differentiation for them. The main goal of the chapter is to introduce analytic functions, which play a central role in complex analysis.

13. FUNCTIONS AND MAPPINGS

Let S be a set of complex numbers. A ***function*** f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the ***value*** of f at z and is denoted by $f(z)$, so that $w = f(z)$. The set S is called the ***domain of definition*** of f.*

It must be emphasized that both a domain of definition and a rule are needed in order for a function to be well defined. When the domain of definition is not mentioned, we agree that the largest possible set is to be taken. Also, it is not always convenient to use notation that distinguishes between a given function and its values.

EXAMPLE 1. If f is defined on the set $z \neq 0$ by means of the equation $w = 1/z$, it may be referred to only as the function $w = 1/z$, or simply the function $1/z$.

Suppose that $u + iv$ is the value of a function f at $z = x + iy$; that is,

$$u + iv = f(x + iy).$$

*Although the domain of definition is often a domain as defined in Sec. 12, it need not be.

Each of the real numbers u and v depends on the real variables x and y, and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y:

$$f(z) = u(x, y) + iv(x, y). \tag{1}$$

EXAMPLE 2. If $f(z) = z^2$, then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy.$$

Hence

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

If the function v in equation (1) always has value zero, then the value of f is always real. Thus f is a ***real-valued function*** of a complex variable.

EXAMPLE 3. A real-valued function that is used to illustrate some important concepts later in this chapter is

$$f(z) = |z|^2 = x^2 + y^2 + i0.$$

If n is a positive integer and if $a_0, a_1, a_2, \ldots, a_n$ are complex constants, where $a_n \neq 0$, the function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a ***polynomial*** of degree n. Note that the sum here has a finite number of terms and that the domain of definition is the entire z plane. Quotients $P(z)/Q(z)$ of polynomials are called ***rational functions*** and are defined at each point z where $Q(z) \neq 0$. Polynomials and rational functions constitute elementary, but important, classes of functions of a complex variable.

If the polar coordinates r and θ are used instead of x and y, then

$$u + iv = f(re^{i\theta})$$

where $w = u + iv$ and $z = re^{i\theta}$. In that case, we may write

$$f(z) = u(r, \theta) + iv(r, \theta). \tag{2}$$

EXAMPLE 4. Consider the function $w = z^2$ when $z = re^{i\theta}$. Here

$$w = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta.$$

Hence

$$u(r, \theta) = r^2 \cos 2\theta \quad \text{and} \quad v(r, \theta) = r^2 \sin 2\theta.$$

A generalization of the concept of function is a rule that assigns more than one value to a point z in the domain of definition. These ***multiple-valued functions*** occur

The transformation $w = z^2$ also maps the upper half plane $r \geq 0, 0 \leq \theta \leq \pi$ onto the entire w plane. However, in this case, the transformation is not one to one since both the positive and negative real axes in the z plane are mapped onto the positive real axis in the w plane.

When n is a positive integer greater than 2, various mapping properties of the transformation $w = z^n$, or $w = r^n e^{in\theta}$, are similar to those of $w = z^2$. Such a transformation maps the entire z plane onto the entire w plane, where each nonzero point in the w plane is the image of n distinct points in the z plane. The circle $r = r_0$ is mapped onto the circle $\rho = r_0^n$; and the sector $r \leq r_0, 0 \leq \theta \leq 2\pi/n$ is mapped onto the disk $\rho \leq r_0^n$, but not in a one to one manner.

Other, but somewhat more involved, mappings by $w = z^2$ appear in Example 1, Sec. 107, and Exercises 1 through 4 Sec. 108.

EXERCISES

1. For each of the functions below, describe the domain of definition that is understood:

(a) $f(z) = \dfrac{1}{z^2+1}$; (b) $f(z) = \text{Arg}\left(\dfrac{1}{z}\right)$;

(c) $f(z) = \dfrac{z}{z+\bar{z}}$; (d) $f(z) = \dfrac{1}{1-|z|^2}$.

