Expression (4) can be useful in finding powers of complex numbers even when they are given in rectangular form and the result is desired in that form.

**EXAMPLE 1.** In order to put $(-1+i)^7$ in rectangular form, write

$$(-1+i)^7 = (\sqrt{2}\,e^{i3\pi/4})^7 = 2^{7/2}e^{i\,21\pi/4} = (2^3 e^{i5\pi})(2^{1/2}e^{i\,\pi/4}).$$

Because

$$2^3 e^{i5\pi} = (8)(-1) = -\,8$$

and

$$2^{1/2}e^{i\,\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 1+i,$$

we arrive at the desired result: $(-1+i)^7 = -\,8\,(1+i)$.

Finally, we observe that if $r = 1$, equation (4) becomes

$$(e^{i\theta})^n = e^{in\theta} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

When written in the form

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (n = 0, \pm 1, \pm 2, \ldots), \tag{6}$$

this is known as ***de Moivre's formula***. The following example uses a special case of it.

**EXAMPLE 2.** Formula (6) with $n = 2$ tells us that

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta,$$

or

$$\cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta.$$

By equating real parts and then imaginary parts here, we have the familiar trigonometric identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \qquad \sin 2\theta = 2\sin\theta\cos\theta.$$

(See also Exercises 10 and 11, Sec. 9.)

## 9. ARGUMENTS OF PRODUCTS AND QUOTIENTS

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, the expression

$$z_1 z_2 = (r_1 r_2)e^{i(\theta_1+\theta_2)} \tag{1}$$

in Sec. 8 can be used to obtain an important identity involving arguments:

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{2}$$

Equation (2) is to be interpreted as saying that if values of two of the three (multiple-valued) arguments are specified, then there is a value of the third such that the equation holds.

We start the verification of statement (2) by letting $\theta_1$ and $\theta_2$ denote any values of $\arg z_1$ and $\arg z_2$, respectively. Expression (1) then tells us that $\theta_1 + \theta_2$ is a value of $\arg(z_1 z_2)$. (See Fig. 9.) If, on the other hand, values of $\arg(z_1 z_2)$ and $\arg z_1$ are specified, those values correspond to particular choices of $n$ and $n_1$ in the expressions

$$\arg(z_1 z_2) = (\theta_1 + \theta_2) + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\arg z_1 = \theta_1 + 2n_1\pi \qquad (n_1 = 0, \pm 1, \pm 2, \ldots).$$

Since

$$(\theta_1 + \theta_2) + 2n\pi = (\theta_1 + 2n_1\pi) + [\theta_2 + 2(n - n_1)\pi],$$

equation (2) is evidently satisfied when the value

$$\arg z_2 = \theta_2 + 2(n - n_1)\pi$$

is chosen. Verification when values of $\arg(z_1 z_2)$ and $\arg z_2$ are specified follows from the fact that statement (2) can also be written

$$\arg(z_2 z_1) = \arg z_2 + \arg z_1.$$

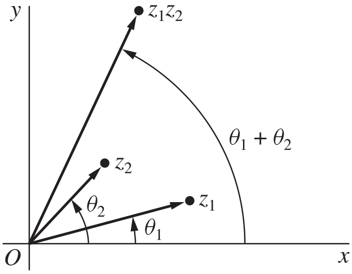

**FIGURE 9**

Statement (2) is sometimes valid when *arg* is replaced everywhere by *Arg* (see Exercise 6). But, as the following example illustrates, that is *not always* the case.

**EXAMPLE 1.** When $z_1 = -1$ and $z_2 = i$,

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(-i) = -\frac{\pi}{2} \quad \text{but} \quad \operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$

If, however, we take the values of $\arg z_1$ and $\arg z_2$ just used and select the value

$$\operatorname{Arg}(z_1 z_2) + 2\pi = -\frac{\pi}{2} + 2\pi = \frac{3\pi}{2}$$

of $\arg(z_1 z_2)$, we find that equation (2) *is* satisfied.