Ans. (a) $z \neq \pm i$; (b) $\text{Re}\, z \neq 0$.

2. In each case, write the function $f(z)$ in the form $f(z) = u(x, y) + iv(x, y)$:

(a) $f(z) = z^3 + z + 1$; (b) $f(z) = \dfrac{\bar{z}^2}{z}$ $(z \neq 0)$.

Suggestion: In part (b), start by multiplying the numerator and denominator by $\bar{z}$.

Ans. (a) $f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$;

(b) $f(z) = \dfrac{x^3 - 3xy^2}{x^2 + y^2} + i\dfrac{y^3 - 3x^2y}{x^2+y^2}$.

3. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions (see Sec. 6)

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}$$

to write $f(z)$ in terms of z, and simplify the result.

Ans. $f(z) = \bar{z}^2 + 2iz$.

4. Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

$$Ans.\ f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

5. By referring to the discussion in Sec. 14 related to Fig. 19 there, find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1$, $u = 2$, $v = 1$, and $v = 2$. (See Fig. 2, Appendix 2.)

6. Find and sketch, showing corresponding orientations, the images of the hyperbolas

$$x^2 - y^2 = c_1\ (c_1 < 0) \quad \text{and} \quad 2xy = c_2\ (c_2 < 0)$$

under the transformation $w = z^2$.

7. Use rays indicated by dashed half lines in Fig. 21 to show that the transformation $w = z^2$ maps the first quadrant onto the upper half plane, as shown in Fig. 21.

8. Sketch the region onto which the sector $r \le 1, 0 \le \theta \le \pi/4$ is mapped by the transformation (a) $w = z^2$; (b) $w = z^3$; (c) $w = z^4$.

9. One interpretation of a function $w = f(z) = u(x, y) + iv(x, y)$ is that of a ***vector field*** in the domain of definition of f. The function assigns a vector w, with components $u(x, y)$ and $v(x, y)$, to each point z at which it is defined. Indicate graphically the vector fields represented by

(a) $w = iz$; (b) $w = \dfrac{z}{|z|}$.

15. LIMITS

Let a function f be defined at all points z in some deleted neighborhood of a point z_0. The statement that $f(z)$ has a ***limit*** w_0 as z approaches z_0, or that

$$\lim_{z \to z_0} f(z) = w_0, \tag{1}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it. We now express the definition of limit in a precise and usable form.

Statement (1) means that for each positive number ε, there is a positive number δ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \tag{2}$$

Geometrically, this definition says that for each ε neighborhood $|w - w_0| < \varepsilon$ of w_0, there is a deleted δ neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε neighborhood (Fig. 22). Note that even though all points in the deleted neighborhood $0 < |z - z_0| < \delta$ are to be considered, their images need not fill up the entire neighborhood $|w - w_0| < \varepsilon$. If f has the constant value w_0, for instance, the image of z is always the center of that neighborhood. Note, too, that once a δ has been found, it can be replaced by any smaller positive number, such as $\delta/2$.

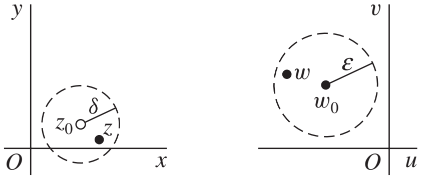

FIGURE 22

The following theorem on uniqueness of limits is central to much of this chapter, especially the material in Sec. 21.

Theorem. *When a limit of a function $f(z)$ exists at a point z_0, it is unique.*

To prove this, we suppose that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} f(z) = w_1.$$

Then, for each positive number ε, there are positive numbers δ_0 and δ_1 such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_0$$

and

$$|f(z) - w_1| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta_1.$$

Since

$$w_1 - w_0 = [f(z) - w_0] + [w_1 - f(z)],$$

the triangle inequality tells us that

$$|w_1 - w_0| \le [f(z) - w_0] + [w_1 - f(z)] = |f(z) - w_0| + |f(z) - w_1|.$$

So if $0 < |z - z_0| < \delta$ where δ is any positive number smaller than δ_0 and δ_1, we find that

$$|w_1 - w_0| < \varepsilon + \varepsilon < 2\varepsilon.$$

But $|w_1 - w_0|$ is a nonnegative constant, and ε can be chosen arbitrarily small. Hence

$$w_1 - w_0 = 0, \quad \text{or} \quad w_1 = w_0.$$

Definition (2) requires that f be defined at all points in some deleted neighborhood of z_0. Such a deleted neighborhood, of course, always exists when z_0 is an interior point of a region on which f is defined. We can extend the definition of limit to the case in which z_0 is a boundary point of the region by agreeing that the first of inequalities (2) need be satisfied by only those points z that lie in both the region *and* the deleted neighborhood.

EXAMPLE 1. Let us show that if $f(z) = i\bar{z}/2$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}, \tag{3}$$

the point 1 being on the boundary of the domain of definition of f. Observe that when z is in the disk $|z| < 1$,

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\overline{z}}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}.$$

Hence, for any such z and each positive number ε (see Fig. 23),

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \quad \text{whenever} \quad 0 < |z-1| < 2\varepsilon.$$

Thus condition (2) is satisfied by points in the region $|z| < 1$ when δ is equal to 2ε or any smaller positive number.

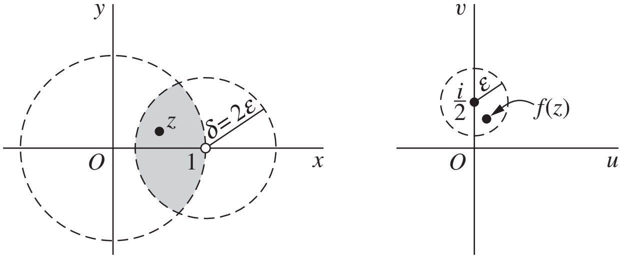

FIGURE 23

If limit (1) exists, the symbol $z \to z_0$ implies that z is allowed to approach z_0 in an arbitrary manner, not just from some particular direction. The next example emphasizes this.

EXAMPLE 2. If

$$f(z) = \frac{z}{\overline{z}}, \tag{4}$$

the limit

$$\lim_{z \to 0} f(z) \tag{5}$$

does not exist. For, if it did exist, it could be found by letting the point $z = (x, y)$ approach the origin in any manner. But when $z = (x, 0)$ is a nonzero point on the real axis (Fig. 24),

$$f(z) = \frac{x + i0}{x - i0} = 1;$$

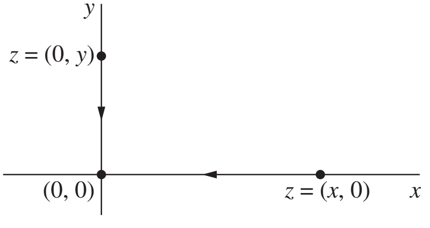

FIGURE 24

and when $z = (0, y)$ is a nonzero point on the imaginary axis,

$$f(z) = \frac{0 + iy}{0 - iy} = -1.$$

Thus, by letting z approach the origin along the real axis, we would find that the desired limit is 1. An approach along the imaginary axis would, on the other hand, yield the limit -1. Since a limit is unique, we must conclude that limit (5) does not exist.

While definition (2) provides a means of testing whether a given point w_0 is a limit, it does not directly provide a method for determining that limit. Theorems on limits, presented in the next section, will enable us to actually find many limits.

16. THEOREMS ON LIMITS

We can expedite our treatment of limits by establishing a connection between limits of functions of a complex variable and limits of real-valued functions of two real variables. Since limits of the latter type are studied in calculus, we may use their definition and properties freely.