Statement (2) tells us that

$$\arg\left(\frac{z_1}{z_2}\right) = \arg\left(z_1 z_2^{-1}\right) = \arg z_1 + \arg\left(z_2^{-1}\right);$$

and, since (Sec. 8)

$$z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2},$$

one can see that

$$\arg\left(z_2^{-1}\right) = -\arg z_2. \tag{3}$$

Hence

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{4}$$

Statement (3) is, of course, to be interpreted as saying that the set of all values on the left-hand side is the same as the set of all values on the right-hand side. Statement (4) is, then, to be interpreted in the same way that statement (2) is.

**EXAMPLE 2.** In order to illustrate statement (4), let us use it to find the principal value of Arg $z$ when

$$z = \frac{i}{-1-i}.$$

We start by writing

$$\arg z = \arg i - \arg(-1-i).$$

Since

$$\operatorname{Arg} i = \frac{\pi}{2} \quad \text{and} \quad \operatorname{Arg}(-1-i) = -\frac{3\pi}{4},$$

one value of $\arg z$ is $5\pi/4$. But this is not a *principal* value $\Theta$, which must lie in the interval $-\pi < \Theta \le \pi$. We can, however, obtain that value by adding some integral multiple, possibly negative, of $2\pi$:

$$\operatorname{Arg}\left(\frac{i}{-1-i}\right) = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}.$$

## EXERCISES

**1.** Find the principal argument Arg $z$ when

$(a)$ $z = \dfrac{-2}{1+\sqrt{3}\,i}$; $(b)$ $z = \left(\sqrt{3} - i\right)^6$.

*Ans.* $(a)$ $2\pi/3$; $(b)$ $\pi$.

**2.** Show that $(a)$ $|e^{i\theta}| = 1$; $(b)$ $\overline{e^{i\theta}} = e^{-i\theta}$.

**3.** Use mathematical induction to show that

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n} = e^{i(\theta_1+\theta_2+\cdots+\theta_n)} \qquad (n = 2, 3, \ldots).$$

**4.** Using the fact that the modulus $|e^{i\theta} - 1|$ is the distance between the points $e^{i\theta}$ and 1 (see Sec. 4), give a geometric argument to find a value of $\theta$ in the interval $0 \le \theta < 2\pi$ that satisfies the equation $|e^{i\theta} - 1| = 2$.

*Ans.* $\pi$.

**5.** By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(*a*) $i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i)$; (*b*) $5i/(2 + i) = 1 + 2i$;

(*c*) $(\sqrt{3} + i)^6 = -64$; (*d*) $(1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i)$.

**6.** Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2,$$

where principal arguments are used.

**7.** Let $z$ be a nonzero complex number and $n$ a negative integer $(n = -1, -2, \ldots)$. Also, write $z = re^{i\theta}$ and $m = -n = 1, 2, \ldots$ . Using the expressions

$$z^m = r^m e^{im\theta} \quad \text{and} \quad z^{-1} = \left(\frac{1}{r}\right) e^{i(-\theta)},$$

verify that $(z^m)^{-1} = (z^{-1})^m$ and hence that the definition $z^n = (z^{-1})^m$ in Sec. 7 could have been written alternatively as $z^n = (z^m)^{-1}$.

**8.** Prove that two nonzero complex numbers $z_1$ and $z_2$ have the same moduli if and only if there are complex numbers $c_1$ and $c_2$ such that $z_1 = c_1 c_2$ and $z_2 = c_1\overline{c_2}$.

*Suggestion:* Note that

$$\exp\left(i\frac{\theta_1 + \theta_2}{2}\right)\exp\left(i\frac{\theta_1 - \theta_2}{2}\right) = \exp(i\theta_1)$$

and [see Exercise 2(*b*)]

$$\exp\left(i\frac{\theta_1 + \theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1 - \theta_2}{2}\right)} = \exp(i\theta_2).$$

**9.** Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \ne 1)$$

and then use it to derive ***Lagrange's trigonometric identity:***

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

*Suggestion:* As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

**10.** Use de Moivre's formula (Sec. 8) to derive the following trigonometric identities:

(*a*) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$;