Theorem 1. *Suppose that*

$$f(z) = u(x, y) + iv(x, y) \quad (z = x + iy)$$

and

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0.$$

If

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0, \tag{1}$$

Then

$$\lim_{z\to z_0} f(z) = w_0; \tag{2}$$

and, conversely, if statement (2) *is true, then so is statement* (1).

To prove the theorem, we first assume that limits (1) hold and obtain limit (2). Limits (1) tell us that for each positive number ε, there exist positive numbers δ_1 and δ_2 such that

$$|u - u_0| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1 \tag{3}$$

and

$$|v - v_0| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2. \tag{4}$$

Let δ be any positive number smaller than δ_1 and δ_2. Since

$$|(u+iv)-(u_0+iv_0)| = |(u-u_0)+i(v-v_0)| \le |u-u_0|+|v-v_0|$$

and

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = |(x-x_0)+i(y-y_0)| = |(x+iy)-(x_0+iy_0)|,$$

it follows from statements (3) and (4) that

$$|(u+iv)-(u_0+iv_0)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

whenever

$$0 < |(x+iy)-(x_0+iy_0)| < \delta.$$

That is, limit (2) holds.

Let us now start with the assumption that limit (2) holds. With that assumption, we know that for each positive number ε, there is a positive number δ such that

$$|(u+iv)-(u_0+iv_0)| < \varepsilon \tag{5}$$

whenever

$$0 < |(x+iy)-(x_0+iy_0)| < \delta. \tag{6}$$

But

$$|u-u_0| \le |(u-u_0)+i(v-v_0)| = |(u+iv)-(u_0+iv_0)|,$$
$$|v-v_0| \le |(u-u_0)+i(v-v_0)| = |(u+iv)-(u_0+iv_0)|,$$

and

$$|(x+iy)-(x_0+iy_0)| = |(x-x_0)+i(y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}.$$

Hence it follows from inequalities (5) and (6) that

$$|u-u_0| < \varepsilon \quad \text{and} \quad |v-v_0| < \varepsilon$$

whenever

$$0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta.$$

This establishes limits (1), and the proof of the theorem is complete.

Theorem 2. *Suppose that*

$$\lim_{z\to z_0} f(z) = w_0 \quad \textit{and} \quad \lim_{z\to z_0} F(z) = W_0. \tag{7}$$

Then

$$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0, \tag{8}$$

$$\lim_{z \to z_0} [f(z)F(z)] = w_0 W_0; \tag{9}$$

and, if $W_0 \neq 0$,

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}. \tag{10}$$

This important theorem can be proved directly by using the definition of the limit of a function of a complex variable. But, with the aid of Theorem 1, it follows almost immediately from theorems on limits of real-valued functions of two real variables.

To verify property (9), for example, we write

$$f(z) = u(x, y) + iv(x, y), \quad F(z) = U(x, y) + iV(x, y),$$

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0, \quad W_0 = U_0 + iV_0.$$

Then, according to hypotheses (7) and Theorem 1, the limits as (x, y) approaches (x_0, y_0) of the functions u, v, U, and V exist and have the values u_0, v_0, U_0, and V_0, respectively. So the real and imaginary components of the product

$$f(z)F(z) = (uU - vV) + i(vU + uV)$$

have the limits $u_0U_0 - v_0V_0$ and $v_0U_0 + u_0V_0$, respectively, as (x, y) approaches (x_0, y_0). Hence, by Theorem 1 again, $f(z)F(z)$ has the limit

$$(u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0)$$

as z approaches z_0; and this is equal to w_0W_0. Property (9) is thus established. Corresponding verifications of properties (8) and (10) can be given.

It is easy to see from definition (2), Sec. 15, of limit that

$$\lim_{z \to z_0} c = c \quad \text{and} \quad \lim_{z \to z_0} z = z_0,$$

where z_0 and c are any complex numbers; and, by property (9) and mathematical induction, it follows that

$$\lim_{z \to z_0} z^n = z_0^n \qquad (n = 1, 2, \ldots).$$

So, in view of properties (8) and (9), the limit of a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

as z approaches a point z_0 is the value of the polynomial at that point:

$$\lim_{z \to z_0} P(z) = P(z_0). \tag{11}$$

Furthermore,

$$\lim_{z\to\infty} \frac{2z^3 - 1}{z^2 + 1} = \infty \quad \text{since} \quad \lim_{z\to 0} \frac{(1/z^2) + 1}{(2/z^3) - 1} = \lim_{z\to 0} \frac{z + z^3}{2 - z^3} = 0.$$

18. CONTINUITY

A function f is ***continuous*** at a point z_0 if all three of the following conditions are satisfied:

(1) $$\lim_{z\to z_0} f(z) \text{ exists},$$

(2) $$f(z_0) \text{ exists},$$

(3) $$\lim_{z\to z_0} f(z) = f(z_0).$$

Observe that statement (3) actually contains statements (1) and (2), since the existence of the quantity on each side of the equation in that statement is needed. Statement (3) says, of course, that for each positive number ε, there is a positive number δ such that

(4) $$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

A function of a complex variable is said to be continuous in a region R if it is continuous at each point in R.

If two functions are continuous at a point, their sum and product are also continuous at that point; their quotient is continuous at any such point if the denominator is not zero there. These observations are direct consequences of Theorem 2, Sec. 16. Note, too, that a polynomial is continuous in the entire plane because of limit (11) in Sec. 16.

We turn now to two expected properties of continuous functions whose verifications are not so immediate. Our proofs depend on definition (4) of continuity, and we present the results as theorems.

Theorem 1. *A composition of continuous functions is itself continuous.*

A precise statement of this theorem is contained in the proof to follow. We let $w = f(z)$ be a function that is defined for all z in a neighborhood $|z - z_0| < \delta$ of a point z_0, and we let $W = g(w)$ be a function whose domain of definition contains the image (Sec. 13) of that neighborhood under f. The composition $W = g[f(z)]$ is, then, defined for all z in the neighborhood $|z - z_0| < \delta$. Suppose now that f is continuous at z_0 and that g is continuous at the point $f(z_0)$ in the w plane. In view of the continuity of g at $f(z_0)$, there is, for each positive number ε, a positive number γ such that

$$|g[f(z)] - g[f(z_0)]| < \varepsilon \quad \text{whenever} \quad |f(z) - f(z_0)| < \gamma.$$

(See Fig. 26.) But the continuity of f at z_0 ensures that the neighborhood $|z - z_0| < \delta$ can be made small enough that the second of these inequalities holds. The continuity of the composition $g[f(z)]$ is, therefore, established.

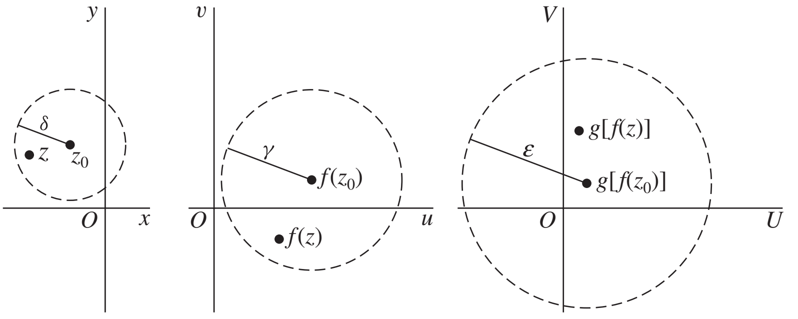

FIGURE 26

Theorem 2. *If a function $f(z)$ is continuous and nonzero at a point z_0, then $f(z) \neq 0$ throughout some neighborhood of that point.*

Assuming that $f(z)$ is, in fact, continuous and nonzero at z_0, we can prove Theorem 2 by assigning the positive value $|f(z_0)|/2$ to the number ε in statement (4). This tells us that there is a positive number δ such that

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2} \quad \text{whenever} \quad |z - z_0| < \delta.$$

So if there is a point z in the neighborhood $|z - z_0| < \delta$ at which $f(z) = 0$, we have the contradiction

$$|f(z_0)| < \frac{|f(z_0)|}{2};$$

and the theorem is proved.