(*b*) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

**11.** (*a*) Use the binomial formula (14), Sec. 3, and de Moivre's formula (Sec. 8) to write

$$\cos n\theta + i\sin n\theta = \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta\,(i\sin\theta)^k \qquad (n = 0, 1, 2, \ldots).$$

Then define the integer $m$ by means of the equations

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

and use the above summation to show that [compare with Exercise 10(*a*)]

$$\cos n\theta = \sum_{k=0}^{m}\binom{n}{2k}(-1)^k\cos^{n-2k}\theta\,\sin^{2k}\theta \qquad (n = 0, 1, 2, \ldots).$$

(*b*) Write $x = \cos\theta$ in the final summation in part (*a*) to show that it becomes a polynomial*

$$T_n(x) = \sum_{k=0}^{m}\binom{n}{2k}(-1)^k x^{n-2k}(1-x^2)^k$$

of degree $n$ ($n = 0, 1, 2, \ldots$) in the variable $x$.

## 10. ROOTS OF COMPLEX NUMBERS

Consider now a point $z = re^{i\theta}$, lying on a circle centered at the origin with radius $r$ (Fig. 10). As $\theta$ is increased, $z$ moves around the circle in the counterclockwise direction. In particular, when $\theta$ is increased by $2\pi$, we arrive at the original point; and the same is true when $\theta$ is decreased by $2\pi$. It is, therefore, evident from Fig. 10 that *two nonzero complex numbers*

$$z_1 = r_1 e^{i\theta_1} \quad \textit{and} \quad z_2 = r_2 e^{i\theta_2}$$

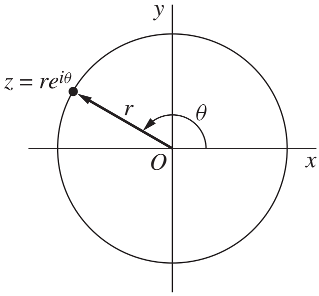

FIGURE 10

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*These are called ***Chebyshev polynomials*** and are prominent in approximation theory.

*are equal if and only if*

$$r_1 = r_2 \quad \textit{and} \quad \theta_1 = \theta_2 + 2k\pi,$$

*where k is any integer* ($k = 0, \pm 1, \pm 2, \ldots$).

This observation, together with the expression $z^n = r^n e^{in\theta}$ in Sec. 8 for integral powers of complex numbers $z = re^{i\theta}$, is useful in finding the $n$th roots of any nonzero complex number $z_0 = r_0 e^{i\theta_0}$, where $n$ has one of the values $n = 2, 3, \ldots$ . The method starts with the fact that an $n$th root of $z_0$ is a nonzero number $z = re^{i\theta}$ such that $z^n = z_0$, or

$$r^n e^{in\theta} = r_0 e^{i\theta_0}.$$

According to the statement in italics just above, then,

$$r^n = r_0 \quad \text{and} \quad n\theta = \theta_0 + 2k\pi,$$

where $k$ is any integer ($k = 0, \pm 1, \pm 2, \ldots$). So $r = \sqrt[n]{r_0}$, where this radical denotes the unique *positive* $n$th root of the positive real number $r_0$, and

$$\theta = \frac{\theta_0 + 2k\pi}{n} = \frac{\theta_0}{n} + \frac{2k\pi}{n} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Consequently, the complex numbers

$$z = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

are $n$th roots of $z_0$. We are able to see immediately from this exponential form of the roots that they all lie on the circle $|z| = \sqrt[n]{r_0}$ about the origin and are equally spaced every $2\pi/n$ radians, starting with argument $\theta_0/n$. Evidently, then, all of the *distinct* roots are obtained when $k = 0, 1, 2, \ldots, n - 1$, and no further roots arise with other values of $k$. We let $c_k$ ($k = 0, 1, 2, \ldots, n - 1$) denote these distinct roots and write

$$(1) \qquad c_k = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad (k = 0, 1, 2, \ldots, n - 1).$$

(See Fig. 11.)