The continuity of a function

$$f(z) = u(x, y) + iv(x, y) \tag{5}$$

is closely related to the continuity of its component functions $u(x, y)$ and $v(x, y)$, as the following theorem indicates.

Theorem 3. *If the component functions u and v in expression* (5) *are continuous at a point $z_0 = (x_0, y_0)$, then so is f. Conversely, if f is continuous at z_0, the same is true of u and v at that point.*

The proof follows immediately from Theorem 1 in Sec. 16, regarding the connection between limits of f and limits of u and v.

The next theorem is extremely important and will be used often in later chapters, especially in applications. Before stating the theorem, whose proof is based on Theorem 3, we recall from Sec. 12 that a region R is *closed* if it contains all of its boundary points and that it is *bounded* if it lies inside some circle centered at the origin.

Theorem 4. *If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that*

$$|f(z)| \le M \qquad \text{for all points } z \text{ in } R, \tag{6}$$

where equality holds for at least one such z.

To prove this, we assume that the function f in equation (5) is continuous and note how it follows that the function

$$\sqrt{[u(x, y)]^2 + [v(x, y)]^2}$$

is continuous throughout R and thus reaches a maximum value M somewhere in R.* Inequality (6) thus holds, and we say that f is ***bounded on R***.

EXERCISES

1. Use definition (2), Sec. 15, of limit to prove that

(*a*) $\lim_{z\to z_0} \operatorname{Re} z = \operatorname{Re} z_0$; (*b*) $\lim_{z\to z_0} \overline{z} = \overline{z_0}$; (*c*) $\lim_{z\to 0} \dfrac{\overline{z}^2}{z} = 0$.

2. Let a, b, and c denote complex constants. Then use definition (2), Sec. 15, of limit to show that

(*a*) $\lim_{z\to z_0} (az + b) = az_0 + b$; (*b*) $\lim_{z\to z_0} (z^2 + c) = z_0^2 + c$;

(*c*) $\lim_{z\to 1-i} [x + i(2x + y)] = 1 + i \quad (z = x + iy)$.

3. Let n be a positive integer and let $P(z)$ and $Q(z)$ be polynomials, where $Q(z_0) \neq 0$. Use Theorem 2 in Sec. 16, as well as limits appearing in that section, to find

(*a*) $\lim_{z\to z_0} \dfrac{1}{z^n} \ (z_0 \neq 0)$; (*b*) $\lim_{z\to i} \dfrac{iz^3 - 1}{z + i}$; (*c*) $\lim_{z\to z_0} \dfrac{P(z)}{Q(z)}$.

Ans. (*a*) $1/z_0^n$; (*b*) 0; (*c*) $P(z_0)/Q(z_0)$.

4. Use mathematical induction and property (9), Sec. 16, of limits to show that

$$\lim_{z\to z_0} z^n = z_0^n$$

when n is a positive integer ($n = 1, 2, \ldots$).

5. Show that the function

$$f(z) = \left(\frac{z}{\overline{z}}\right)^2$$

has the value 1 at all nonzero points on the real and imaginary axes, where $z = (x, 0)$ and $z = (0, y)$, respectively, but that it has the value -1 at all nonzero points on the line $y = x$, where $z = (x, x)$. Thus show that the limit of $f(z)$ as z tends to 0 does

*See, for instance, A. E. Taylor and W. R. Mann, "*Advanced Calculus*," 3d ed., pp. 125–126 and p. 529, 1983.