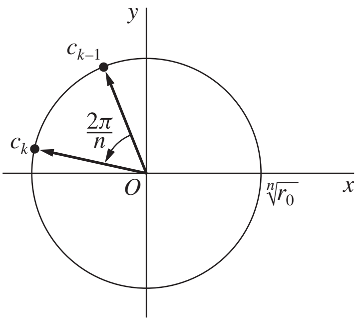

**FIGURE 11**

The number $\sqrt[n]{r_0}$ is the length of each of the radius vectors representing the $n$ roots. The first root $c_0$ has argument $\theta_0/n$; and the two roots when $n = 2$ lie at the opposite ends of a diameter of the circle $|z| = \sqrt[n]{r_0}$, the second root being $-c_0$. When $n \geq 3$, the roots lie at the vertices of a regular polygon of $n$ sides inscribed in that circle.

We shall let $z_0^{1/n}$ denote the *set* of $n$th roots of $z_0$. If, in particular, $z_0$ is a positive real number $r_0$, the symbol $r_0^{1/n}$ denotes the entire set of roots; and the symbol $\sqrt[n]{r_0}$ in expression (1) is reserved for the one positive root. When the value of $\theta_0$ that is used in expression (1) is the principal value of $\arg z_0$ $(-\pi < \theta_0 \leq \pi)$, the number $c_0$ is referred to as the ***principal root.*** Thus when $z_0$ is a positive real number $r_0$, its principal root is $\sqrt[n]{r_0}$.

Observe that if we write expression (1) for the roots of $z_0$ as

$$c_k = \sqrt[n]{r_0}\exp\left(i\frac{\theta_0}{n}\right)\exp\left(i\frac{2k\pi}{n}\right) \qquad (k = 0, 1, 2, \ldots, n-1),$$

and also write

$$\omega_n = \exp\left(i\frac{2\pi}{n}\right), \tag{2}$$

it follows from property (5), Sec. 8, of $e^{i\theta}$ that

$$\omega_n^k = \exp\left(i\frac{2k\pi}{n}\right) \qquad (k = 0, 1, 2, \ldots, n-1) \tag{3}$$

and hence that

$$c_k = c_0\omega_n^k \quad (k = 0, 1, 2, \ldots, n-1). \tag{4}$$

The number $c_0$ here can, of course, be replaced by any particular $n$th root of $z_0$, since $\omega_n$ represents a counterclockwise rotation through $2\pi/n$ radians.

Finally, a convenient way to remember expression (1) is to write $z_0$ in its most general exponential form (compare with Example 2 in Sec. 7)

$$z_0 = r_0\, e^{i(\theta_0 + 2k\pi)} \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{5}$$

and to *formally* apply laws of fractional exponents involving real numbers, keeping in mind that there are precisely $n$ roots:

$$c_k = \left[r_0\, e^{i(\theta_0+2k\pi)}\right]^{1/n} = \sqrt[n]{r_0}\exp\left[\frac{i(\theta_0 + 2k\pi)}{n}\right] = \sqrt[n]{r_0}\exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]$$
$$(k = 0, 1, 2, \ldots, n-1).$$

The examples in the next section serve to illustrate this method for finding roots of complex numbers.

## 11. EXAMPLES

In each of the examples here, we start with expression (5), Sec. 10, and proceed in the manner described just after it.

**EXAMPLE 1.** Let us find all four values of $(-16)^{1/4}$, or all of the fourth roots of the number $-16$. One need only write

$$-16 = 16\exp[i(\pi + 2k\pi)] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

to see that the desired roots are

$$c_k = 2\exp\left[i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right] \qquad (k = 0, 1, 2, 3). \tag{1}$$

They lie at the vertices of a square, inscribed in the circle $|z| = 2$, and are equally spaced around that circle, starting with the principal root (Fig. 12)

$$c_0 = 2\exp\left[i\left(\frac{\pi}{4}\right)\right] = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1+i).$$

Without any further calculations, it is then evident that

$$c_1 = \sqrt{2}(-1+i), \quad c_2 = \sqrt{2}(-1-i), \quad \text{and} \quad c_3 = \sqrt{2}(1-i).$$

Note how it follows from expressions (2) and (4) in Sec. 10 that these roots can be written

$$c_0,\ c_0\omega_4,\ c_0\omega_4^2,\ c_0\omega_4^3 \qquad \text{where} \qquad \omega_4 = \exp\left(i\frac{\pi}{2}\right).$$

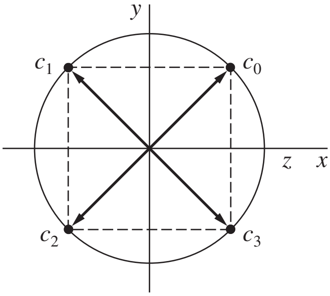

FIGURE 12

**EXAMPLE 2.** In order to determine the $n$th roots of unity, we start with

$$1 = 1\exp[i(0 + 2k\pi)] \qquad (k = 0, \pm 1, \pm 2 \ldots)$$

and find that

$$c_k = \sqrt[n]{1}\exp\left[i\left(\frac{0}{n} + \frac{2k\pi}{n}\right)\right] = \exp\left(i\frac{2k\pi}{n}\right) \qquad (k = 0, 1, 2, \ldots, n-1). \tag{2}$$

When $n = 2$, these roots are, of course, $\pm 1$. When $n \geq 3$, the regular polygon at whose vertices the roots lie is inscribed in the unit circle $|z| = 1$, with one vertex corresponding to the principal root $z = 1$ $(k = 0)$. In view of expression (3), Sec. 10, these roots are simply

$$1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1} \quad \text{where} \quad \omega_n = \exp\left(i\frac{2\pi}{n}\right).$$

See Fig. 13, where the cases $n = 3, 4$, and 6 are illustrated. Note that $\omega_n^n = 1$.

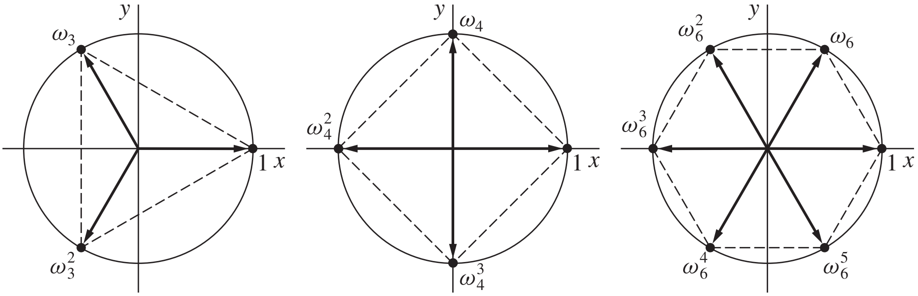

**FIGURE 13**

**EXAMPLE 3.** Let $a$ denote any positive real number. In order to find the two square roots of $a + i$, we first write

$$A = |a + i| = \sqrt{a^2 + 1} \qquad \text{and} \qquad \alpha = \operatorname{Arg}(a + i).$$

Since

$$a + i = A \exp[i(\alpha + 2k\pi)] \quad (k = 0, \pm 1, \pm 2, \ldots),$$

the desired square roots are

$$c_k = \sqrt{A} \exp\left[i\left(\frac{\alpha}{2} + k\pi\right)\right] \quad (k = 0, 1). \tag{3}$$

Because $e^{i\pi} = -1$, these two values of $(a + i)^{1/2}$ reduce to

$$c_0 = \sqrt{A}\, e^{i\alpha/2} \quad \text{and} \quad c_1 = -c_0. \tag{4}$$

Euler's formula tells us that

$$c_0 = \sqrt{A}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right). \tag{5}$$

Because $a + i$ lies above the real axis, we know that $0 < \alpha < \pi$; and so

$$\cos\frac{\alpha}{2} > 0 \quad \text{and} \quad \sin\frac{\alpha}{2} > 0.$$

Hence, in view of the trigonometric identities

$$\cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2}, \qquad \sin^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{2},$$

expression (5) can be put in the form

$$c_0 = \sqrt{A}\left(\sqrt{\frac{1+\cos\alpha}{2}} + i\sqrt{\frac{1-\cos\alpha}{2}}\right). \tag{6}$$

But $\cos\alpha = a/A$, and so

$$\sqrt{\frac{1\pm\cos\alpha}{2}} = \sqrt{\frac{1\pm(a/A)}{2}} = \sqrt{\frac{A\pm a}{2A}}. \tag{7}$$

Consequently, it follows from expression (6) and (7), as well as the relation $c_1 = -\,c_0$, that the two square roots of $a + i(a > 0)$ are (see Fig. 14)

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a} + i\sqrt{A-a}\right). \tag{8}$$

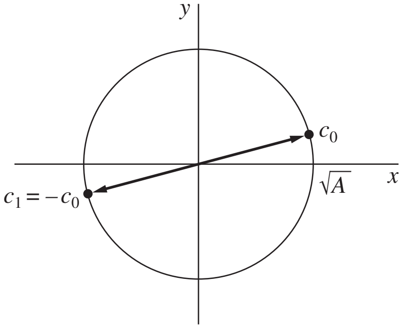

FIGURE 14

## EXERCISES

1. Find the square roots of (*a*) $2i$; (*b*) $1-\sqrt{3}i$ and express them in rectangular coordinates.

   *Ans.* (*a*) $\pm(1+i)$; (*b*) $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

2. Find the three cube roots $c_k\,(k = 0, 1, 2)$ of $-8i$, express them in rectangular coordinates, and point out why they are as shown in Fig. 15.

   *Ans.* $\pm\sqrt{3} - i,\ 2i$.

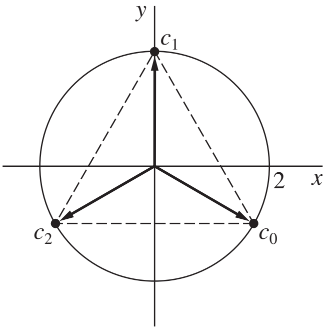

FIGURE 15

**3.** Find $(-8-8\sqrt{3}i)^{1/4}$, express the roots in rectangular coordinates, exhibit them as the vertices of a certain square, and point out which is the principal root.

*Ans.* $\pm(\sqrt{3}-i),\ \pm(1+\sqrt{3}i)$.

**4.** In each case, find all of the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

(*a*) $(-1)^{1/3}$; (*b*) $8^{1/6}$.

*Ans.* (*b*) $\pm\sqrt{2},\ \pm\dfrac{1+\sqrt{3}i}{\sqrt{2}},\ \pm\dfrac{1-\sqrt{3}i}{\sqrt{2}}$.

**5.** According to Sec. 10, the three cube roots of a nonzero complex number $z_0$ can be written $c_0, c_0\omega_3, c_0\omega_3^2$ where $c_0$ is the principal cube root of $z_0$ and

$$\omega_3 = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}.$$

Show that if $z_0 = -4\sqrt{2}+4\sqrt{2}i$, then $c_0 = \sqrt{2}(1+i)$ and the other two cube roots are, in rectangular form, the numbers

$$c_0\omega_3 = \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}, \quad c_0\omega_3^2 = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}.$$

**6.** Find the four zeros of the polynomial $z^4+4$, one of them being

$$z_0 = \sqrt{2}\,e^{i\pi/4} = 1+i.$$

Then use those zeros to factor $z^2+4$ into quadratic factors with real coefficients.

*Ans.* $(z^2+2z+2)(z^2-2z+2)$.

**7.** Show that if $c$ is any $n$th root of unity other than unity itself, then

$$1+c+c^2+\cdots+c^{n-1}=0.$$

*Suggestion:* Use the first identity in Exercise 9, Sec. 9.

**8.** (*a*) Prove that the usual formula solves the quadratic equation

$$az^2+bz+c=0 \qquad (a \neq 0)$$

when the coefficients $a$, $b$, and $c$ are complex numbers. Specifically, by completing the square on the left-hand side, derive the ***quadratic formula***

$$z = \frac{-b+(b^2-4ac)^{1/2}}{2a},$$

where both square roots are to be considered when $b^2-4ac \neq 0$,

(*b*) Use the result in part *(a)* to find the roots of the equation $z^2+2z+(1-i)=0$.

*Ans.* (*b*) $\left(-1+\dfrac{1}{\sqrt{2}}\right)+\dfrac{i}{\sqrt{2}},\quad \left(-1-\dfrac{1}{\sqrt{2}}\right)-\dfrac{i}{\sqrt{2}}$.

**9.** Let $z = re^{i\theta}$ be a nonzero complex number and $n$ a negative integer ($n = -1, -2, \ldots$). Then define $z^{1/n}$ by means of the equation $z^{1/n} = (z^{-1})^{1/m}$ where $m = -n$. By showing that the $m$ values of $(z^{1/m})^{-1}$ and $(z^{-1})^{1/m}$ are the same, verify that $z^{1/n} = (z^{1/m})^{-1}$. (Compare with Exercise 7, Sec. 9.)

## 12. REGIONS IN THE COMPLEX PLANE

In this section, we are concerned with sets of complex numbers, or points in the $z$ plane, and their closeness to one another. Our basic tool is the concept of an $\varepsilon$ **neighborhood**

$$|z - z_0| < \varepsilon \tag{1}$$

of a given point $z_0$. It consists of all points $z$ lying inside but not on a circle centered at $z_0$ and with a specified a positive radius $\varepsilon$ (Fig. 16). When the value of $\varepsilon$ is understood or immaterial in the discussion, the set (1) is often referred to as just a neighborhood. Occasionally, it is convenient to speak of a ***deleted neighborhood***, or punctured disk,

$$0 < |z - z_0| < \varepsilon \tag{2}$$

consisting of all points $z$ in an $\varepsilon$ neighborhood of $z_0$ except for the point $z_0$ itself.

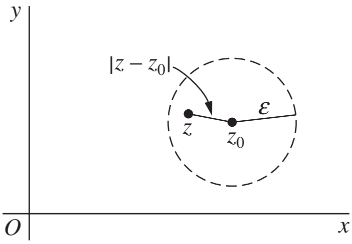

**FIGURE 16**

A point $z_0$ is said to be an ***interior point*** of a set $S$ whenever there is some neighborhood of $z_0$ that contains only points of $S$; it is called an ***exterior point*** of $S$ when there exists a neighborhood of it containing no points of $S$. If $z_0$ is neither of these, it is a ***boundary point*** of $S$. A boundary point is, therefore, a point all of whose neighborhoods contain at least one point in $S$ and at least one point not in $S$. The totality of all boundary points is called the ***boundary*** of $S$. The circle $|z| = 1$, for instance, is the boundary of each of the sets

$$|z| < 1 \quad \text{and} \quad |z| \le 1. \tag{3}$$

A set is ***open*** if it does not contain any of its boundary points. It is left as an exercise to show that a set is open if and only if each of its points is an interior point. A set is ***closed*** if it contains all of its boundary points, and the ***closure*** of a set $S$ is the closed set consisting of all points in $S$ together with the boundary of $S$. Note that the first of sets (3) is open and that the second is its closure.

Some sets are, of course, neither open nor closed. For a set $S$ to be not open there must be a boundary point that is contained in the set, and for $S$ to be not closed there must be a boundary point not in it. Observe that the punctured disk $0 < |z| \leq 1$ is neither open nor closed. The set of all complex numbers is, on the other hand, both open and closed since it has no boundary points.

An open set $S$ is ***connected*** if each pair of points $z_1$ and $z_2$ in it can be joined by a ***polygonal line***, consisting of a finite number of line segments, joined end to end, that lies entirely in $S$. The open set $|z| < 1$ is connected. The annulus $1 < |z| < 2$ is, of course open and it is also connected (see Fig. 17). A nonempty open set that is connected is called a ***domain***. Note that any neighborhood is a domain. A domain together with some, none, or all of its boundary points is usually referred to as a ***region***.

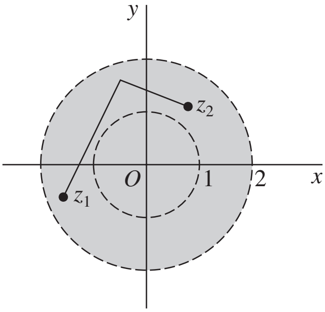

**FIGURE 17**

A set $S$ is ***bounded*** if every point in $S$ lies inside some circle $|z| = R$; otherwise, it is unbounded. Both of the sets (3) are bounded regions, and the half plane $\operatorname{Re} z \geq 0$ is unbounded.

**EXAMPLE.** Let us sketch the set

$$\operatorname{Im}\left(\frac{1}{z}\right) > 1 \tag{4}$$

and identify a few of the properties just described.

First of all, except when $z = 0$,

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} \qquad (z = x + iy).$$

Inequality (4) then becomes

$$\frac{-y}{x^2 + y^2} > 1,$$

or

$$x^2 + y^2 + y < 0.$$

By completing the square, we arrive at

$$x^2 + \left(y^2 + y + \frac{1}{4}\right) < \frac{1}{4}.$$

So inequality (4) represents the region interior to the circle (Fig. 18)

$$(x-0)^2 + \left(y + \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$$

centered at $z = -i/2$ and with radius $1/2$.

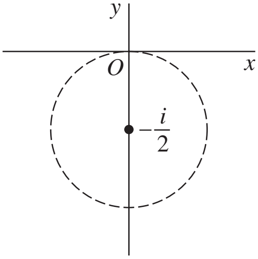

**FIGURE 18**

A point $z_0$ is said to be an ***accumulation point***, or limit point, of a set $S$ if each deleted neighborhood of $z_0$ contains at least one point of $S$. It follows that if a set $S$ is closed, then it contains each of its accumulation points. For if an accumulation point $z_0$ were not in $S$, it would be a boundary point of $S$; but this contradicts the fact that a closed set contains all of its boundary points. It is left as an exercise to show that the converse is, in fact, true. Thus a set is closed if and only if it contains all of its accumulation points.

Evidently, a point $z_0$ is *not* an accumulation point of a set $S$ whenever there exists some deleted neighborhood of $z_0$ that does not contain at least one point in $S$. Note that the origin is the only accumulation point of the set

$$z_n = \frac{i}{n} \quad (n = 1, 2, \ldots).$$

## EXERCISES

1. Sketch the following sets and determine which are domains:

   (*a*) $|z - 2 + i| \le 1$; (*b*) $|2z + 3| > 4$;

   (*c*) $\operatorname{Im} z > 1$; (*d*) $\operatorname{Im} z = 1$;

   (*e*) $0 \le \arg z \le \pi/4$ $(z \ne 0)$; (*f*) $|z - 4| \ge |z|$.

   *Ans.* (*b*), (*c*) are domains.

2. Which sets in Exercise 1 are neither open nor closed?

   *Ans.* (*e*).

3. Which sets in Exercise 1 are bounded?

   *Ans.* (*a*).

**4.** In each case, sketch the closure of the set:

(*a*) $-\pi < \arg z < \pi \ (z \neq 0)$; (*b*) $|\operatorname{Re} z| < |z|$;

(*c*) $\operatorname{Re}\left(\dfrac{1}{z}\right) \leq \dfrac{1}{2}$; (*d*) $\operatorname{Re}(z^2) > 0$.

**5.** Let $S$ be the open set consisting of all points $z$ such that $|z| < 1$ or $|z - 2| < 1$. State why $S$ is not connected.

**6.** Show that a set $S$ is open if and only if each point in $S$ is an interior point.

**7.** Determine the accumulation points of each of the following sets:

(*a*) $z_n = i^n \ (n = 1, 2, \ldots)$; (*b*) $z_n = i^n/n \ (n = 1, 2, \ldots)$;

(*c*) $0 \leq \arg z < \pi/2 \ (z \neq 0)$; (*d*) $z_n = (-1)^n(1+i)\dfrac{n-1}{n} \ (n = 1, 2, \ldots)$.

*Ans*. (*a*) None; (*b*) 0; (*d*) $\pm(1 + i)$.

**8.** Prove that if a set contains each of its accumulation points, then it must be a closed set.

**9.** Show that any point $z_0$ of a domain is an accumulation point of that domain.

**10.** Prove that a finite set of points $z_1, z_2, \ldots, z_n$ cannot have any accumulation points.